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QFT Basics

Problem Set #2

Consider a CFT in 2d with complex coordinate $z = x + i\tau, \bar{z} = x - i\tau$. In Euclidean signature (real τ), z and $\bar{z}$ are complex conjugates, but in Lorentzian signature they are independent.

1. Show that the metric

$$ds^2 = dx^2 + d\tau^2 \tag{0.1}$$

is invariant, up to an overall local rescaling, under any *holomorphic* change of coordinates:

$$z \to z' = z'(z), \quad \bar{z} \to \bar{z}' = \bar{z}'(\bar{z}) \tag{0.2}$$

It follows that the 2d conformal algebra is infinite dimensional. (It is called the Virasoro algebra.)

SOLUTION: In complex notation,

$$ds^2 = dz d\bar{z} \ . \tag{0.3}$$

Under the holomorphic coordinate change written,

$$ds^2 = dz d\bar{z} = \frac{\partial z}{\partial z'}\frac{\partial \bar{z}}{\partial \bar{z}'} dz' d\bar{z}' \tag{0.4}$$

which is the same as $dz' d\bar{z}'$ up to an overall factor. Therefore this is a conformal transformation.

2. Show that the exponential mapping $z = e^{2\pi w/\beta}$, $\bar{z} = e^{2\pi \bar{w}/\beta}$ maps the plane to the cylinder. It follows that results in 2d CFT on a cylinder are related to results on the plane; for example, 2pt functions on the cylinder are entirely fixed by conformal symmetry. (This is not true in higher dimensions). What do the circles $|z|$ =const map to on the cylinder?

SOLUTION: Define real coordinates (σ_1, σ_2) by $w = \sigma_1 + i\sigma_2$, $\bar{w} = \sigma_1 - i\sigma_2$. Under

$$\sigma_2 \to \sigma_2 + \beta \ , \tag{0.5}$$

the point on the z-plane that we map to is $z \to e^{2\pi i z} = z$ — it's the same point. So in the σ coordinates, σ_2 parameterizes a circle of size β. σ_1 can be anything, so together, (σ_1, σ_2) parameterize a Euclidean cylinder.

The circle $|z| = A$ maps to the circle

$$\sigma_1 = \frac{\beta}{2\pi} \log A, \quad \sigma_2 \in [0, \beta] \tag{0.6}$$

This is a circle going around the cylinder. So the various circles around the cylinder map to concentric cylinders on the complex z-plane, centered at the origin.

3. Under a general conformal transformation (0.2), a correlation function of local primary scalar operators transforms as

$$\langle O_1'(z_1', \bar{z}_1') O_2'(z_2', \bar{z}_2') \cdots \rangle = \left(\frac{dz_1}{dz_1'}\right)^{\Delta_1/2} \left(\frac{d\bar{z}_1}{d\bar{z}_1'}\right)^{\Delta_1/2} \langle O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) \cdots \rangle \tag{0.7}$$

Use this formula, together with your results of parts (1) and (2), to derive the (Euclidean) thermal two-point function

$$G_\beta(x_1, \tau_1; x_2, \tau_2) = \operatorname{tr} e^{-\beta H} O(x_1, \tau_1) O(x_2, \tau_2) \tag{0.8}$$

(The answer to this question was the starting point on problem set #1.)

SOLUTION: The trace formula states, according to the results of lecture 1, that the thermal 2-point function is equal to the correlator on a cylinder of radius β. So (trivially shifting one point to the origin),

$$G_\beta(x, \tau; 0, 0) = \langle O'(w = x + i\tau, \bar{w} = x - i\tau) O'(0) \rangle \tag{0.9}$$

where O' denotes an operator on the cylinder. Using $z' = w$ in (0.7),

$$\begin{aligned} G_\beta(x, \tau; 0, 0) &= \left(\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}}\right)^{\Delta/2} \Big|_{z=e^{2\pi(x+i\tau)/\beta}} \left(\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}}\right)^{\Delta/2} \Big|_{z=1} && (0.10) \\ &\quad \times \langle O(z = e^{2\pi(x+i\tau)/\beta}, \bar{z} = e^{2\pi(x-i\tau)/\beta}) O(z = 1, \bar{z} = 1) \rangle \\ &= \left(\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}}\right)^{\Delta/2} \Big|_{z=e^{2\pi(x+i\tau)/\beta}} \left(\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}}\right)^{\Delta/2} \Big|_{z=1} \times (e^{4\pi x/\beta})^{-\Delta} && (0.11) \\ &= \left(\frac{1}{\sinh(\pi(x + i\tau)/\beta) \sinh(\pi(x - i\tau)/\beta)}\right)^{\Delta} && (0.12) \end{aligned}$$

4. The stress tensor is conserved, $\bar{\partial}T = 0$, so we can write it as just a function of z (not $\bar{z}$): $T = T_{zz}(z)$. The stress tensor is not primary: it has an extra, anomalous term in its transformation law. The correct transformation of the stress tensor turns out to be

$$T'(z') = \left(\frac{dz'}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{z', z\}\right] \tag{0.13}$$

where the brackets denote a 'Schwarzian derivative,'

$$\{f(z), z\} \equiv \frac{f'''}{f'} - \frac{3}{2}\frac{(f'')^2}{(f')^2} \tag{0.14}$$

The extra term in (0.13) is called the conformal anomaly, and the constant c is called the central charge. Conformal invariance on the plane implies $\langle T(z)\rangle_{plane} = 0$. Use the anomalous transformation law and mapping to the cylinder to determine the energy density of a 2d CFT in its vacuum state on a spatial circle of size L. (This is often called the Casimir energy.)

SOLUTION: Since we are talking about a circle of size L, we can use the exponential mapping above but with $\beta \to L$. The schwarzian for the exponential mapping is

$$\{w(z), z\} = \{\frac{L}{2\pi}\log z, z\} \tag{0.15}$$

$$= \frac{1}{2z^2}\ . \tag{0.16}$$

So

$$T_{cyl}(w) = \left(\frac{dz}{dw}\right)^2\left[T_{plane}(z) - \frac{c}{12}\frac{1}{2z^2}\right] \tag{0.17}$$

$$= -\frac{c}{24}\left(\frac{2\pi}{L}\right)^2 \tag{0.18}$$

5. By swapping your interpretation of 'space' and 'Euclidean time' on the cylinder in the previous question, find the energy density of a 2d CFT on a line at inverse temperature β.

SOLUTION: The calculation is the same, up to a tricky sign. Energy density means T_{tt}. When we interpret the Euclidean cylinder as defining a thermal state, we think of the relation between real and complex coordinates as $w = x + i\tau$. In Lorenzian language, $w = x - t$, $\bar{w} = x + t$. Because of the minus sign , T_{ww} gives the *negative* of the energy density. So we find

$$\text{energy density} = T_{tt} = -(T_{ww} + T_{\bar{w}\bar{w}}) = \frac{c}{12}\left(\frac{2\pi}{\beta}\right)^2 . \tag{0.19}$$