Problem Set #4
Quantum Error Correction
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Due Tuesday, March 10, 2020

Problem #1. Correctness with weaker versions of fault tolerance
For this problem, we will consider a fault-tolerant protocol for a QECC capable of correcting $t$ errors.

a) Suppose we consider weaker versions of the ECRP and the GPP. The weak ECRP has a $2s$-filter on the right-hand-side for the output instead of an $s$-filter when there are $s$ faults in the EC gadget, with $2s \leq t$. The weak GPP (for a single-block gate) has an $(r + 2s)$-filter instead of an $(r + s)$-filter for the output on the right-hand-side when the input passes an $r$-filter and there are $s$ faults in the circuit, with $r + 2s \leq t$. The correctness conditions are unchanged. For a single-block gate exRec, how many faults can we allow within the exRec and still guarantee that the exRec is correct?

b) Consider the weak version of the ECRP and a very weak version of the GPP, with a $(2r + 2s)$-filter for the output on the right-hand-side instead of a $(r + 2s)$-filter, for $2r + 2s \leq t$. With these properties, how many faults can we allow within the exRec and still guarantee that the exRec is correct?

Problem #2. Pseudo-thresholds
For this problem, use the FTEC circuit for the 7-qubit code for which we computed the threshold in class, which contains 64 CNOT locations and 104 single-qubit locations (waits, Hadamards, measurements, and $|0\rangle$ preparations), and again ignore corrections due to post-selection. Assume all single-qubit locations have the same error rate $p_{\text{single}}$, but CNOT locations may have a different error rate $p_{\text{CNOT}}$.

a) Write down formulas for the logical error rate (i.e., after level reduction) for a single level of the 7-qubit code for logical CNOT and logical single-qubit locations in terms of $p_{\text{single}}$ and $p_{\text{CNOT}}$. Lump together different types of single-block extended rectangles; the largest single-block extended rectangle is for the Hadamard.

b) Now imagine that the physical error rate for single-qubit locations is $p_{\text{single}} = 0$. Calculate the pseudo-threshold for CNOT gates: $p_{PT}$ is the CNOT error rate at which the logical CNOT error rate after one level of the QECC is less than or equal to $p_{\text{CNOT}}$.

c) Show that after two levels of concatenation, the logical CNOT error rate is greater than $p_{PT}$ when the error rates on physical locations are $p_{\text{single}} = 0$, $p_{\text{CNOT}} = p_{PT}$. 