The AdS/CFT Correspondence
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1 Lecture 2

Q1. The simplest black hole spacetime we can describe in an asymptotically AdS geometry is the famous BTZ black hole in three spacetime dimensions. Let us consider the global geometry, whose metric is parameterized as

$$ds^2 = -\frac{r^2 - r_+^2}{\ell_{\text{AdS}}^2}\, dt^2 + \frac{\ell_{\text{AdS}}^2\, dr^2}{r^2 - r_+^2} + r^2\, d\varphi^2\,. \tag{1.1}$$

There are various things we can do analytically in this geometry, which make it a useful intuition building example.

(i) Compute the temperature of this solution analyzing the Euclidean geometry obtained by analytically continuing the timelike Killing field $t \to i\, t_{\text{E}}$. Obtain the entropy of the solution using the Bekenstein-Hawking formula and thence the specific heat.

(ii) Having obtained the Euclidean geometry, argue that it is topologically is a filled two-torus $\mathbf{T}^2$ (i.e., a handlebody), with one non-contractible one cycle. Further, argue that there are two distinct solutions obtained upon analytic continuation.

(iii) The solution (1.1) satisfies Einstein's equations obtained from

$$\mathcal{S}_{grav} = \frac{1}{16\pi\, G_N^{(3)}} \int d^3x \sqrt{-g} \left(R + \frac{2}{\ell_{\text{AdS}}^2}\right) + \frac{1}{8\pi\, G_N^{(3)}} \int d^2x\, \sqrt{-\gamma}\, \left(K + \Lambda_\partial\right) \tag{1.2}$$

where we have included a boundary term involving the extrinsic curvature, the Gibbons-Hawking term, to ensure that Einstein's equations derive from a variational principle. The second contribution involving Λ_∂ is a finite counter-term which we fix to ensure that the action is finite. Compute the on-shell action for the BTZ spacetime and fix Λ_∂ to obtain a finite on-shell action.

(iv) Compare the action computed in the previous part with that for the thermal AdS geometry, viz., the spacetime obtained by setting $r_+ = i\, \ell_{\text{AdS}}$, $t \to i\, t_{\text{E}}$, and requiring $t_{\text{E}} = t_{\text{E}} + \beta$ in the metric

$$ds^2 = -(\frac{r^2}{\ell_{\text{AdS}}^2} + 1)\, dt^2 + \frac{dr^2}{\frac{r^2}{\ell_{\text{AdS}}^2} + 1} + r^2\, d\varphi^2\,. \tag{1.3}$$

Requiring that the temperature or Euclidean periodicity is the same, and interpreting the on-shell value of the action as the saddle point value of the thermal free energy, decide which of the two geometries dominates the thermal physics.

(v) How do spacelike geodesics behave in the BTZ spacetime. For simplicity you can first consider geodesics on the $t = 0$ slice. You should however be able to get a closed form expression for the curve in the spacetime.

(vi) Describe the geometry when $t_+ = i\,\ell_{\text{AdS}}(1-\mu)$ for $\mu \in (0,1)$. Can you infer what the geodesics in the spatial sections look like?

Soln # 1.

Euclidean geometry: The Euclidean BTZ solution is simply

$$\begin{aligned} ds^2 &= \frac{r^2 - r_+^2}{\ell_{\text{AdS}}^2}\, dt_{\text{E}}^2 + \frac{\ell_{\text{AdS}}^2\, dr^2}{r^2 - r_+^2} + r^2\, d\varphi^2 \\ &= \frac{r_+^2}{\ell_{\text{AdS}}^2} \sinh^2\left(\frac{\rho}{\ell_{\text{AdS}}}\right) dt_{\text{E}}^2 + d\rho^2 + r_+^2 \cosh^2\left(\frac{\rho}{\ell_{\text{AdS}}}\right) d\varphi^2\,, \\ &\qquad \text{where} \quad r = r_+ \cosh\left(\frac{\rho}{\ell_{\text{AdS}}}\right) \end{aligned} \tag{1.4}$$

