It from Qubit: Entanglement Theory Monday Problems

Schmidt decomposition of a pure bipartite state.

In a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ it is always possible to write a ket $|\psi\rangle$ as a double sum

$$|\psi\rangle = \sum_{i,j} a_{ij} \left|\phi_A^{(i)}\right\rangle \left|\phi_B^{(j)}\right\rangle, \tag{1}$$

involving a set of basis kets $\left\{\left|\phi_A^{(i)}\right\rangle\right\}$ in $\mathcal{H}_A$ and a set of basis kets $\left\{\left|\phi_B^{(j)}\right\rangle\right\}$ in $\mathcal{H}_B$. Those basis kets can be chosen to be orthonormal,

$$\left\langle\phi_A^{(i)}|\phi_A^{(j)}\right\rangle = \delta^{ij},$$
$$\left\langle\phi_B^{(i)}|\phi_B^{(j)}\right\rangle = \delta^{ij}.$$

This problem deals with a different kind of expansion of a ket, an expansion involving only *one* sum,

$$|\psi\rangle = \sum_{k} c_k \left|\chi_A^{(k)}\right\rangle \left|\chi_B^{(k)}\right\rangle, \tag{2}$$

using a set of kets $\left\{\left|\chi_A^{(k)}\right\rangle\right\}$ in $\mathcal{H}_A$ and a set of kets $\left\{\left|\chi_B^{(k)}\right\rangle\right\}$ in $\mathcal{H}_B$. This is called a *biorthogonal* or *Schmidt decomposition* if we have

$$\left\langle\chi_A^{(k)}|\chi_A^{(l)}\right\rangle = \delta^{kl},$$
$$\left\langle\chi_B^{(k)}|\chi_B^{(l)}\right\rangle = \delta^{kl}.$$

You will show in parts (a) and (b) below that such a decomposition is always possible. Clearly *any* orthogonal sets of basis kets $\left\{\left|\phi_A^{(i)}\right\rangle\right\}$ in $\mathcal{H}_A$ and $\left\{\left|\phi_B^{(j)}\right\rangle\right\}$ in $\mathcal{H}_B$ can be used to perform the expansion (1), and the same sets of basis kets can be used for any ket $|\psi\rangle$. In (2) that is not true, and the sets of kets $\left\{\left|\chi_A^{(k)}\right\rangle\right\}$ in $\mathcal{H}_A$ and $\left\{\left|\chi_B^{(k)}\right\rangle\right\}$ in $\mathcal{H}_B$ will in general be different for each $|\psi\rangle$.

(a) Let $\left\{\left|\chi_A^{(k)}\right\rangle\right\}$ be an orthogonal basis in $\mathcal{H}_A$ that includes the eigenstates of $\rho_A = \mathrm{Tr}_B\left(|\psi\rangle\langle\psi|\right)$. For the ket $|\psi\rangle$ of interest, write

$$|\psi\rangle = I_A|\psi\rangle = \sum_k \left|\chi_A^{(k)}\right\rangle \left\langle\chi_A^{(k)}|\psi\right\rangle.$$

Now find c_k and $\left|\chi_B^{(k)}\right\rangle$ such that

$$|\psi\rangle = \sum_k c_k \left|\chi_A^{(k)}\right\rangle \left|\chi_B^{(k)}\right\rangle$$

with each $\left|\chi_B^{(k)}\right\rangle$ normalized.

(b) Then, with the ket explicitly in this form, take $\rho_A = \mathrm{Tr}_B\left(|\psi\rangle\langle\psi|\right)$ and use the resulting expression to show that $\left\langle\chi_B^{(k)}|\chi_B^{(l)}\right\rangle = \delta^{kl}$.

(c) On the basis of this "proof by construction," what can you say about when, if ever, the Schmidt decomposition will be unique?

(d) To work out a particular example, consider the ket

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|E_1\rangle + |-\rangle|E_2\rangle\right)$$

in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\{|+\rangle, |-\rangle\}$ are an orthonormal basis of $\mathcal{H}_A$, while

$$\langle E_1|E_1\rangle = \langle E_2|E_2\rangle = 1,$$
$$\langle E_1|E_2\rangle = \frac{1}{2}.$$

Find the biorthogonal decomposition of this particular ket.

Quantum teleportation.

