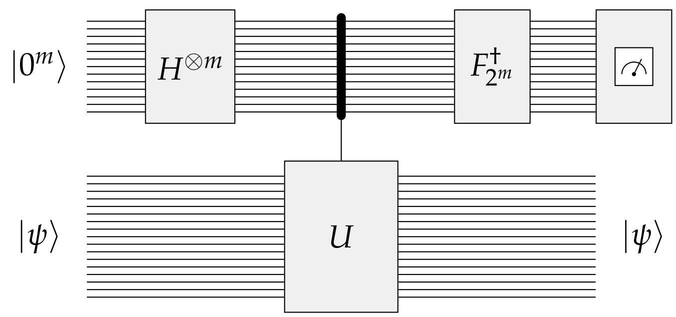
It From Qubit

Problems on Phase Estimation (with solutions)

1. Here is a picture of the phase estimation procedure, for an arbitrary unitary operator U, an eigenvector $|\psi\rangle$ of U, and a number of qubits m that are used to control the precision of the procedure:



Suppose that $|\psi\rangle$ and U happen to satisfy

$$U|\psi\rangle = e^{2\pi i/3}|\psi\rangle .$$

Calculate the probabilities for all possible measurement outcome for $m = 1, 2$, and 3, and compare the results with the best m-bit approximation to $\theta = 1/3$.

Solution. When the phase estimation procedure is run in the case that

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle ,$$

the probability p_j to obtain each outcome $j \in \{0, \ldots, 2^m - 1\}$ is given by this formula:

$$p_j = \left| \frac{1}{2^m} \sum_{k=0}^{2^m-1} e^{2\pi i k(\theta - j/2^m)} \right|^2 .$$

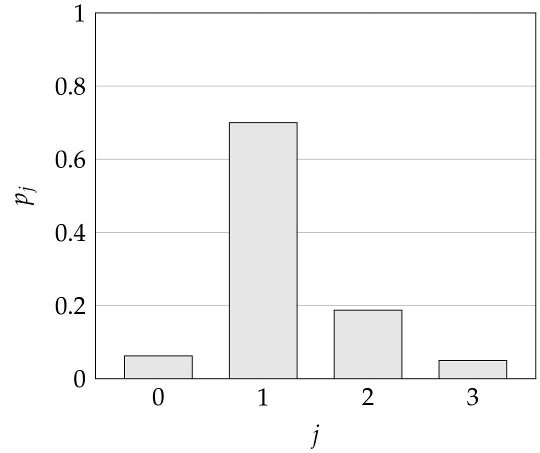
These probabilities are as follows.

(a) For $m = 1$ we have $p_0 = 0.25$ and $p_1 = 0.75$.

(b) For $m = 2$ we have (rounded to 4 digits)

$$\begin{aligned} p_0 &= 0.0625 \\ p_1 &= 0.7000 \\ p_2 &= 0.1875 \\ p_3 &= 0.0500 \end{aligned}$$

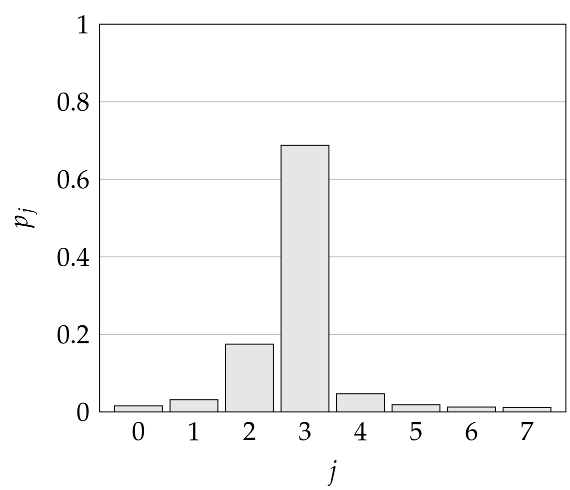
A plot illustrating this distribution is as follows:



(c) For $m = 3$ we have (rounded to 4 digits)

$$
\begin{aligned}
p_0 &= 0.0156 \\
p_1 &= 0.0316 \\
p_2 &= 0.1749 \\
p_3 &= 0.6878 \\
p_4 &= 0.0469 \\
p_5 &= 0.0186 \\
p_6 &= 0.0126 \\
p_7 &= 0.0119
\end{aligned}
$$

A plot illustrating this distribution is as follows:



In all three cases, the distribution has a noticible peak at the best m-bit approximation to $1/3$.