To get the temperature look at the solution near $r = r_+$ or equivalently $\rho = 0$. The circle parameterized by φ stays of finite size always for $r > r_+$. The local geometry near the horizon however looks like

$$ds^2 = \frac{r_+^2}{\ell_{\text{AdS}}^4}\, \rho^2\, dt_{\text{E}}^2 + d\rho^2 + r_+^2\, d\varphi^2 \tag{1.5}$$

Since the (t_{E}, ρ) part of the geometry looks like a copy of $\mathbb{R}^2$ in the neighbourhood of the origin in polar coordinates; regularity of the coordinate chart demands that $t_{\text{E}} \to t_{\text{E}} + 2\pi \frac{\ell_{\text{AdS}}^2}{r_+}$. This convinces us that t_{E} parameterizes a circle, which we will call the Euclidean thermal circle. The period of this circle is the inverse temperature from usual statistical mechanics, so we find:

$$T = \frac{r_+}{2\pi\, \ell_{\text{AdS}}^2}\,, \qquad \beta = \frac{2\pi\, \ell_{\text{AdS}}^2}{r_+}\,. \tag{1.6}$$

The entropy of the solution is given by

$$S_{bh} = \frac{\text{Area(horizon)}}{4\, G_N^{(3)}} = \frac{2\pi\, r_+}{4\, G_N^{(3)}} = \frac{2\pi^2}{3}\, c\, T\, \ell_{\text{AdS}}\,. \tag{1.7}$$

where we used the Brown-Henneaux formula $\frac{\ell_{\text{AdS}}}{4G_N^{(3)}} = \frac{2}{3}\, c$, relating the gravitational data to the field theory central charge. The specific heat is simply

$$C = \frac{T\, \partial S}{\partial T} = \frac{2\pi^2}{3}\, c\, T\, \ell_{\text{AdS}} \geq 0\,. \tag{1.8}$$

The Euclidean solution is indeed topologically a $\mathbf{T}^2 = \mathbf{S}^1_{t_{\text{E}}} \times \mathbf{S}^1_{\varphi}$. The t_{E} circle shrinks to zero at the horizon, so the in the bulk one of the homology cycles of the torus is indeed contractible.

Note that (1.4) admits an interpretation as thermal AdS_3 if we simply switch the identification of time and space. Under $\{\frac{r_+}{\ell_{\rm AdS}}\, t_{\rm E}, r_+\,\varphi\} \mapsto \{\tilde{\varphi}, \tilde{t}_{\rm E}\}$ we make the spatial circle contractible but keep the thermal on non-contractible, for we get

$$ds^2 = \sinh^2\left(\frac{\rho}{\ell_{\rm AdS}}\right) d\tilde{\varphi}^2 + d\rho^2 + \cosh^2\left(\frac{\rho}{\ell_{\rm AdS}}\right) d\tilde{t}_{\rm E}^2 \tag{1.9}$$

which is indeed the global AdS_3 geometry. Now since $\tilde{t}_{\rm E}$ is non-contractible we have the freedom to pick its period to be arbitrary.

Saddle point of the Euclidean quantum gravity: The BTZ solution has constant curvature since it secretly is just AdS_3 in disguise.[1]

$$R = -\frac{6}{\ell_{\rm AdS}^2}\,, \tag{1.10}$$

One can also compute the induced metric and extrinsic curvature of the constant r hypersurface

$$\begin{aligned}
\gamma_{\mu\nu}\, dx^\mu\, dx^\nu &= \frac{r^2 - r_+^2}{\ell_{\rm AdS}^2}\, dt_{\rm E}^2 + r^2\, d\varphi^2\,, \\
K_{\mu\nu} dx^\mu\, dx^\nu &= \frac{r}{\ell_{\rm AdS}^3}\sqrt{r^2 - r_+^2}\left(-dt^2 + \ell_{\rm AdS}^2\, d\varphi^2\right), \\
K &= \frac{2\,r^2 - r_+^2}{r\,\ell_{\rm AdS}\sqrt{r^2 - r_+^2}}
\end{aligned} \tag{1.11}$$

