It from Qubit Summer School
Problems for Tensor Networks

G. Vidal

July 2016

1 Matrix Product State

Consider a lattice made of N sites, where each site is described by a two-dimensional vector space $\mathbb{C}_2$ so that the total vector space is $(\mathbb{C}_2)^{\otimes N}$. A matrix product state (MPS) representation of a pure state $|\Psi\rangle \in (\mathbb{C}_2)^{\otimes N}$ is of the form (see Fig. 1)

$$|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_n} \mathbf{v}_l^\dagger \cdot A_{i_1} \cdot A_{i_2} \cdot \cdots \cdot A_{i_n} \cdot \mathbf{v}_r \ |i_1 i_2 \cdots i_N\rangle \,, \tag{1}$$

where each index i takes two values $i = 0, 1$, $\mathbf{v}_l$ and $\mathbf{v}_r$ and χ-dimensional vectors, and A_0 and A_1 are $\chi \times \chi$ matrices.

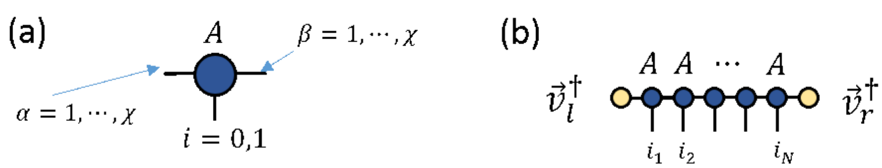

Figure 1: (a) A tensor A of an MPS has three indices: a physical index i which is used to label a basis of the Hilbert space of one lattice site (in our example, $i = 0, 1$) and two bond indices α and β that take χ different values. Upon fixing the value $i = 0, 1$ of the physical index, we obtain a $\chi \times \chi$ matrix (matrices A_0 and A_1), hence the name *matrix product state*. (b) A certain class of MPS for N lattice sites can be produced by multiplying N copies of the same tensor A, together with a left vector $\mathbf{v}_l$ and a right vector $\mathbf{v}_r$.

1.a Check that the GHZ state

$$|GHZ\rangle \equiv \frac{1}{\sqrt{2}} \left(|00\cdots 0\rangle + |11\cdots 1\rangle \right) \tag{2}$$

can be represented by an MPS with $\chi = 2$ and with vectors and matrices given by

$$\mathbf{v}_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3}$$

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

1.b Write an MPS with $\chi = 1$ for each of the product states

$$|prod_1\rangle = |00\cdots 0\rangle, \tag{5}$$

$$|prod_2\rangle = |11\cdots 1\rangle, \tag{6}$$

$$|prod_3\rangle = |\phi\phi\cdots\phi\rangle, \qquad |\phi\rangle \equiv a\,|0\rangle + b\,|1\rangle, \tag{7}$$

$$|prod_4\rangle = \left|\phi^\perp\phi^\perp\cdots\phi^\perp\right\rangle, \qquad \left|\phi^\perp\right\rangle \equiv b^*\,|0\rangle - a^*\,|1\rangle, \tag{8}$$

where $a, b \in \mathbb{C}$.

1.c Write an MPS with $\chi = 2$ for the generalized GHZ state

$$\left|GHZ'\right\rangle = \alpha\,|\phi\phi\cdots\phi\rangle + \beta\left|\phi^\perp\phi^\perp\cdots\phi^\perp\right\rangle, \tag{9}$$

where $\alpha, \beta \in \mathbb{C}$.