2. When the phase estimation procedure is applied to a unitary operator U, an eigenvector $|\psi\rangle$ with $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$, and a given choice of m, the probability of measuring the best (or one of the two best) possible m-bit approximations to $\theta \in [0,1)$ is at least $4/\pi^2$. Prove that this is true.

   Solution. For the best possible m-bit approximation $j/2^m$ to θ we have

   $$e^{2\pi i\theta} = e^{2\pi i(j/2^m+\varepsilon)}$$

   for some real number ε with $|\varepsilon| \leq 2^{-(m+1)}$. Assuming that j satisfies this equation we may prove a lower bound on p_j as follows.

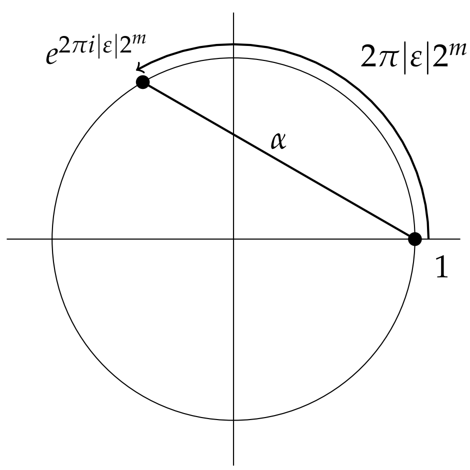
   First, we use the formula for a sum of a geometric series to obtain

   $$p_j = \left| \frac{1}{2^m} \sum_{k=0}^{2^m-1} e^{2\pi ik(\theta-j/2^m)} \right|^2 = \frac{1}{2^{2m}} \frac{\alpha^2}{\beta^2}$$

   for

   $$\begin{aligned} \alpha &= \left| e^{2\pi i(2^m\theta-j)} - 1 \right| = \left| e^{2\pi i\varepsilon 2^m} - 1 \right|, \\ \beta &= \left| e^{2\pi i(\theta-j/2^m)} - 1 \right| = \left| e^{2\pi i\varepsilon} - 1 \right|. \end{aligned}$$

   To obtain a lower bound on p_j we need a lower bound on α and an upper bound on β. To get a lower bound on α, consider the following picture:



   The ratio of the minor arc length to the chord length is at most $\pi/2$, so

   $$\frac{2\pi|\varepsilon|2^m}{a} \leq \frac{\pi}{2},$$

   which implies

   $$\alpha \geq 4|\varepsilon|2^m.$$

Along similar lines, we may consider β along with the fact that the ratio of arc length to chord length is at least 1. We obtain

$$\frac{2\pi|\varepsilon|}{\beta} \geq 1$$

so

$$\beta \leq 2\pi|\varepsilon|.$$

Putting the two bounds together, we obtain

$$p_j \geq \frac{1}{2^{2m}} \frac{16|\varepsilon|^2 2^{2m}}{4\pi^2|\varepsilon|^2} = \frac{4}{\pi^2}.$$

There are, of course, other ways to obtain this bound.

3. This is a follow-up question to the previous question. Prove that no better constant bound than $4/\pi^2$ is possible for the probability to obtain a best m-bit approximation to θ using the phase estimation procedure. That is, prove that for every $\delta > 0$, there exists a choice of m and θ so that the probability of obtaining a best m-bit approximation to θ using the phase estimation procedure is smaller than $4/\pi^2 + \delta$.

   Solution. Let us take

$$\theta = \frac{1}{2^{m+1}},$$

   so that θ is halfway between 0 and $1/2^m$ (which means that $j = 0$ and $j = 1$ are equally good estimates). We could equally well take any other choice of θ that is halfway between $j/2^m$ and $(j+1)/2^m$ for any other choice of j.

   Following a similar analysis to what we have for question 2, we see that the probability to obtain a best estimate ($j = 0$ or $j = 1$) is

$$\frac{4}{2^{2m}\left|1 - e^{2\pi i/2^{m+1}}\right|^2}.$$

   Using the fact that

$$\lim_{\eta \downarrow 0} \frac{\left|1 - e^{2\pi i\eta}\right|}{2\pi\eta} = 1$$

   (which intuitively corresponds to the observation that the chord length and the arc length become closer and closer as the angle becomes small), we find that

$$\lim_{m\to\infty} \frac{4}{2^{2m}\left|1 - e^{2\pi i/2^{m+1}}\right|^2} = \frac{4}{\pi^2}.$$

   For a suitably large choice of m, the probability to obtain a best estimate must therefore be smaller than $4/\pi^2 + \delta$ for any given $\delta > 0$.