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QFT Basics

Problem Set #1

SOLUTIONS

Consider a QFT in 2 spacetime dimensions, with a field ϕ and Euclidean action $S[\phi]$. Denote by $O(x)$ some scalar operator in this theory (it could be ϕ itself, or some other scalar, such at ϕ^2). We will assume the theory is *conformal*. One consequence of conformal symmetry is that correlation functions behave nicely under rescaling all the spacetime coordinates:

$$\langle O(x_1)O(x_2)\cdots O(x_n)\rangle = \lambda^{n\Delta}\langle O(\lambda x_1)O(\lambda x_2)\cdots O(\lambda x_n)\rangle \tag{0.1}$$

where $\Delta > 0$ is called the *scaling dimension* of the operator O.

1. Find the Euclidean 2-point function on a plane,

$$G_E(x_1, x_2) = \langle O(x_1)O(x_2)\rangle \tag{0.2}$$

by assuming a power law form. Here x_i labels a vector in R^2.

SOLUTION: By translation invariance, it is a function only of the separation: $G_E = G_E(|x_1 - x_2|)$. Assuming a power law $G_E = |x_1 - x_2|^a$ and imposing the scale symmetry (0.1) gives

$$G_E(x_1, x_2) = |x_1 - x_2|^{-2\Delta}\ . \tag{0.3}$$

2. Now consider the Lorentzian theory with space an infinite line. This is related to the Euclidean theory on a plane by analytic continuation in one of the two coordinates (ie, Wick rotation). Analytically continue your answer from part (1) to find the Lorentzian correlator

$$G_L(t_1, y_1; t_2, y_2) = \langle O(t_1, y_1)O(t_2, y_2)\rangle \tag{0.4}$$

where (t_i, y_i) labels a point in 2d Minkowski spacetime.[1]

[1] Your answer will have a branch-cut ambiguity when point 2 is causally connected to point 1, ie $(t_2 - t_1)^2 > (y_2 - y_1)^2$. This ambiguity corresponds to a choice of operator ordering in the Lorentzian correlator: $\langle O(t_1, y_1)O(t_2, y_2)\rangle$ vs $\langle O(t_2, y_2)O(t_1, y_1)\rangle$. Don't worry about this ambiguity for this problem, just pick one.

SOLUTION: First write $G_E = [(\tau_1 - \tau_2)^2 + (y_1 - y_2)^2]^{-\Delta}$ where τ is Euclidean time. Then take $\tau \to it$, giving

$$G_L = [-(t_1 - t_2)^2 + (y_1 - y_2)^2]^{-\Delta} \tag{0.5}$$

Note that for timelike separated points, there is an ambiguity from the branch cut in defining the power:

$$G_L = |(t_1^2 - t_2)^2 - (y_1 - y_2)^2|^{-\Delta} e^{\pm i\pi\Delta} \tag{0.6}$$

The choice of sign in the exponent corresponds to a choice of operator ordering (time-ordered or anti-time-ordered).

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Now we'll consider the same theory on a Euclidean cylinder with period β. The coordinates are $y \in (-\infty, \infty)$ (space) and $t_E \in [0, \beta)$ with $t_E \sim t_E + \beta$ (Euclidean time). It is convenient to label points instead by the complex combinations $w = y + it_E$, $\bar{w} = y - it_E$. It turns out that in 2d, the correlators on the cylinder are still entirely fixed by conformal symmetry. The 2-point function is

$$\langle O(w_1, \bar{w}_1) O(w_2, \bar{w}_2) \rangle_{cylinder} = \left(\frac{1}{\sinh(\pi(w_1 - w_2)/\beta) \sinh(\pi(\bar{w}_1 - \bar{w}_2)/\beta)} \right)^{\Delta} . \tag{0.7}$$

3. Write the path integral expression that would in principle compute (0.7), and draw a picture of this path integral (like the pictures in lecture).