We integrate the curvature term over the bulk, in Euclidean signature taking $t_{\rm E} \in (0, \beta)$ and $\varphi \in (0, 2\pi\ell_{\rm AdS})$, with a radial cut-off $r_+ \leq r \leq R_\epsilon$. Then

$$\begin{aligned}
\frac{1}{16\pi\, G_N^{(3)}} \int d^3x\sqrt{-g}\left(R + \frac{2}{\ell_{\rm AdS}^2}\right) &= \frac{1}{16\pi\, G_N^{(3)}} \int_0^\beta dt_{\rm E} \int_0^{2\pi\ell_{\rm AdS}} d\varphi \int_{r_+}^{R_\epsilon} \left(-\frac{4}{\ell_{\rm AdS}^2}\right) r\, dr \\
&= -\,\frac{2\pi\,\beta}{8\pi\, G_N^{(3)}\, \ell_{\rm AdS}}\, \left(R_\epsilon^2 - r_+^2\right) \\
\frac{1}{8\pi\, G_N^{(3)}} \int d^2x\, \sqrt{-\gamma}\, K &= \frac{2\pi\, \ell_{\rm AdS}\, \beta}{8\pi\, G_N^{(3)}} \left(\frac{R_\epsilon}{\ell_{\rm AdS}}\sqrt{R_\epsilon^2 - r_+^2}\right) \left(\frac{2\, R_\epsilon^2 - r_+^2}{R_\epsilon\, \ell_{\rm AdS}\sqrt{R_\epsilon^2 - r_+^2}}\right) \\
&= 2\, \frac{2\pi\,\beta}{8\pi\, G_N^{(3)}\, \ell_{\rm AdS}} (R_\epsilon^2 - \frac{1}{2}\, r_+^2) \\
\frac{1}{8\pi\, G_N^{(3)}} \int d^2x\, \sqrt{-\gamma}\, \Lambda_\partial &= \Lambda_\partial\, \frac{2\pi\,\beta}{8\pi\, G_N^{(3)}}\, R_\epsilon^2\sqrt{1 - \frac{r_+^2}{R_\epsilon^2}}
\end{aligned} \tag{1.12}$$

[1] One can prove that all solutions to Einstein's equations with negative cosmological constant following from (1.2) are all locally AdS_3, and have the same curvature data.

Adding up the contributions we find:

$$
\begin{aligned}
\mathcal{S}_{on-shell} &= \frac{2\pi\,\beta}{8\pi\,G_N^{(3)}\,\ell_{\rm AdS}}\left[-R_\epsilon^2 + r_+^2 + 2R_\epsilon^2 - r_+^2 + \Lambda_\partial\,R_\epsilon^2\,\ell_{\rm AdS} - \frac{1}{2}\Lambda_\partial\,r_+^2\,\ell_{\rm AdS}\right] \\
&= \frac{2\pi\,\beta}{8\pi\,G_N^{(3)}\,\ell_{\rm AdS}}\left(-\frac{1}{2}\,r_+^2\right)\,, \qquad \Lambda_\partial \equiv \frac{1}{\ell_{\rm AdS}}
\end{aligned}
\tag{1.13}
$$

A similar computation of the gravitational action for the global AdS spacetime an be done. The answers can be obtained either by redoing the computation, using:

$$
\begin{aligned}
\gamma_{\mu\nu}\,dx^\mu\,dx^\nu &= \left(\frac{r^2}{\ell_{\rm AdS}^2}+1\right)\,dt_{\rm E}^2 + r^2\,d\varphi^2\,, \\
K_{\mu\nu}dx^\mu\,dx^\nu &= \frac{r}{\ell_{\rm AdS}}\,\sqrt{1+\frac{r^2}{\ell_{\rm AdS}^2}}\,\left(-dt^2+d\varphi^2\right)\,, \\
K &= \frac{\ell_{\rm AdS}^2 + 2\,r^2}{r\,\ell_{\rm AdS}\,\sqrt{r^2+\ell_{\rm AdS}^2}}
\end{aligned}
\tag{1.14}
$$

or realizing that we we simply can set $r_+ = i\,\ell_{\rm AdS}$ in the BTZ expression. We obtain:

$$
S_{on-shell}\big|_{\rm BTZ} - S_{on-shell}\big|_{\rm AdS} = -\frac{\pi\,\beta\,\ell_{\rm AdS}}{8\pi\,G_N^{(3)}}\left(\frac{r_+^2}{\ell_{\rm AdS}^2}-1\right)\,, \tag{1.15}
$$

