1 Evaluating integrals using the residue theorem

Recall the residue theorem. If \( f(z) \) has singularities at \( z_1, z_2, \ldots, z_k \) which are enclosed by a contour \( C \) then

\[
\oint_C f(z) \, dz = 2\pi i \sum_{i=1}^{k} \text{Res of } f(z) \text{ at } z = z_i
\]

Figure 1: Using the residue theorem
2 Examples

2.1 Trigonometric Integrals

First we consider integrals of the form

\[ I = \int_{0}^{2\pi} \frac{1}{a + b \cos \theta} \, d\theta, \]  

with \( a > b > 0 \) and \((a, b) \in \mathbb{R}\).

Since this is an integral on the circle we can choose our contour \( C \) to be the unit circle on the complex plane. We set

\[ z = e^{i\theta} \]

which implies

\[ \frac{-idz}{z} = d\theta. \]

However, we want as our integrand to be a meromorphic function whose residues are easy to compute. So let us choose

\[ \cos \theta = \frac{z + 1/z}{2}. \]

Then our integral becomes

\[ I = -2i \oint_{C} \frac{dz}{2za + b + bz^2}. \]

So we have to find the zeros of the polynomial

\[ P(z) = bz^2 + 2za + b \]

that lie inside the unit circle. The two zeros are

\[ z_{\pm} = \frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}. \]

Note that since \( a > b \), we have \(-\frac{a}{b} < -1\) as well as \(a^2 - b^2 > 0\). Thus we see that \(z_{-} < -1\) and hence outside the unit circle. On the other hand

\[ z_+ z_- = \left( \frac{-a}{b} + \frac{\sqrt{a^2 - b^2}}{b} \right) \left( \frac{-a}{b} - \frac{\sqrt{a^2 - b^2}}{b} \right) = 1. \]

This implies

\[ |z_+| < 1. \]

Thus the residue of \( I(z) \) at \( z_+ \) is given by

\[ \lim_{z \to z_+} \frac{(z - z_+)}{(z - z_+)(z - z_-)} = \frac{-2i}{(z_+ - z_-)} = \frac{-2i}{\sqrt{a^2 - b^2}}. \]

Thus we have

\[ I = \frac{2\pi b}{\sqrt{a^2 - b^2}}. \]
2.2 Semi-circular Contours

Before we evaluate integrals that use semi-circular contours let us state a useful theorem:

Suppose we \( f(z) \) satisfies

\[
|f(z)| \leq \frac{M}{R^k}
\]

on a contour \( \Gamma \) parametrized by

\[
z = Re^{i\theta}
\]

for either \( 0 \leq \theta \leq \pi \) or \( \pi \leq \theta \leq 2\pi \) and, \( k > 1 \) and \( M \) are real positive constants. Then

\[
\left| \int_{\Gamma} f(z)dz \right| \leq \frac{M\pi}{R^{k-1}}
\]

and this vanishes as \( R \to \infty \) for \( k > 1 \).

2.3 An Application

Integrals of the form

\[
\int_{0}^{\infty} \frac{dx}{(a^2 + x^2)(b^2 + x^2)}
\]

can be evaluated by using semi-circular contours centred about the origin. Because the integrand is an even function, this integral is equal to

\[
\frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(a^2 + x^2)(b^2 + x^2)}.
\]

The complex version of the integrand

\[
\frac{1}{2} \frac{1}{(a^2 + z^2)(b^2 + z^2)}
\]

Figure 2: Semi-circular contour that closes on the upper half-plane.
has poles in the upper half-plane at 

\[ z = ia \text{ and } ib. \]

In the limit \( R \to \infty \) the contour integral reduces to real integral that we want to evaluate since the contribution from the semicircular part goes to zero very rapidly. Thus we have

\[
I = \frac{1}{2} \int_{\Gamma} dz \left( \frac{1}{(z^2 + a^2)(z^2 + b^2)} + \lim_{z \to ib} \frac{1}{(z^2 + a^2)(z + ib)} \right)
\]

= \frac{\pi}{2ab(a + b)}.

In the above calculation we assumed that \( a \neq b \). In case \( a = b \) we have double poles at \( z = ia \) and \( z = -ia \). In this case the integral can be evaluated using the formula for higher order pole that we saw in the last lecture:

\[
\frac{1}{2} 2\pi i \lim_{z \to ia} \frac{d}{dz} (z - ia) \frac{1}{(z - ia)^2(z + ia)} = \frac{\pi}{4a^3}.
\]

Note that, care should be taken in taking the limit \( R \to \infty \). In doing so we are evaluating the Cauchy principle value of the integral \( I \). If the two sides of the semicircle were taken to infinity in different ways we would get different results.

Another thing to note is that in the examples above we closed the contour in the upper half-plane, but we could have easily chosen the contour to close in the lower-half plane and we would have been able to evaluate the integral. This is because in the Cauchy principle value limit both contours reduce to the integral that we want to evaluate. In the next example we will see that we have to choose one particular contour.

