Problems for Black Hole Information Paradox lectures

(1) Free scalar quantum field theory in d spacetime dimensions has a Hilbert space spanned by a set of orthonormal states labeled by spatial field configurations $|\phi\rangle$, a scalar field operator $\Phi(\vec{x})$ obeying $\Phi(\vec{x})|\phi\rangle = \phi(\vec{x})|\phi\rangle$, a canonical conjugate field $\Pi(\vec{x})$ obeying the algebra

$$\begin{aligned}[\phi(\vec{x}), \Pi(\vec{x}')] &= i\delta^{d-1}(\vec{x} - \vec{x}') \\ [\phi(\vec{x}), \phi(\vec{x}')] &= 0 \\ [\Pi(\vec{x}), \Pi(\vec{x}')] &= 0,\end{aligned}$$

and the Hamiltonian

$$H = \frac{1}{2}\int d^{d-1}x \left(\Pi^2 + \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi + m^2\Phi^2\right).$$

(a) Show that the Heisenberg field $\Phi(t, \vec{x}) \equiv e^{iHt}\Phi(\vec{x})e^{-iHt}$ obeys $\dot{\Phi} = \Pi$, and also that it obeys the equation of motion

$$\Box\Phi \equiv \partial_\mu\partial^\mu\Phi = \left(-\frac{d^2}{dt^2} + \vec{\nabla}\cdot\vec{\nabla}\right)\Phi = m^2\Phi.$$

(b) Check that

$$f_{\vec{k}}(t, \vec{x}) \equiv \frac{1}{\sqrt{2\omega_k}} e^{i\vec{k}\cdot\vec{x} - i\omega_k t}$$

solve the equation of motion provided that we take $\omega_k = \sqrt{k^2 + m^2}$, and use the canonical commutation relations given above to show that, if we expand the field in terms of these solutions as

$$\Phi(t, \vec{x}) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \left(f_{\vec{k}}(t, \vec{x})a_{\vec{k}} + f^*_{\vec{k}}(t, \vec{x})a^\dagger_{\vec{k}}\right),$$

then the operators $a_{\vec{k}}$, $a^\dagger_{\vec{k}}$ obey

$$\begin{aligned}[a_{\vec{k}}, a_{\vec{k}'}] &= 0 \\ [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'} &= 0 \\ [a_{\vec{k}}, a^\dagger_{\vec{k}'}] &= (2\pi)^{d-1}\delta(\vec{k} - \vec{k}').\end{aligned}$$

(c) Show that we can rewrite the Hamiltonian as

$$H = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \omega_k a^\dagger_{\vec{k}} a_{\vec{k}} + C,$$

where C is a constant that is proportional to the identity operator. What is its value? How should we interpret this?

(d) Briefly describe the structure of the spectrum of this Hamiltonian, and comment on what changes in the limit $m^2 \to 0$.

(2) Say we have coordinates $(t, x, \vec{y})$ on Minkowski space, with metric $ds^2 = -dt^2 + dx^2 + d\vec{y}^2$. We can define new coordinates via

$$\begin{aligned}x &= e^\xi \cosh\tau \\ t &= e^\xi \sinh\tau.\end{aligned}$$

(a) Argue that if we take $-\infty < \xi < \infty$, $-\infty < \tau < \infty$, these coordinates cover the right Rindler wedge, with $x^2 > t^2$ and $x > 0$.

(b) Show that in these coordinates the metric has the form

$$ds^2 = e^{2\xi}\left(-d\tau^2 + d\xi^2\right) + d\vec{y}^2.$$

(3) In curved spacetime, or with general coordinates in flat spacetime, we can write the wave equation as

$$\Box\Phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = m^2\Phi.$$

Here g is the determinant of the metric.

(a) Show that in the (τ, ξ) coordinates of the previous problem, the equation of motion becomes

$$\left(-\partial_\xi^2 + e^{2\xi}(m^2 - \partial_y^2) + \partial_\tau^2\right)\Phi = 0.$$

(b) Given a candidate solution of the form

$$f_{\omega,\vec{k}}(\tau, \xi, \vec{y}) = e^{i\vec{k}\cdot\vec{y} - i\omega\tau}\psi_{\omega,k}(\xi),$$

find the ordinary differential equation that $\psi_{\omega k}$ must obey. Does it look familiar?

(4) The Schwarzschild metric in $3+1$ dimensions is given (after setting $2GM = 1$) by

$$ds^2 = -\frac{r-1}{r}dt^2 + \frac{r}{r-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(a) Show that in the "tortoise" coordinate $r_* \equiv r + \log(r-1)$, this becomes

$$ds^2 = \frac{r-1}{r}(-dt^2 + dr_*^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(b) If we define a candidate solution

$$f_{\omega\ell m}(t, r, \Omega) = \frac{1}{r}Y_{\ell m}(\Omega)e^{-i\omega t}\psi_{\omega\ell}(r)$$

of the free scalar equation, show that we have

$$\left(-\frac{d^2}{dr_*^2} + V(r) - \omega^2\right)\psi_{\omega\ell} = 0,$$

and find the explicit form of $V(r)$.

(5) The Unruh temperature seen by an accelerating observer in quantum field theory is given by $k_B T = \frac{\hbar a}{2\pi c}$. Compute this temperature in Kelvin for one g of acceleration, and also compute it for the acceleration felt by the electron in the hydrogen atom. The Hawking temperature of a black hole is given by $k_B T = \frac{\hbar c^3}{8\pi GM}$. Compute this temperature in Kelvin for a solar-mass black hole and an earth-mass black hole, and find the mass in kg of a black hole whose temperature is $300K$.

(6) Say that we have a Hilbert space with a direct sum decomposition $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_{\overline{A}}$, with dimensionalities $|A|$ and $|\overline{A}|$ respectively, and say that $|\psi(U)\rangle = U|0\rangle$, with $|0\rangle$ an arbitrary state and U a random unitary. Intuitively we expect that if $|A| \ll |\overline{A}|$, then the projection of $|\psi(U)\rangle$ onto $\mathcal{H}_A$ should be small. We can formalize this as the statement that the trace distance of $|\psi(U)\rangle$ and $P_{\overline{A}}|\psi(U)\rangle$ should be small. Show that

$$\begin{aligned}\int dU \left|\left| |\psi(U)\rangle\langle\psi(U)| - \frac{1}{\langle\psi(U)|P_{\overline{A}}|\psi(U)\rangle}P_{\overline{A}}|\psi(U)\rangle\langle\psi(U)|P_{\overline{A}}\right|\right|_1 \\ = 2\int dU\sqrt{\langle\psi(U)|P_A|\psi(U)\rangle} \\ \leq 2\sqrt{\frac{|A|}{|A| + |\overline{A}|}}.\end{aligned}$$

You might want to refer section 5.3 of hep-th/1409.1231 for help.