This quantity is the gravitational free energy; it is negative for $r_+ > \ell_{\rm AdS}$ but positive for $r_+ < \ell_{\rm AdS}$. This suggests a phase transition with the BTZ geometry dominating the thermal partition function for large black holes or high temperature, and the thermal AdS geometry taking over at low temperatures.

Geodesics: The geodesics in a spacetime with metric

$$
ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\varphi^2 \tag{1.16}
$$

can be simply obtained by exploiting the two Killing symmetries. Take a geodesic with tangent vector:

$$
\xi^A = \frac{dt}{d\lambda}\left(\frac{\partial}{\partial t}\right)^A + \frac{dr}{d\lambda}\left(\frac{\partial}{\partial r}\right)^A + \frac{d\varphi}{d\lambda}\left(\frac{\partial}{\partial\varphi}\right)^A \tag{1.17}
$$

and realize that the projection of the tangent along the Killing directions parameterized by t and φ is conserved under parallel transport along the geodesic.[2] Denoting the conserved charges as the ‘energy' and 'angular momentum' of the geodesic we find for the geometry:

$$
\frac{dt}{d\lambda} = -\frac{E}{f(r)}\,, \qquad \frac{d\varphi}{d\lambda} = \frac{J}{r^2} \tag{1.18}
$$

[2] If you haven't seen this try to prove it in general using the Killing equation, which for a vector field K reads: $\mathcal{L}_K g_{AB} = 2\,\mathcal{D}_{(A}\,K_{B)} = 0$, and the equation for an affinely parameterized geodesic: $\xi^B\mathcal{D}_B\,\xi_A = 0$.

which then leads to a simple first order ODE for the radial profile by realizing that the tangent vector has fixed normalization:

$$\frac{1}{f(r)}\left(\frac{dr}{d\lambda}\right)^2 - \frac{E^2}{f(r)} + \frac{J^2}{r^2} = \kappa\,, \qquad \kappa = \begin{cases} +1\,, & \text{spacelike} \\ 0\,, & \text{null} \\ -1\,, & \text{timelike} \end{cases} \tag{1.19}$$

We just need to integrate this expression for $f(r) = \frac{r^2 - r_+^2}{\ell_{\text{AdS}}^2}$ for both the BTZ and the family of conical singularities. This can be done, and the answer expressed nicely in terms of the end-points of the geodesic.

For geodesics confined to a single spatial slice say $t = 0$ the result can be expressed as:

$$r = r_+ \left(1 - \frac{\cosh^2(\frac{r_+}{\ell_{\text{AdS}}}\,\varphi)}{\cosh^2(\frac{r_+}{\ell_{\text{AdS}}}\,\varphi_0)}\right)^{-\frac{1}{2}} \tag{1.20}$$

where we have assumed that the end-points of the geodesic are at $\varphi = \pm\varphi_0$ at $t = 0$. The result for the conical defect geometries immediately follows by suitably analytic continuing r_+ as written.

Q2. Let us now compute how the BTZ black hole reacts to being perturbed. Take a free scalar field with some mass m whose action is given to be

$$\mathcal{S}_{matter} = \frac{1}{16\,\pi G_N^{(3)}} \int d^3x\,\sqrt{-g}\,\left(\frac{1}{2}\,\nabla_A\phi\,\nabla^A\phi - \frac{1}{2}\,m^2\,\phi^2\right) \tag{1.21}$$

Write down the wave equation for the scalar field in the BTZ geometry (1.1) and mode decompose it using the symmetries. We wish to treat the black hole as an open system, so we will solve the Schrödinger equation resulting from the above wave equation, with the following boundary conditions.

