# Problem Set 8

Quantum Error Correction, 2018 spring
Instructor: Beni Yoshida
Due: At the beginning of the next lecture

**Problem 1. Circuit complexity of the GHZ state**

In the lecture, we showed that a constant-depth quantum circuit cannot create a ground state of the toric code from a product state $|0\rangle^{\otimes n}$. Prove a similar statement for an $n$-qubit GHZ state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\cdots 0\rangle + |1\cdots 1\rangle). \tag{1}$$

**Problem 2. Hypercube code and multi-qubit control-$Z$ gates**

This problem is a simpler version of section 4 of arXiv:1503.02065. Consider the following single-qubit phase operator:

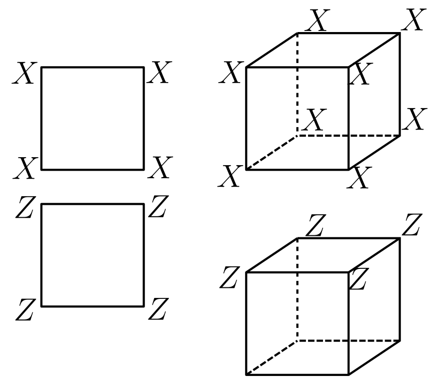
$$\mathcal{R}_m = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{2\pi}{2^m}} \end{bmatrix} = |0\rangle\langle 0| + e^{i\frac{2\pi}{2^m}}|1\rangle\langle 1| \qquad m \text{ is a non-negative integer.} \tag{2}$$

Here $\mathcal{R}_0 = I$, $\mathcal{R}_1 = Z$, $\mathcal{R}_2 = S$ and $\mathcal{R}_3 = T$. In the lecture, we learned that the $D$-dimensional topological color code with certain boundaries has a single logical qubit ($k = 1$) with the following transversal logical operator:

$$\overline{\mathcal{R}_D} = (\mathcal{R}_D)^{\otimes n_{\mathrm{odd}}} \otimes (\mathcal{R}_D)^{\otimes n_{\mathrm{even}}} \tag{3}$$

where $n_{\mathrm{odd}}$ and $n_{\mathrm{even}}$ are the number of qubits at odd and even sites when the lattice is viewed as a bipartite graph. Namely, we showed that $\overline{\mathcal{R}_D}$ acts as a logical $\mathcal{R}_D$ (or $\mathcal{R}_D^{\dagger}$) operator. We also learned that the smallest realization is the so-called $D$-th level Reed-Muller code.

In this problem, we treat the cases where the color code has multiple logical qubits. The code below is the smallest realization of the $D$-dimensional topological color code with $k = D$ logical qubits. Consider a stabilizer code defined on a $d$-dimensional hypercube with $n = 2^D$ qubits living on vertices. The code has only one $X$-type stabilizer generator, $X^{\otimes n}$, acting on all the qubits, while $Z$-type stabilizer generators are four-body and are defined on each two-dimensional face. Two-dimensional and three-dimensional examples are shown below:




(4)

In three dimensions, there are six $Z$-type stabilizers. But not all of them are independent!

(The three-dimensional code has eight qubits, and has a transversal non-Clifford gate as we show below. To the best of my knowledge, this is the smallest qubit stabilizer code with such a property).

(a) Let us define a *commutator* of two unitary operators as follows:

$$\mathcal{K}(V, W) = VWV^\dagger W^\dagger. \tag{5}$$

Show that

$$\mathcal{K}(\mathcal{R}_m, X) \propto \mathcal{R}_{m-1} \qquad \text{for all } m \geq 1. \tag{6}$$

(b) Consider a Hilbert space of $m$ qubits. Let $X_1$ be a Pauli-$X$ acting on the first qubit. Let us define a multi-qubit Control-$Z$ gate as follows:

$$\mathrm{C}^{\otimes m-1}Z|j_1, \ldots, j_d\rangle = (-1)^{j_1 \cdots j_m}|j_1, \ldots, j_d\rangle \qquad j_m = 0, 1. \tag{7}$$

Here $j_1 \cdots j_m$ means a product of $j_1, \ldots, j_m$. Compute the commutator $\mathcal{K}(\mathrm{C}^{\otimes m-1}Z, X_1)$.

(c) Show that the code has $D$ logical qubits. Show that the code distance (minimal weight of a non-trivial logical operator) is two.

(d) Show that $\overline{\mathcal{R}_d} = (R_d)^{\otimes n}$ is a logical operator of the code. Also show that it acts as a logical $\mathrm{C}^{\otimes d-1}Z$ gate. If you find this problem difficult, you can do the $D = 3$ case only.