1.d Write an MPS with $\chi = 2$ for the W state,

$$|W\rangle \equiv \frac{1}{\sqrt{N}} \left(|10\cdots 0\rangle + |01\cdots 0\rangle + \cdots + |00\cdots 1\rangle \right) \tag{10}$$

and for the generalized W state

$$\left|W'\right\rangle \equiv \alpha\,|00\cdots 0\rangle + \beta\,|W\rangle \tag{11}$$

2 Tree Tensor Network

A tree tensor network (TTN) can be regarded as a simplified version of the multi-scale entanglement renormalization ansatz (MERA) where the disentanglers are trivial (equal to the identity operator). For $N = 4$ sites, a TTN is of the form (see Fig. 2a)

$$|\Psi\rangle = \sum_{i's} \sum_{\alpha,\beta's} v_\alpha A^{\alpha}_{\beta_1\beta_2} B^{\beta_1}_{i_1 i_2} B^{\beta_2}_{i_3 i_4} |i_1 i_2 i_3 i_4\rangle \tag{12}$$

where v is a χ-dimensional vector, and A and B are rank 3 tensors that can be regarded as linear maps from one to two sites

$$\begin{aligned} A &: \mathbb{C}^{|\alpha|} \to \mathbb{C}^{|\beta|} \otimes \mathbb{C}^{|\beta|}, &(13)\\ B &: \mathbb{C}^{|\beta|} \to \mathbb{C}^{|i|} \otimes \mathbb{C}^{|i|}, &(14)\end{aligned}$$

where

$$\alpha = 1, \cdots, |\alpha|, \quad \beta = 1, \cdots, |\beta|, \quad i = 1, \cdots, |i|.$$

2.a Check that the generalized W state $|W'\rangle$ of Eq. 11 on $N = 2^Q$ sites (for $Q > 0$ an integer) can be represented with a TTN with Q layers of identical tensors $A = B = C = \ldots$ (see Figs. 2b-2c), with bond dimension $|\alpha| = |\beta| = |\gamma| = \chi = 2$ and vector c and tensor A given by

$$\begin{aligned} v &= \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &(15)\\ A &= |00\rangle\langle 0| + \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} &(16)\end{aligned}$$

by noticing first that,

$$A \cdot v = \alpha|00\rangle + \frac{\beta}{\sqrt{2}}\left(|10\rangle + |01\rangle\right), \tag{17}$$

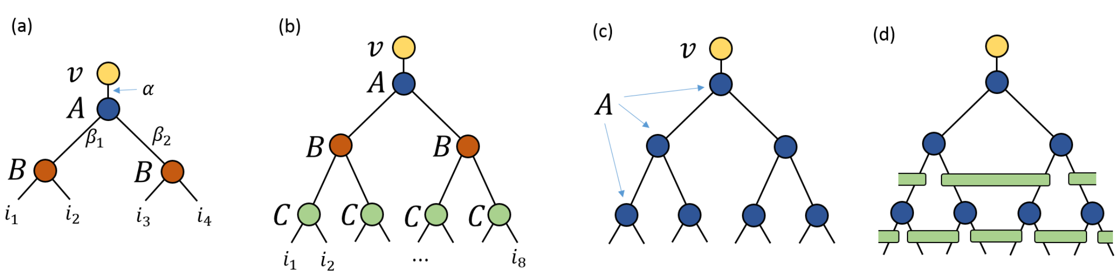

Figure 2: (a) Diagrammatic representation of the TTN for 4 sites of Eq. 12. (b) TTN for 8 sites, with different tensors A, B, and C in different layers. (c) TTN for 8 sites, with the same tensor A on all the layers. (d) MERA, which is obtained from a TTN by adding extra tensors known as disentanglers.

but also that

$$(A \otimes A) \cdot A \cdot v = \alpha |0000\rangle + \frac{\beta}{\sqrt{4}} (|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle) . \quad (18)$$

2.b Write a TTN for the product states $|prod_1\rangle, |prod_2\rangle, |prod_3\rangle, |prod_4\rangle$ of Eqs. 5-8.

2.c Write a TTN for the GHZ state $|GHZ\rangle$ of Eq. 2 and for the generalized GHZ state $|GHZ'\rangle$ of Eq. 9.