- The modes are ingoing at the horizon.
- The modes die off fast enough to be normalizable at infinity.

To get an operational sense of the ingoing boundary conditions, view the horizon as a one-way membrane and come up with a definition of right/left moving waves across this membrane.

Use these boundary conditions to determine the eigenvalues of the frequency; these are the quasinormal modes.

Soln # 2.

For simplicity let $f(r) = \frac{1}{\ell_{\text{AdS}}^2}\,(r^2 - r_+^2)$. With the ansatz:

$$\phi(t, r, \varphi) = e^{-i\,\omega\,t + i\,p\,\phi}\,\frac{1}{\sqrt{r}}\,\chi(r) \tag{1.22}$$

Then the scalar wave equation evaluates in the background to:

$$\begin{aligned}\mathcal{D}_A\,\mathcal{D}^A\phi &= \frac{1}{\sqrt{-g}}\partial_A\left(\sqrt{-g}\,g^{AB}\,\partial_B\;\phi\right)=0\\ &=\frac{1}{r}\frac{\partial}{\partial r}\left(r\,f(r)\,\frac{\partial}{\partial r}\left(\frac{\chi(r)}{\sqrt{r}}\right)\right)+\left[\frac{\omega^2}{f(r)}-\frac{p^2}{r^2}-m^2\right]\frac{\chi(r)}{\sqrt{r}}=0\end{aligned}\tag{1.23}$$

We can bring this wave equation into a more familiar Schrödinger form by a suitable change of variable. Define the tortoise coordinate, which runs from $-\infty$ at the horizon to ∞ at the spacetime asymptopia:

$$r_* = \int dr\,\frac{1}{f(r)} = \frac{\ell_{\rm AdS}^2}{2\,r_+}\,\log\left(\frac{r-r_+}{r+r_+}\right) = -r_+\,\coth\left(\frac{r_*\,r_+}{\ell_{\rm AdS}^2}\right)\tag{1.24}$$

and do some manipulations to bring the wave equation to the form:

$$\frac{d^2\chi(r_*)}{dr_*^2}+\omega^2\,\chi(r_*)-V_0(r_*)\,\chi(r_*)=0\tag{1.25}$$

A little algebra should convince you that the potential can be written in a nice form:

$$\begin{aligned}V_0(r_*) &= f(r)\,\left[\frac{p^2}{r^2}+\frac{1}{4}\left(-\frac{f(r)}{r^2}+2\,\frac{f'(r)}{r}\right)+m^2\right]\,,\\ &=\frac{r^2-r_+^2}{4\,\ell_{\rm AdS}^2}\left(\frac{4\,p^2+r_+^2\,\ell_{\rm AdS}^{-2}}{r^2}+3\ell_{\rm AdS}^{-2}+4\,m^2\right)\,,\\ &=\frac{r_+^2}{4\,\ell_{\rm AdS}^2}\,{\rm csch}^2\left(\frac{r_*\,r_+}{\ell_{\rm AdS}^2}\right)\left[\frac{4\,p^2+r_+^2\,\ell_{\rm AdS}^{-2}}{r_+^2\,\coth^2\left(\frac{r_*\,r_+}{\ell_{\rm AdS}^2}\right)}+3\,\ell_{\rm AdS}^{-2}+4\,m^2\right]\end{aligned}\tag{1.26}$$

