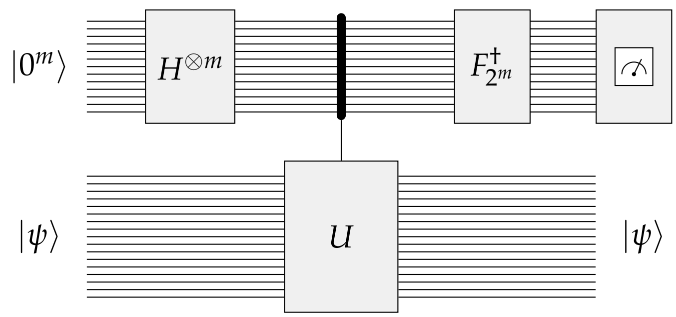
It From Qubit

Problems on Phase Estimation

1. Here is a picture of the phase estimation procedure, for an arbitrary unitary operator U, an eigenvector $|\psi\rangle$ of U, and a number of qubits m that are used to control the precision of the procedure:



Suppose that $|\psi\rangle$ and U happen to satisfy

$$U|\psi\rangle = e^{2\pi i/3}|\psi\rangle .$$

Calculate the probabilities for all possible measurement outcome for $m = 1, 2$, and 3, and compare the results with the best m-bit approximation to $\theta = 1/3$.

2. When the phase estimation procedure is applied to a unitary operator U, an eigenvector $|\psi\rangle$ with $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$, and a given choice of m, the probability of measuring the best (or one of the two best) possible m-bit approximations to $\theta \in [0, 1)$ is at least $4/\pi^2$. Prove that this is true.

3. This is a follow-up question to the previous question. Prove that no better constant bound than $4/\pi^2$ is possible for the probability to obtain a best m-bit approximation to θ using the phase estimation procedure. That is, prove that for every $\delta > 0$, there exists a choice of m and θ so that the probability of obtaining a best m-bit approximation to θ using the phase estimation procedure is smaller than $4/\pi^2 + \delta$.