2.d Write a TTN for the product of entangled pairs (see Fig. 3b)

$$|pairs_1\rangle \equiv |\psi_{12}\rangle \otimes |\psi_{34}\rangle \otimes |\psi_{56}\rangle \otimes |\psi_{78}\rangle , \quad (19)$$

$$|\psi_{AB}\rangle \equiv \frac{1}{\sqrt{2}} (|0_A 0_B\rangle + |1_A 1_B\rangle) . \quad (20)$$

You should be able to use bond dimension $\chi = 1$: in joining sites 1 and 2 together (and sites 3 and 4 together, etc), the entanglement disappears and we end up with a product, unentangled state.

2.e The lowest layer of tensors (tensors C) in the above example can be understood as a coarse-graining transformation that maps the state $|pairs_1\rangle$ of a lattice made of 8 sites into the product state $|0000\rangle$ of an effective lattice made of 4 sites (Fig. 3. Try to similarly coarse-grain the state

$$|pairs_2\rangle \equiv |\psi_{81}\rangle \otimes |\psi_{23}\rangle \otimes |\psi_{45}\rangle \otimes |\psi_{67}\rangle . \quad (21)$$

into some state $|\Psi'\rangle$ of an effective lattice of 4 sites, by a suitable choice of tensors C, such that $|pairs_2\rangle = (C \otimes C \otimes C \otimes C) |\Psi'\rangle$ (see Fig. 3c). [**Hint**: this time, in joining sites 1 and 2 together, the entanglement between sites 8 and 1 (and between 2 and 3) is not eliminated and we cannot produce a product, unentangled state; tensor C cannot compress at all the dimension of the vector space $\mathbb{C}^2 \otimes \mathbb{C}^2$ of two sites].

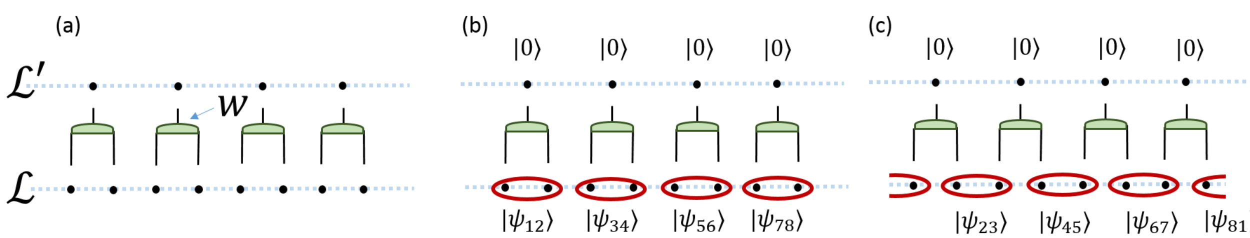

Figure 3: (a) Coarse-graining transformations defined by a row of tensors of a TTN. (b) Can we use this coarse-graining transformation to (reversibly) map the state $|pairs_1\rangle$ to the product state $|0000\rangle$? (c) Can we use this coarse-graining transformation to (reversibly) map the state $|pairs_2\rangle$ to the product state $|0000\rangle$?

3 Multi-scale entanglement renormalization ansatz

The multi-scale entanglement renormalization ansatz (MERA) (see Fig. 2d) contains double rows of tensors. Each double row consists of a row of disentanglers u followed by a row of isometries w, and can be regarded as a coarse-graining transformation (see Fig. 4a). Disentanglers u are linear maps between two sites,

$$u : \mathbb{C}^\chi \otimes \mathbb{C}^\chi \to \mathbb{C}^\chi \otimes \mathbb{C}^\chi, \tag{22}$$

whereas isometries w are linear maps from one site into two sites

$$w : \mathbb{C}^\chi \to \mathbb{C}^\chi \otimes \mathbb{C}^\chi. \tag{23}$$

3.a Try to use a double row of MERA tensors to transform the GHZ state of Eq. 2 into the product state $|prod_1\rangle$ of Eq. 5. Why can you not succeed?

Conclusion 1: The GHZ state contains long-range entanglement: entanglement that cannot be removed or introduced just through a *local* re-organization of degrees of freedom. As a result, a local coarse-graining transformation cannot remove the entanglement in the GHZ state.

