It from Qubit 2016
QFT Basics
Problem Set #1

Consider a QFT in 2 spacetime dimensions, with a field ϕ and Euclidean action $S[\phi]$. Denote by $O(x)$ some scalar operator in this theory (it could be ϕ itself, or some other scalar, such at ϕ^2). We will assume the theory is *conformal*. One consequence of conformal symmetry is that correlation functions behave nicely under rescaling all the spacetime coordinates:

$$\langle O(x_1)O(x_2)\cdots O(x_n)\rangle = \lambda^{n\Delta}\langle O(\lambda x_1)O(\lambda x_2)\cdots O(\lambda x_n)\rangle \tag{0.1}$$

where $\Delta > 0$ is called the *scaling dimension* of the operator O.

1. Find the Euclidean 2-point function on a plane,

$$G_E(x_1, x_2) = \langle O(x_1)O(x_2)\rangle \tag{0.2}$$

by assuming a power law form. Here x_i labels a vector in R^2.

2. Now consider the Lorentzian theory with space an infinite line. This is related to the Euclidean theory on a plane by analytic continuation in one of the two coordinates (ie, Wick rotation). Analytically continue your answer from part (1) to find the Lorentzian correlator

$$G_L(t_1, y_1; t_2, y_2) = \langle O(t_1, y_1)O(t_2, y_2)\rangle \tag{0.3}$$

where (t_i, y_i) labels a point in 2d Minkowski spacetime.[1]

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Now we'll consider the same theory on a Euclidean cylinder with period β. The coordinates are $y \in (-\infty, \infty)$ (space) and $t_E \in [0, \beta)$ with $t_E \sim t_E + \beta$ (Euclidean time). It is convenient to label points instead by the complex combinations $w = y + it_E$,

[1] Your answer will have a branch-cut ambiguity when point 2 is causally connected to point 1, ie $(t_2 - t_1)^2 > (y_2 - y_1)^2$. This ambiguity corresponds to a choice of operator ordering in the Lorentzian correlator: $\langle O(t_1, y_1)O(t_2, y_2)\rangle$ vs $\langle O(t_2, y_2)O(t_1, y_1)\rangle$. Don't worry about this ambiguity for this problem, just pick one.

$\bar{w} = y - it_E$. It turns out that in 2d, the correlators on the cylinder are still entirely fixed by conformal symmetry. The 2-point function is

$$\langle O(w_1, \bar{w}_1) O(w_2, \bar{w}_2) \rangle_{cylinder} = \left(\frac{1}{\sinh(\pi(w_1 - w_2)/\beta) \sinh(\pi(\bar{w}_1 - \bar{w}_2)/\beta)} \right)^{\Delta} . \quad (0.4)$$

3. Write the path integral expression that would in principle compute (0.4), and draw a picture of this path integral (like the pictures in lecture).

4. Translate your path integral into operator language by slicing the cylinder 'the short way', that is, defining states on slices of fixed y.

5. Now recast your path integral into operator language by slicing the cylinder 'the long way' (states defined at fixed t_E).

6. Check that your answer in (5) agrees with the operator definition of a thermal correlator derived in lecture, and check that it has the required periodicity in imaginary time. Thus real-time physics at finite temperature is related by analytic continuation to physics on the Euclidean cylinder.

7. Analytically continue $t_E \to it$ to compute the finite-temperature 2-point function in the real-time (Lorentzian) theory. [2]

8. Find the late-time thermal 2-point function, $t_2 \gg t_1$. It should decay exponentially. (This is related to the fact that things thermalize: at late times, you should not be able to tell that a thermal state was perturbed initially.)

9. Now suppose that instead of a line, we choose space to be a circle, so that $y \in (0, R)$ with $y \sim y + R$. Draw a picture of the path integral that would in principle compute the thermal 2-point function.

[2] Again, don't worry about the branch cut, which is a choice of operator ordering. The most obvious choice will compute the time-ordered Lorentzian correlator.