**It from Qubit 2016**

**QFT Basics**

**Problem Set #2**

Consider a CFT in 2d with complex coordinate $z = x + i\tau, \bar{z} = x - i\tau$. In Euclidean signature (real $\tau$), $z$ and $\bar{z}$ are complex conjugates, but in Lorentzian signature they are independent.

1. Show that the metric

$$ds^2 = dx^2 + d\tau^2 \tag{0.1}$$

is invariant, up to an overall local rescaling, under any *holomorphic* change of coordinates:

$$z \to z' = z'(z), \quad \bar{z} \to \bar{z}' = \bar{z}'(\bar{z}) \tag{0.2}$$

It follows that the 2d conformal algebra is infinite dimensional. (It is called the Virasoro algebra.)

2. Show that the exponential mapping $z = e^{2\pi w/\beta}$, $\bar{z} = e^{2\pi \bar{w}/\beta}$ maps the plane to the cylinder. It follows that results in 2d CFT on a cylinder are related to results on the plane; for example, 2pt functions on the cylinder are entirely fixed by conformal symmetry. (This is not true in higher dimensions). What do the circles $|z|$ =const map to on the cylinder?

3. Under a general conformal transformation (0.2), a correlation function of local primary scalar operators transforms as

$$\langle O_1'(z_1', \bar{z}_1') O_2'(z_2', \bar{z}_2') \cdots \rangle = \left(\frac{dz_1}{dz_1'}\right)^{\Delta_1/2} \left(\frac{d\bar{z}_1}{d\bar{z}_1'}\right)^{\Delta_1/2} \langle O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) \cdots \rangle \tag{0.3}$$

Use this formula, together with your results of parts (1) and (2), to derive the (Euclidean) thermal two-point function

$$G_\beta(x_1, \tau_1; x_2, \tau_2) = \operatorname{tr} e^{-\beta H} O(x_1, \tau_1) O(x_2, \tau_2) \tag{0.4}$$

(The answer to this question was the starting point on problem set #1.)

4. The stress tensor is conserved, $\bar{\partial} T = 0$, so we can write it as just a function of $z$ (not $\bar{z}$): $T = T_{zz}(z)$. The stress tensor is not primary: it has an extra, anomalous term

in its transformation law. The correct transformation of the stress tensor turns out to be

$$T'(z') = \left(\frac{dz'}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{z, z'\}\right] \tag{0.5}$$

where the brackets denote a 'Schwarzian derivative,'

$$\{f(z), z\} \equiv \frac{f'''}{f'} - \frac{3}{2}\frac{(f'')^2}{(f')^2} \tag{0.6}$$

The extra term in (0.5) is called the conformal anomaly, and the constant $c$ is called the central charge. Conformal invariance on the plane implies $\langle T(z)\rangle_{plane} = 0$. Use the anomalous transformation law and mapping to the cylinder to determine the energy density of a 2d CFT in its vacuum state on a spatial circle of size $L$. (This is often called the Casimir energy.)

5. By swapping your interpretation of 'space' and 'Euclidean time' on the cylinder in the previous question, find the energy density of a 2d CFT on a line at inverse temperature $\beta$.