It from Qubit: Entanglement Theory Monday Problem Solutions

Schmidt decomposition of a pure bipartite state.
(a)

$$\begin{aligned} c_k|\chi_B^{(k)}\rangle &= \langle\chi_A^{(k)}|\psi\rangle \\ &= \sum_{i,j} a_{ij}\langle\chi_A^{(k)}|\phi_A^{(i)}\rangle|\phi_B^{(j)}\rangle. \end{aligned} \tag{1}$$

By requiring that $|\chi_B^{(k)}\rangle$ be a normalized state, we obtain

$$c_k = \sqrt{\sum_{i,j} |a_{ij}\langle\chi_A^{(k)}|\phi_A^{(i)}\rangle|^2}. \tag{2}$$

$$|\chi_B^{(k)}\rangle = \frac{1}{c_k}\sum_{i,j} a_{ij}\langle\chi_A^{(k)}|\phi_A^{(i)}\rangle|\phi_B^{(j)}\rangle. \tag{3}$$

(b)

$$\rho_A = \mathrm{Tr}_B\left(|\psi\rangle\langle\psi|\right) \tag{4}$$

$$= \mathrm{Tr}_B\left(\sum_{k,k'} c_k c_{k'}^* |\chi_A^{(k)}\rangle\langle\chi_A^{(k')}| \otimes |\chi_B^{(k)}\rangle\langle\chi_B^{(k')}|\right) \tag{5}$$

$$= \sum_{k,k'} c_k c_{k'}^* \langle\chi_B^{(k')}|\chi_B^{(k)}\rangle|\chi_A^{(k)}\rangle\langle\chi_A^{(k')}| \tag{6}$$

But because $\{|\chi_A^{(k)}\rangle\}$ is, by definition, the basis that diagonalizes ρ_A, it follows that we must have $\langle\chi_B^{(k')}|\chi_B^{(k)}\rangle = \delta_{k,k'}$.

(c) the Schmidt decomposition of $|\psi\rangle$ is unique only if there are no degeneracies among the eigenvalues of its reduced density operator.

(d)

$$\rho_B = \frac{1}{2}|E_1\rangle\langle E_1| + \frac{1}{2}|E_2\rangle\langle E_2| \tag{7}$$

For notational simplicity, take $|E_1\rangle = |0\rangle$ and $|E_2\rangle = \sqrt{1}2(|0\rangle + |1\rangle)$. Relative to the $\{|0\rangle, |1\rangle\}$ basis,

$$\rho_B = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}. \tag{8}$$

The eigenvalues of ρ_B are $\lambda_1 = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ and $\lambda_2 = \frac{1}{2} - \frac{1}{2\sqrt{2}}$ with associated eigenvectors $|\chi_B^{(1)}\rangle = \frac{1}{N_1}\left[(1+\sqrt{2})|0\rangle + |1\rangle\right]$ and $|\chi_B^{(2)}\rangle = \frac{1}{N_2}\left[(1-\sqrt{2})|0\rangle + |1\rangle\right]$ where N_1 and N_2 are normalization factors. If one defines $|\chi_A^{(i)}\rangle = \frac{1}{M_i}\langle\chi_B^{(i)}|\psi\rangle$ where M_i is a normalization factor, then

$$|\psi\rangle = (\frac{1}{2} + \frac{1}{2\sqrt{2}})|\chi_A^{(1)}\rangle|\chi_B^{(1)}\rangle + (\frac{1}{2} + \frac{1}{2\sqrt{2}})|\chi_A^{(2)}\rangle|\chi_B^{(2)}\rangle. \tag{9}$$

Quantum teleportation
(a) We denote the Pauli operators by I, X, Y, Z. We make use of the fact that

$$\left|\Phi^+\right\rangle = (I \otimes I)\left|\Phi^+\right\rangle \tag{10}$$

$$\left|\Phi^-\right\rangle = (I \otimes Z)\left|\Phi^+\right\rangle \tag{11}$$

$$\left|\Psi^+\right\rangle = (I \otimes X)\left|\Phi^+\right\rangle \tag{12}$$

$$\left|\Psi^-\right\rangle = (I \otimes iY)\left|\Phi^+\right\rangle, \tag{13}$$

to conclude that for each element of the Bell basis, the corresponding collapsed state of B is $|\psi\rangle, Z|\psi\rangle, X|\psi\rangle$, and $Y|\psi\rangle$ respectively.

(b) The four unitaries are I, Z, X, and Y resptively.

(c) Under the von-Neumann-Luders collapse rule (i.e. the projection postulate), the final state for A' is the reduced density operator on A' of the Bell basis state corresponding to the outcome of the Bell basis measurement. But in each case, this reduced density operator is the maximally mixed state, hence independent of $|\psi\rangle$. As such, teleportation is not in conflict with the no-cloning theorem.

