Area law for entanglement entropy

The aim of this problem is to compute the area law of entropy for a free scalar field in vacuum, reduced to a spherical region. This calculation was first done by Srednicki in http://arxiv.org/abs/hep-th/9303048. We will use the formula for the entropy of bosonic Gaussian states in terms of correlation functions and apply it to vacuum state for a scalar field with a particular radial lattice discretization.

a) The Hamiltonian of a free massless scalar field is

$$H = \frac{1}{2}\int d^{d-1}x \; \left(\pi^2(x) + (\nabla\phi(x))^2\right) \,, \tag{1}$$

and

$$[\phi(x), \pi(y)] = i\delta(x-y)\,. \tag{2}$$

Consider the case of four spacetime dimensions $d = 4$ (the case of two spatial dimensions, $d = 3$ can be treated in a similar way). Expand field and momentum in spherical waves using

$$\begin{aligned} \phi_{lm}(r) &= r\int d\Omega\, \phi(x)\, Y_{lm}(\theta,\varphi)\,, &(3)\\ \pi_{lm}(r) &= r\int d\Omega\, \pi(x)\, Y_{lm}(\theta,\varphi)\,, &(4) \end{aligned}$$

where $Y_{lm}(\theta,\varphi)$ are the spherical harmonics. Show that the problem decomposes en each l, m and you have

$$\begin{aligned} [\phi_{lm}(r), \pi_{l'm'}(r')] &= i\delta_{ll'}\delta_{mm'}\delta(r-r')\,, &(5)\\ H &= \sum_{lm} H_{lm}\,, &(6)\\ H_{lm} &= \frac{1}{2}\int_0^\infty dr\, \left(\pi^2_{lm}(r) + r^2\left[\frac{\partial}{\partial r}\left(\frac{\phi_{lm}(r)}{r}\right)\right]^2 + \frac{l(l+1)}{r^2}\phi^2_{lm}(r)\right)\,. &(7) \end{aligned}$$

b) Discretize the radial variable in integer numbers, and we will also need to put the system in a spherical box to make it finite. Then consider the radius can take values $0, 1, ..., M$, where M is the size of the box. The discrete variables and Hamiltonian obey

$$\begin{aligned} [\phi_{lm,j}, \pi_{l'm',j'}] &= i\delta_{ll'}\delta_{mm'}\delta_{jj'}\,, &(8)\\ H_{lm} &= \frac{1}{2}\sum_{j=1}^{M}\left(\pi^2_{lm,j} + (j+1/2)^2\left(\frac{\phi_{lm,j}}{j} - \frac{\phi_{lm,j+1}}{j+1}\right)^2 + \frac{l(l+1)}{j^2}\phi^2_{lm,j}\right)\,, &(9) \end{aligned}$$

where $\phi_{lm,M+1} = 0$.

c) Now you are in position of computing numerically the entropy of a spherical region V of size R by computing the entropy of regions formed by the points $j = 1, ..., n$ in each of the modes l, m, and summing over l, m. The radius R can be conveniently expressed as $R = n + 1/2$. For doing this calculation you should:

i) For fixed l, m write down the $M\times M$ matrix $K_{(l)}$ of the expression $H = \frac{1}{2}\sum \pi^2_{lm,i} + \frac{1}{2}\sum_{ij}\phi_{lm,i}K^{ij}_{(l)}\phi_{lm,j}$ for a quadratic Hamiltonian (see the problem "entropy for Gaussian states") by extracting it from eq. (9).

ii) Compute the $M\times M$ correlation matrices for field and momentum $X_{(l)} = 1/2K^{-1/2}_{(l)}$, $P_{(l)} = 1/2K^{1/2}_{(l)}$.

iii) Reduce these matrices to the ones $X^V_{(l)}, P^V_{(l)}$ in the region V by taking the first $n\times n$ blocks of the correlation matrices.

iv) Compute the entropy $S(n,l)$ using formula (this entropy does not depend on m)

$$C^V_{(l)} = \sqrt{X^V P^V}\,, \tag{10}$$

$$S_{n,l} = \mathrm{tr}\left((C^V_{(l)}+1/2)\log(C^V_{(l)}+1/2)-(C^V_{(l)}-1/2)\log(C^V_{(l)}-1/2)\right)\,. \tag{11}$$

(Warning: some eigenvalues may be very near 1/2, and numerical error can give complex values in (11). Notice that these eigenvalues do not contribute much to the entropy and you can simply choose to eliminate these eigenvalues rather than computing them with higher precision.)
v) The entropy is given by

$$S(R) = \sum_{l=0}^{\infty}(2l+1)S_{n,l}\,. \tag{12}$$

You are not going to sum over an infinite number of l. Check that $(2l+1)S(n,l)$ goes to zero as $l^{-3}\log(l)$ for large l. You can either cut the sum at sufficiently large l or fit the behavior for large l and sum the fit of the large l tail to infinity. Analytic perturbative calculation of $S(n,l)$ for large l can also be done (see the paper by Srednicki).

Then you can change the value of R and plot the entropy as a function of R. You should find an area law behavior $S(R) \simeq cR^2$. The constant $c \sim 0.29$. You should check this result is not much modified by changing the infrared cutoff M. Taking $M = 100$ and $R \in (1, 50)$ and summing for $l < 1000$ is good enough.

Remarks and generalizations

0- Please see Mark Mezei's focus lecture this afternoon (Thursday, July 21), as he will describe this problem and its solution.

1- At your leisure, you can investigate the case of dimension $d = 3$, and $d = 5$. It is known that the sum over angular momentum does not converge for $d > 4$ (see http://arxiv.org/abs/quant-ph/0605112), and radial discretization is then not enough to get a finite entropy. What about mutual information $I(R_1, R_2)$ between a sphere of radius R_1 and the exterior of another sphere of radius $R_2 > R_1$? Does the sum over angular momentum for this mutual information converge? (I do not know the answer to this last question, by my guess is yes).

2- Doing the same calculation for a scalar field in $d = 4$ with more precision you should be able to see a subleading logarithmic term,

$$S(R) = cR^2 + a\log(R) + \mathrm{const}\,. \tag{13}$$

The number $a = -1/90$ can be computed analytically by other methods, and is proportional to the trace anomaly of the theory. It has information on the theory in the continuum limit and is independent on the way the entropy was regularized. Because of this it is called a universal term. This numerical calculation is however tricky. It is not necessary to have more numerical precision but larger M and summing for more l is important, as well as having a good approximation for the tail of large l. It is also important to treat the low l with larger M, or to fit the results for increasing values of M. See http://arxiv.org/abs/arXiv:0911.4283

3- The method can be applied to other regularizations such as square lattices, and, with small modifications, to other fields, as far as they are free (quadratic Hamiltonian), such as fermions (see for example http://arxiv.org/abs/arXiv:0905.2562) and gauge fields (http://arxiv.org/abs/arXiv:1406.2991).