SOLUTION:

The path integral expression for this correlator is just the usual path integral $\int [D\phi] e^{-S[\phi]} O(x_1) O(x_2)$ subject to periodic boundary conditions. The boundary conditions can be implemented for example by integrating on the strip of size β with a boundary condition ϕ_0, then summing over all ϕ_0:

$$\int d\phi_0(y) \int_{\phi(t_E=0,y)=\phi_0(y)}^{\phi(t_E=\beta,y)=\phi_0(y)} [D\phi] e^{-S[\phi]} O(w_1, \bar{w}_1) O(w_2, \bar{w}_2) \tag{0.8}$$

The corresponding picture is simply an infinite cylinder, of period β, with insertions at w_1 and w_2.

4. Translate your path integral into operator language by slicing the cylinder 'the short way', that is, defining states on slices of fixed y.

SOLUTION: In this case we interpret y as 'time' and t_E as 'space'. That is, the Hilbert space of states is defined on the t_E-circles at fixed y. The first seminfinite cylinder prepares the state $|0\rangle_{S^1_\beta}$, meaning the vacuum state on a circle of size β. Then we have an operator insertion O; then we evolve some more; then another operator insertion; then the other semi-infinite cylinder produces the vacuum ket. Writing this in operator language,

$$\langle O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)\rangle_{cylinder} = {}_{S^1_\beta}\langle 0|O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)|0\rangle_{S^1_\beta} \tag{0.9}$$

5. Now recast your path integral into operator language by slicing the cylinder 'the long way' (states defined at fixed t_E).

SOLUTION: Now we want to turn the path integral (0.8) into operator language. For a given value of the boundary condition ϕ_0, the rest of the path integral computes a transition amplitude, so can write (0.8) as

$$\int d\phi_0(y)\langle\phi_0|e^{-\beta H}O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)|\phi_0\rangle\ . \tag{0.10}$$

The remaining sum over boundary conditions is the definition of an operator trace:

$$\langle O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)\rangle_{cylinder} = \mathrm{tr}\, e^{-\beta H}O(w_1,\bar{w}_1)O(w_2,\bar{w}_2) \tag{0.11}$$

6. Check that your answer in (5) agrees with the operator definition of a thermal correlator derived in lecture, and check that it has the required periodicity in imaginary time. Thus real-time physics at finite temperature is related by analytic continuation to physics on the Euclidean cylinder.

SOLUTION: It's the same. The sinh function is periodic, $\sinh x = \sinh(x + i\pi)$, which means (0.7) is periodic under $t_E \to t_E + \beta$ as required.

7. Analytically continue $t_E \to it$ to compute the finite-temperature 2-point function in the real-time (Lorentzian) theory. [2]

[2] Again, don't worry about the branch cut, which is a choice of operator ordering. The most obvious choice will compute the time-ordered Lorentzian correlator.

SOLUTION: In (0.7) we simply set

$$w_i = y_i - t_i, \quad \bar{w}_i = y_i + t_i \ . \tag{0.12}$$

Note that w and $\bar{w}$ are no longer complex conjugates in Lorentzian signature. This holds generally. It is always important to *first* compute, *then* analytically continue to non-conjugate coordinates, not the other way around!

8. Find the late-time thermal 2-point function, $t_2 \gg t_1$. It should decay exponentially. (This is related to the fact that things thermalize: at late times, you should not be able to tell that a thermal state was perturbed initially.)

SOLUTION: At late times, we use

$$\sinh(x) \approx \frac{1}{2} e^{|x|} \tag{0.13}$$

to find

$$\langle O(t_1, y_1) O(t_2, y_2) \rangle_{cylinder} \approx 4^{\Delta} \exp\left[-\frac{2\pi\Delta}{\beta}(t_2 - t_1) \right] \ . \tag{0.14}$$

After a time of order β, the perturbation of the initial operator insertion becomes almost imperceptible.

9. Now suppose that instead of a line, we choose space to be a circle, so that $y \in (0, R)$ with $y \sim y + R$. Draw a picture of the path integral that would in principle compute the thermal 2-point function.

SOLUTION: Now both space and Euclidean time are periodic, so thermal correlators are computed by doing the path integral on a torus with periods (β, R).