3.b Use one double row of MERA tensors (with $\chi = 2$) to transform the product of entangled pairs $|pairs_1\rangle$ of Eq. 19 into the product state $|prod_1\rangle$ of Eq. 5 (see Fig. 4b). [**Hint**: choose trivial disentanglers $u = \mathbb{I}_\chi \otimes \mathbb{I}_\chi$ and the isometries of the TTN in exercise 2.d].

3.c Use one double-row of MERA tensors (with $\chi = 2$) to transform the product of entangled pairs $|pairs_2\rangle$ of Eq. 21 into the product state $|prod_1\rangle$ of Eq. 5 (see Fig. 4c). [**Hint**: this time you will have to use the disentangler u_{23} to transform the entangled pair $|\psi_{23}\rangle$ into product state $|00\rangle_{23}$.

Conclusion 2: The states $|pairs_1\rangle$ and $|pairs_2\rangle$ only contain short-range entanglement [specifically, nearest neighbour entanglement]. A local coarse-graining transformation (based on either a layer of TTN or of MERA) can remove the short-range entanglement from $|pairs_1\rangle$. However, disentanglers are required to remove the short-range entanglement from $|pairs_2\rangle$. Therefore, disentanglers u play the key role of allowing the removal of short-range entanglement that isometries w alone would not be able to remove.

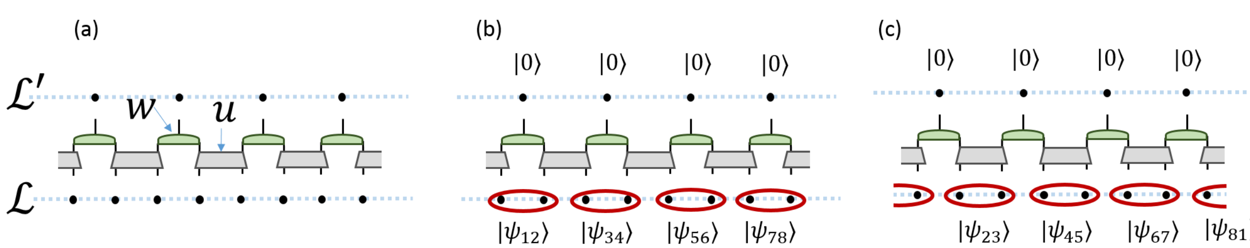

Figure 4: (a) Coarse-graining transformations defined by a double row of tensors a MERA (b) Can we use this coarse-graining transformation to map the state $|pairs_1\rangle$ to the product state $|0000\rangle$? (c) Can we use this coarse-graining transformation to map the state $|pairs_2\rangle$ to the product state $|0000\rangle$?

4 Scaling of entanglement in an MPS

Consider an MPS for the state of a 1D lattice made of $N \to \infty$ sites, given by (infinitely many) copies of a tensor $A_i^{\alpha\beta}$ and some boundary vectors $\mathbf{v}_l$ and $\mathbf{v}_r$ which will not play a role in this discussion (see Fig. 1.

4.a Using diagrammatic notation, build a tensor network representation of the reduced density matrix ρ_L for a region made of L adjacent sites (see Fig. 5).

4.b What is the smallest number of bond indices that you need to break in order to split the tensor network for ρ_L into two pieces, one containing the L "bra" indices and the other containing the L "ket" indices? If the bond dimension is χ, give an upper bond for the rank (number of non-zero eigenvalues) of ρ_L.

4.c Recall that the von Neumann entropy

$$S(\rho_L) \equiv -\mathrm{tr}\left(\rho_L \log_2\left(\rho_L\right)\right) \tag{24}$$

is equal to the Shannon entropy

$$S(\{p_\mu\}) \equiv -\sum_\mu p_\mu \log_2(p_\mu) \tag{25}$$

of the eigenvalues $\{p_\mu\}$ of ρ_L. In addition, the Shannon entropy of a probability distribution $\{p_\mu\}_{\mu=1,\cdots,|\mu|}$ with $|\mu|$ non-zero probabilities is maximized by the flat distribution $|\mu|^{-1}(1,1,\cdots,1)$, which has entropy $-|\mu| \times 1/|\mu| \log_2(1/|\mu|) = \log_2(|\mu|)$. Therefore we have the following upper bound for the entanglement entropy of ρ_L: $S(\rho_L) \leq \log_2(|\mu|)$. Use this bound to show that the entanglement entropy of ρ_L for an MPS is at most

$$S(\rho_L) \leq 2\log_2(\chi). \tag{26}$$

This upper bound is compatible with the *area law* in $D = 1$ spatial dimensions.