### 2.4 Other applications of semicircular contours

Let’s consider the integral

\[
I = \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + x + 1} dx.
\]

We can consider this integral to be the imaginary part of

\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + x + 1} dx.
\]

So consider

\[
\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2 + z + 1} dz.
\]

We want to choose a contour that closes in the upper half-plane, because

\[
e^{i(x+iy)} = e^{ix}e^{-y} \to 0 \text{ in the upper half-plane.}
\]

\[
e^{ix}e^{-y} \to \infty \text{ in the lower half-plane.}
\]

\( z^2 + z + 1 \) has poles at

\[ z_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i. \]

So we evaluate the integral by computing the residue at \( z_{+} \):

\[
I = -\frac{2e^{-\sqrt{3}/2}\pi \sin \frac{1}{2}}{\sqrt{3}}.
\]
2.5 Mousehole Contours

Mousehole contours are useful for evaluating integrals where the integrand has a simple pole at the origin. Before we consider a concrete example of an integral let us prove a simple and useful theorem:

![Figure 3: A theorem for mouse holes](image)

Figure 3: A theorem for mouse holes

Suppose $f(z)$ has a simple pole at $z = a$ with residue $\sigma$ then consider the contour

$$\phi_\epsilon = a + \epsilon e^{it}, \ a \leq t \leq \beta,$$

then

$$\lim_{\epsilon \to 0} \int_{\phi_\epsilon} f(z) \, dz = (\beta - \alpha)i\sigma.$$

**Proof:** Define the function $g(z)$ which is holomorphic at $z = a$:

$$f(z) = g(z) + \frac{\sigma}{z - a}.$$

Now consider

$$\lim_{\epsilon \to 0} \int_{\phi_\epsilon} f(z) \, dz.$$

By continuity the contribution from the $g(z)$ term is zero and we have the desired result.

2.6 Example

Consider the integral

$$\int_0^{\infty} \frac{x - \sin x}{x^3} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} \, dx.$$

Now consider the function

$$f(z) = \frac{iz - e^{iz}}{z^3} = \frac{1}{z^3} \left[ i\partial - 1 - i\partial + \frac{z^2}{2!} - \cdots \right]$$

The imaginary part $f(z)$ on the real line coincides with our integrand but $f(z)$ has a triple pole at $z = 0$. So let us consider instead

$$h(z) = \frac{iz - e^{iz}}{z^3} + \frac{1}{z^3}.$$

In defining $h(z)$ we have kept the imaginary part of the function on the real line unchanged but we have subtracted the triple pole at $z = 0$. The subleading singularity in (3) is a simple pole and we can use our theorem above. We choose the contour below:
The integral then becomes
\[
\left( \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \right) h(z)dz = 0.
\]
In the limit \( R \to \infty \) the contribution from \( C_4 \) drops out and we have
\[
\int_{C_1} h(z)dz + \int_{C_2} h(z)dz = -\int_{C_3} h(z)dz
\]
\[
= \frac{(\pi - 0)i}{2}
\]
\[
= \frac{i\pi}{2}.
\]
So
\[
\int_0^\infty \frac{x - \sin x}{x^3} dx = \frac{\pi}{4}.
\]

2.7 Integrals involving logarithms

Sometimes we can take advantage of the fact that the log function is mutiple valued to evaluate certain integrals. For example, we can compute integrals of the form
\[
\int_0^\infty f(x)dx
\]
if \( f(z) \) has no singularities along the positive real axis. Let us consider the integral
\[
\oint_C f(z) \log z dz
\]
where we choose the contour \( C \) to be the one shown in figure. Such contours are often called keyhole contours. The integrals along \( C_2 \) and \( C_4 \) are
\[
\int_0^\infty f(x) \log x dx + \int_0^\infty f(x) (\log x + 2\pi i) dx
\]
\[
= -\int_0^\infty 2\pi i f(x) dx.
\]
Let us assume that in the limit
\[
R \to \infty \text{ the contribution from } C_1 \to 0
\]
\[
r \to 0 \text{ the contribution from } C_3 \to 0.
\]
This implies

$$\int_{0}^{\infty} f(x)dx = -\sum_{i} \text{Res of } f(z) \log z \text{ inside } C.$$  

As an example consider

$$\int_{0}^{\infty} \frac{1}{x^3+1} dx.$$  

As we saw in class the cube roots of $-1$ are at

$$e^{i\pi/3}, e^{i\pi} \text{ and } e^{5i\pi/3}.$$  

On the other hand we have on the unit circle $\log z = \log e^{i\theta} = i\theta = \text{Arg}(z)$. Thus the residues of the function

$$\frac{\log z}{z^3 + 1}$$  

on the unit circle will be $i\text{Arg}(z)$ divided by the derivative of the denominator:

$$\text{Res}(z) = \frac{i\text{Arg}(z)}{3z^2}.$$  

You can verify that the three residues at

$$e^{i\pi/3}, e^{i\pi} \text{ and } e^{5i\pi/3}.$$  

are

$$\left( -\frac{i\pi}{18} + \frac{\pi}{6\sqrt{3}} \right), \frac{i\pi}{3} \text{ and } -\frac{5i\pi}{18} - \frac{5\pi}{6\sqrt{3}},$$  

respectively.

which leads us to the result

$$\int_{0}^{\infty} \frac{1}{x^3+1} dx = \frac{2\pi}{3\sqrt{3}}.$$  
