Mass term in entanglement entropy

The aim of this problem is to obtain terms induced by a field mass in EE. For an interval in $1+1$ dimensions and a massless scalar we expect an EE having a logarithmic divergence $S = 1/3\log(R/\epsilon)$[1]. We are using that the central charge of the scalar field is $c = 1$. One can understand that half of the divergence comes from each of the two end-points of the interval, where short wavelength entanglement accumulates. This gives a $S \sim -1/6\log(\epsilon)$ divergence for each boundary point. If we turn on a mass m for the field, for large Rm, the EE will stop increasing with R since the modes of long wavelength $\sim R$ that can sense the full size of the interval are highly suppressed by the mass. Hence, in this limit entanglement across the boundaries of the intervals will happen independently in each of the boundaries. However, we expect the same $\log(\epsilon)$ dependence, since UV short wavelength modes will not notice the mass. Hence, dimensional analysis gives

$$S = -\frac{1}{3}\log(m\epsilon) + \text{const}\,, \tag{1}$$

for $Rm \gg 1$.

This gives a contribution $-\frac{1}{6}\log(m\epsilon)$ for each boundary point. You are going to compute this using a direct calculation of the partition function in the replica trick[2]. Since we know the result depends on only one boundary we can set the interval size to infinity. The replica manifold is then an Euclidean cone of opening angle $2\pi n$. In polar coordinates $r \in (0,\infty)$ and $\theta \in (0, 2\pi n)$ with periodic identification $\theta \equiv \theta + 2\pi n$. At the end of these pages there are several formulas about Bessel functions that you may find useful for solving this exercise.

a) Show using the path integral representation that

$$\frac{d}{dm^2}\log Z = -\frac{1}{2}\int d^2x\, G(x,x)\,, \tag{2}$$

where $G(x,y)$ is the Euclidean Green function of the scalar field. This satisfies the differential equation

$$(-\nabla_x^2 + m^2)G(x,y) = \delta^2(x-y)\,. \tag{3}$$

Then you will need to evaluate the Green function at coincidence points for computing (2).

b) Equation (3) tells the Green function is the inverse of the differential operator $(-\nabla^2 + m^2)$. To obtain it you can diagonalize this operator first. That is, you can solve the eigenvalue equation (in polar coordinates)

$$(-\nabla^2 + m^2)\psi(r,\theta) = \lambda\psi(r,\theta)\,. \tag{4}$$

This can be done by proposing a separable function $\psi(r,\theta) = f(r)g(\theta)$. You have to choose solutions that obey the periodicity condition on the angle and that are regular at $r = 0$. You should find a set of eigenfunctions

$$\psi_{k,\nu}(r,\theta) = N e^{i\theta\frac{k}{n}} J_{|k/n|}(\nu r)\,, \tag{5}$$

where k is any integer, J is the Bessel function, $\nu \in (0,\infty)$, N is a normalization factor, and $\lambda = \nu^2 + m^2$. Then the eigenfunctions are described by a discrete parameter k and a continuous parameter ν.

c) Normalize the Eigenfunctions such that

$$\int d^2x\, \psi^*_{k,\nu}(r,\theta)\psi_{k',\nu'}(r,\theta) = \delta_{k,k'}\,\delta(\nu-\nu')\,. \tag{6}$$

Use this spectral decomposition to write the Green function as

$$G(r,\theta,r',\theta') = \sum_{k=-\infty}^{\infty}\int_0^\infty d\nu\, \frac{1}{\nu^2+m^2}\psi_{k,\nu}(r,\theta)\psi^*_{k,\nu}(r',\theta')\,. \tag{7}$$

[1] For a massless scalar in $d = 2$ there are other terms that depend on an infrared regulator. These will not enter the calculation for a massive field we are interested in here.