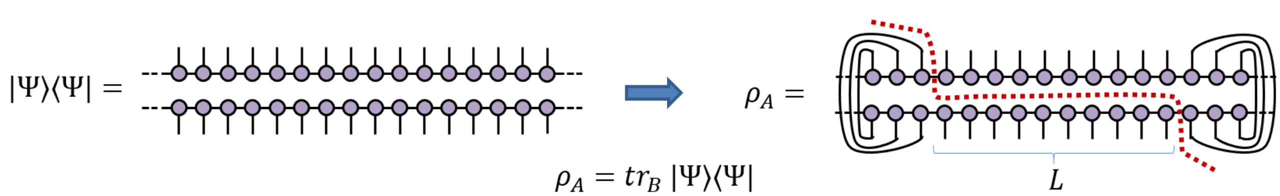

Figure 5: Upper bound for the entanglement entropy in a MPS.

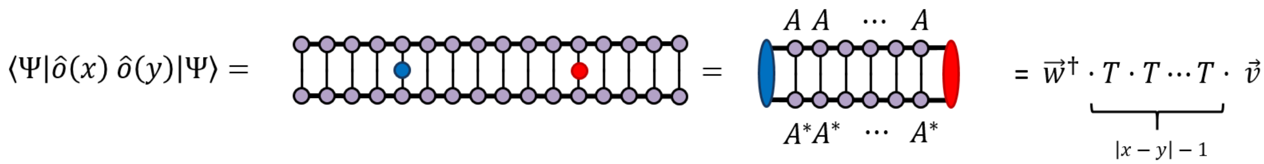

Figure 6: Scaling of correlations in a MPS.

5 Scaling of correlations in an MPS

Consider the same MPS in an infinite 1D lattice. Explain why two-point correlation function $c(x,y) \equiv \langle\Psi|\, \hat{o}(x)\hat{o}(y)\,|\Psi\rangle$ for any local operator $\hat{o}$ decays exponentially at large distances,

$$c(x,y) \approx e^{-\frac{|x-y|}{\xi}}, \qquad |x-y| >> \xi, \tag{27}$$

for some correlation length ξ. [**Hint**: Use Fig. 6 as a guide; assume that the normalization $\langle\Psi|\Psi\rangle = 1$ implies that all eigenvalues λ_α of the transfer matrix T fulfil $|\lambda_\alpha| \leq 1$].

6 Some bibliography

You can learn more about MERA in

- *Entanglement renormalization*, Phys. Rev. Lett. 99, 220405 (2007), arXiv: cond-mat/0512165
- *A class of quantum many-body states that can be efficiently simulated*, Phys. Rev. Lett. 101, 110501 (2008), arXiv: quant-ph/0610099
- *Entanglement Renormalization: an introduction* (book chapter), arXiv: 0912.1651

The relation between MERA and holography was proposed in

- Brian Swingle, *Entanglement renormalization and holography*, Phys. Rev. D 86, 065007 (2012), arXiv:0905.1317;
- Brian Swingle, *Constructing holographic spacetimes using entanglement renormalization*, arXiv:1209.3304

Swingle's proposal related MERA with a time slice of AdS_3 (namely the hyperbolic plane). More recently, it has been argued that MERA is related instead to the integral transform of a time slice of AdS_3 (which happens to be de Sitter dS_2), namely to kinematic space:

- Bartlomiej Czech, Lampros Lamprou, Samuel McCandlish, James Sully, *Tensor Networks from Kinematic Space*, arXiv:1512.01548