(d) No measurement can determine the quantum state because nonorthogonal quantum states cannot be discriminated with certainty.

(e) Because a pure quantum state of a qubit is specified by two real parameters, it takes an infinite number of bits to describe such a state. But only 2 bits are transmitted in the quantum teleportation protocol.

Deterministic conversion under LOCC and entanglement catalysis

(a) We consider the two states to be defined on a pair of systems each with Hilbert space dimension 4. The eigenvalues of the reduced density oprator of $|\psi_1\rangle$ are $(0.5, 0.25, 0.25, 0)$ in nonincreasing order, while for $|\psi_2\rangle$ they are $(0.4, 0.4, 0.1, 0.1)$. The second vector does not majorize the first because $0.4 < 0.5$, and the first vector does not majorize the second because $0.5 + 0.25 < 0.4 + 0.4$.

(b) The eigenvalues of the reduced density oprator of $|\psi_1\rangle|\phi\rangle$ are $(0.30, 0.20, 0.15, 0.15, 0.10, 0.10, 0, 0))$ in nonincreasing order, while for $|\psi_2\rangle|\phi\rangle$ they are $(0.24, 0.24, 0.16, 0.16, 0.06, 0.06, 0.04, 0.04)$. The first vector can be verified to majorize the second.

(c) A maximally entangled state has the form $|\mu\rangle = \sum_{i=1}^{d} \frac{1}{\sqrt{d}} |i\rangle|i\rangle$. Let the eigenvalues of the reduction of $|\psi_1\rangle$, be denoted $(\lambda_1, ..., \lambda_n)$ in nonincreasing order and let those of the reduction of $|\psi_2\rangle$ be denoted by $(\kappa_1, ..., \kappa_n)$. Catalysis using $|\mu\rangle$ requires that the transition $|\psi_1\rangle \to |\psi_2\rangle$ be impossible by LOCC, but $|\psi_1\rangle|\mu\rangle \to |\psi_2\rangle|\mu\rangle$ be possible. Nielsen's theorem states that if $|\psi_1\rangle \to |\psi_2\rangle$ is impossible, then $(\lambda_1, ..., \lambda_n)$ is not majorized by $(\kappa_1, ..., \kappa_n)$, which implies that there is some $1 \leq l \leq n$ such that $\sum_{k=1}^{l} \lambda_k > \sum_{k=1}^{l} \kappa_k$. To have catalysis, we require that $\left(\frac{1}{d}\lambda_1, ...\frac{1}{d}\lambda_1, ..., \frac{1}{d}\lambda_n, ...\frac{1}{d}\lambda_n\right)$ (where each term is d-fold degenerate) is majorized by $\left(\frac{1}{d}\kappa_1, ...\frac{1}{d}\kappa_1, ..., \frac{1}{d}\kappa_m, ...\frac{1}{d}\kappa_m\right)$. However, this majorization relation fails to hold because the condition $\sum_{k=1}^{l} \lambda_k > \sum_{k=1}^{l} \kappa_k$ implies that $\sum_{k=1}^{dl} \frac{1}{d}\lambda_k > \sum_{k=1}^{dl} \frac{1}{d}\kappa_k$.

(d) Composing $|\phi\rangle|\psi_1\rangle \to |\phi\rangle|\psi_2\rangle$ and $|\eta\rangle|\psi_2\rangle \to |\eta\rangle|\psi_1\rangle$, we have $|\phi\rangle|\eta\rangle|\psi_1\rangle \leftrightarrow |\phi\rangle|\eta\rangle|\psi_2\rangle$. The latter implies that the Schmidt coefficients of $|\phi\rangle|\eta\rangle|\psi_1\rangle$ are equal to those of $|\phi\rangle|\eta\rangle|\psi_2\rangle$. This implies that the Schmidt coefficients of $|\psi_1\rangle$ are equal to those of $|\psi_2\rangle$, which in turn implies that they are equivalent up to local unitaries.

(e) For nontrivial catalysis to be possible, it must be that the two states are not reversibly catalytically interconvertible, which is to say that the catalysis facilitates interconversion in only one direction. Suppose, for instance, that we have $|\psi_1\rangle \not\to |\psi_2\rangle$ while $|\psi_1\rangle \to_C |\psi_2\rangle$, where $\to_C$denotes interconversion by catalyst. The one-way nature of the catalysis implies that $|\psi_2\rangle \not\to_C |\psi_1\rangle$ which in turn implies that $|\psi_2\rangle \not\to |\psi_1\rangle$, so that $|\psi_1\rangle$ and $|\psi_2\rangle$ are incomparable.