[2] This calculation is done in P. Calabrese and J. L. Cardy, "Entanglement entropy and quantum field theory," J. Stat. Mech. **0406**, P06002 (2004) [hep-th/0405152].

d) Evaluate the Green function at coincidence points and integrate over θ and ν. Plot the resulting integrand in r (which also contains the volume factor r) for different values of k/n. If you subtract a constant that is independent of k/n from the integrand you should be able to integrate in r to get

$$\frac{d}{dm^2}\log Z = -\frac{1}{2}\int d^2x\, G(x,x) = \frac{1}{m^2}\sum_{k=-\infty}^{\infty}\left(\frac{|k|}{4n} + \mathrm{U}\right). \tag{8}$$

The constant U in the parenthesis is divergent, but does not depend on k or n. You can make it finite thinking in a cone a finite size R, but this constant will anyway disappear from the final result. You get a divergent result also because of the sum in k. This is natural, since the Green function at coincidence points is divergent, and the divergence is also present in the plane without conical singularities.

e) Remember that Renyi entropies depend on the normalized partition function $Z(n)/Z(1)^n$,

$$S_n = (1-n)^{-1}(\log Z(n) - n\log Z(1)). \tag{9}$$

Hence a sensible quantity to compute is the combination

$$\frac{d}{dm^2}(\log Z(n) - n\log Z(1)). \tag{10}$$

Divergences should cancel in this subtraction. However, in order that this cancelation happens we have to regularize "in the same way" both terms. The best way to do it is to think in a physical regularization. The sum you have is in the integers k. The ratio k/n gives the momentum conjugated to the angular variable θ. Then if we imagine discretizing the continuous angle θ in small pieces $\delta\theta$ this will correspond to some cutoff in $|k/n|$. Hence, using the same cutoff for $n \neq 1$ and $n = 1$ means using the same cutoff function $F(|k/n|)$ both for $n \neq 1$ and $n = 1$. This cutoff function has to go to 1 for small values of $|k/n|$ and go to zero for large values of $|k/n|$. Here the choice of this function can be highly arbitrary. For example, you can write the sum as a sum over positive integers, and take $e^{-s|k/n|}$ as a cutoff function. After performing the sums and the calculation (10) let $s \to 0$. Other regularizing functions should give the same result. A general argument uses the Euler-MacLaurin formula for converting sums into integrals, see the paper by Calabrese and Cardy cited in the footnote above.

You should get

$$\frac{d}{dm^2}(\log Z(n) - n\log Z(1)) = \frac{1}{24m^2}\left(n - \frac{1}{n}\right). \tag{11}$$

This gives, inserting the UV cutoff to match dimensions,

$$S_n = -\frac{1}{12}\frac{n+1}{n}\log(m\epsilon). \tag{12}$$

Setting $n = 1$ gives the result for the entropy.

Useful formulas

In these formulas $a \geq 0$.

$$\int_0^\infty dr\, r\, J_a(\nu r) J_a(\nu' r) = \frac{1}{\nu}\delta(\nu - \nu'). \tag{13}$$

$$\int_0^\infty d\nu\, J_a(\nu r)^2 \frac{\nu}{\nu^2 + m^2} = I_a(rm) K_a(rm), \tag{14}$$

where I_a and K_a are the modifyed Bessel Functions.

$$\int_0^\infty dr\,(r I_a(r) K_a(r) - 1/2) = -\frac{a}{2}. \tag{15}$$

Remarks

This term in the entropy can be thought as a "dressing" of the area term as we go from the UV (small regions) to the IR (large regions). Similar terms appear in any dimensions. You can try to compute them in more dimensions by decomposing the fields in higher dimensions in Fourier modes in the direction parallel to the entangling surface. Each of these modes will contribute similarly to the two dimensional case, but with a modifyed mass. See the paper by Calabrese and Cardy and also M. P. Hertzberg and F. Wilczek, "Some Calculable Contributions to Entanglement Entropy," Phys. Rev. Lett. **106**, 050404 (2011) doi:10.1103/PhysRevLett.106.050404 [arXiv:1007.0993 [hep-th]], for a calculation using the heat kernel technique. Notice that if black hole entropy is to be interpreted as some entanglement entropy across the horizon, this dressing of the area term due to low energy physics should also be reflected in a corresponding renormalization of Newton's constant. This was proposed in L. Susskind and J. Uglum, Black hole entropy in canonical quantum gravity and superstring theory, Phys. Rev. D 50, 2700 (1994) [hep-th/9401070]. For further references on this relation and some recent developments see H. Casini, F. D. Mazzitelli and E. Test, "Area terms in entanglement entropy," Phys. Rev. D **91**, no. 10, 104035 (2015) [arXiv:1412.6522 [hep-th]].