The boundary conditions are:

$$\begin{aligned}&\text{normalizable:}\qquad \phi(r)\to\frac{1}{r^\Delta}\,,\qquad r\to\infty\\ &\text{ingoing:}\qquad \phi(r_*,t)\to e^{i\,\omega(t+r_*)}\,,\qquad r\to r_+\end{aligned}\tag{1.27}$$

where $\Delta = 1+\sqrt{1+m^2\,\ell_{\rm AdS}^2}$. One finds a solution in terms of the Hypergeometric functions:

$$\begin{aligned}\chi(r) = r^{\frac{1}{2}-i\mathfrak{p}}\left(r^2-r_+^2\right)^{-\frac{i}{2}\mathfrak{w}}&\left[A\;{}_2F_1\left(1-\frac{1}{2}i(\mathfrak{p}-i\Delta\,\mathfrak{w}),-\frac{1}{2}i(\mathfrak{p}+i\Delta\mathfrak{w});1-i\mathfrak{p};\frac{r^2}{r_+^2}\right)\right.\\ &\left.+B\left(\frac{r}{r_+}\right)^{2i\mathfrak{p}}{}_2F_1\left(\frac{1}{2}i(\mathfrak{p}+i(\Delta+i\mathfrak{w}-2)),\frac{1}{2}i(\mathfrak{p}-i\Delta\mathfrak{w});i\mathfrak{p}+1;\frac{r^2}{r_+^2}\right)\right]\end{aligned}\tag{1.28}$$

We have dropped some obvious factors of $\ell_{\rm AdS}$ for convenience and write $\mathfrak{p}=\frac{p}{r_+}$ and $\mathfrak{w}=\frac{\omega}{r_+}$ to avoid clutter.

Imposing the boundary conditions, we find the eigenfrequencies:

$$\omega = \pm p - i\,\frac{r_+}{\ell_{\rm AdS}^2}\,(2\,s+\Delta)\;,\qquad n\in\mathbb{Z}_+\;,p\in\mathbb{Z}\tag{1.29}$$

where s is a mode number and p is the spatial momentum (quantized because the φ circle is 2π periodic). Note that $2\pi\, T = \frac{r_+}{\ell_{\text{AdS}}^2}$ from Problem 1. You could have gotten this answer from the result in the first problem set (Q3 there); can you figure out how?

Q3. We can repeat much of problem 1 for the Schwarzschild-AdS$_{d+1}$ black hole both in planar and global geometries. The metrics in the two cases are:

$$\begin{aligned}
&\text{Global}: \qquad ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega_{d-1}^2\,, \qquad f(r) = \frac{r^2}{\ell_{\text{AdS}}^2} + 1 - \frac{r_+^{d-2}}{r^{d-2}}\left(1 + \frac{r_+^2}{\ell_{\text{AdS}}^2}\right) \\
&\text{Planar}: \qquad ds^2 = \frac{\ell_{\text{AdS}}^2}{z^2}\left(-f(z)\, dt^2 + \frac{dz^2}{f(z)} + d\mathbf{x}_{d-1}^2\right)\,, \qquad f(z) = 1 - \frac{z^d}{z_+^d}
\end{aligned} \tag{1.30}$$

Try to in particular,

(i) Work out the temperature and specific heat for the two solutions.

(ii) Understand the topology of the solution.

Soln # 3. To get the temperature we simply look at the geometry near the horizon. Ignore the spatial directions and write in general:

$$\begin{aligned}
ds_{nh}^3 &= -(r - r_+)\, f'(r_+) dt^2 + \frac{dr^2}{(r - r_+)\, f'(r_+)} \\
&= \frac{f'(r_+)^2}{4}\, r_\epsilon^2\, dt_{\text{E}}^2 + dr_\epsilon^2\,, \qquad r_\epsilon = \frac{2}{\sqrt{f'(r_+)}}\, \sqrt{r - r_+}
\end{aligned} \tag{1.31}$$

which gives

$$\begin{aligned}
&\text{global}: \qquad T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi}\left(\frac{d\, r_+}{\ell_{\text{AdS}}^2} + \frac{(d-2)}{r_+}\right) \\
&\text{planar}: \qquad T = \frac{d}{4\pi}\, \frac{1}{z_+}
\end{aligned} \tag{1.32}$$

In the global case the topology is that of a two-ball (a disc) in the $r - t$ plane combined with a non-contractible $\mathbf{S}^{d-2}$. In the planar case we still have a disc, but the spatial topology is now $\mathbb{R}^{d-2}$.