Alice and Bob are spatially separated, but have in their possession one qubit each, denoted A and B respectively. The quantum state of the pair AB is the maximally entangled state $\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$. Victor gives Alice another qubit, denoted A', which he prepares in the quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Neither Alice nor Bob know anything about the identity of $|\psi\rangle$. Alice would like to send this quantum state to Bob, but they do not share a quantum channel. As it turns out, however, by use of a classical channel and their entangled state, they can simulate a single use of a quantum channel, that is, Alice can transfer the quantum state $|\psi\rangle$ to Bob with certainty.

(a) The protocol asks Alice to measure the "Bell basis"

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right) \tag{3}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |1\rangle|1\rangle\right) \tag{4}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|0\rangle\right) \tag{5}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right) \tag{6}$$

on the pair of qubits, A' and A, that in her possession. What are the quantum states to which she collapses Bob's qubit B for each possible outcome of this measurement?

(b) The protocol asks Alice to communicate to Bob, through the classical channel, two bits of information specifying which of the four outcomes she obtained in her measurement. Show that by using this information, Bob can implement one of four unitary transformations on qubit B which has the effect of leaving it in the quantum state $|\psi\rangle$.

(c) What is the final quantum state of qubits A' and A? Is teleportation consistent with the no-cloning theorem?

(d) Why can't Alice teleport the quantum state of A' as follows: she determines $|\psi\rangle$ by a measurement, then sends a classical description of $|\psi\rangle$ to Bob, who then prepares it himself.

(e) How much information does it take to provide a classical description of the quantum state $|\psi\rangle$? In other words, how much classical information would someone who knew $|\psi\rangle$ need to send Bob such that Bob could prepare the quantum state $|\psi\rangle$ locally? Your answer should exceed the amount of information that is communicated in the teleportation protocol. How then is enough information about $|\psi\rangle$ transferred from Alice to Bob in the that protocol? (This is the aspect of quantum teleportation that is typically judged to be mysterious.)

Deterministic conversion under LOCC and entanglement catalysis

(a) Two bipartite pure states, $|\psi_1\rangle$ and $|\psi_2\rangle$, are said to be *incomparable* if neither $|\psi_1\rangle \to |\psi_2\rangle$ nor $|\psi_2\rangle \to |\psi_1\rangle$ are possible by LOCC.

Using Nielsen's theorem relating deterministic state conversion to majorization, show that the following two states are incomparable:

$$\begin{aligned} |\psi_1\rangle &= \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.25}|22\rangle, \\ |\psi_2\rangle &= \sqrt{0.4}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle + \sqrt{0.1}|33\rangle. \end{aligned} \tag{7}$$

(b) If the conversion $|\psi_1\rangle \rightarrow |\psi_2\rangle$ is impossible by LOCC, then it may still be possible with the help of a *catalyst*. This occurs if there is another bipartite pure state $|\phi\rangle$, called the catalyst, such that $|\phi\rangle|\psi_1\rangle \rightarrow |\phi\rangle|\psi_2\rangle$.

Show that the bipartite state $|\phi\rangle = \sqrt{0.6}|00\rangle + \sqrt{0.4}|11\rangle$ catalyzes the conversion $|\psi_1\rangle \rightarrow |\psi_2\rangle$ for the incomparable states of Eq. (7).

(c) Show that bipartite pure states that are maximally entangled (i.e. for which the reduced density operator is maximally mixed) are not useful as entanglement catalysts, which is to say that if the transformation $|\psi_1\rangle \rightarrow |\psi_2\rangle$ is not possible by LOCC, then $|\phi\rangle|\psi_1\rangle \rightarrow |\phi\rangle|\psi_2\rangle$ is also not possible by LOCC if $|\phi\rangle$ is maximally entangled.

(d) Suppose that two pure bipartite states $|\psi_1\rangle$ and $|\psi_2\rangle$ are reversibly interconvertible catalytically, that is, suppose that there exists a catalyst $|\phi\rangle$ such that $|\phi\rangle|\psi_1\rangle \rightarrow |\phi\rangle|\psi_2\rangle$ under LOCC and that there exists a catalyst $|\eta\rangle$ such that $|\eta\rangle|\psi_2\rangle \rightarrow |\eta\rangle|\psi_1\rangle$ under LOCC. Show that it follows that the state $|\psi_1\rangle$ and $|\psi_2\rangle$ are equivalent up to local unitary transformations.

(e) Using the result from the previous question, show that the only pairs of pure bipartite states $|\psi_1\rangle$ and $|\psi_2\rangle$ for which conversion from one to the other requires a catalyst are incomparable states.