The AdS/CFT Correspondence
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1 Lecture 1

Q1. Large N expansion: Consider the following Lagrangian for a zero dimensional field theory (matrix model):

$$\mathcal{L} = \frac{1}{g^2} \left(\text{Tr}\,(\Phi\Phi) + \text{Tr}\,(\Phi\Phi\Phi) + \text{Tr}\,(\Phi\Phi\Phi\Phi) \right) \tag{1.1}$$

where Φ is an adjoint valued field of the group $U(N)$.

(i) Write down the Feynman diagrams for this theory.

(ii) Inspection of the Feynman diagrams should tell you that it is useful to regard the matrix indices of the adjoint field Φ explicitly. Writing $\Phi = \phi^a_{\ b}$, express the Feynman diagrams in a double line notation, where the two lines correspond to the upper and lower matrix index of Φ.

(iii) Draw the general Feynman diagram which contributes to the free energy of the matrix model and show that the free energy of the theory has a perturbation expansion as a double sum:

$$\mathcal{F} = \sum_{g=0}^{\infty} N^{2-2\,g}\, f_g(\lambda)\ , \qquad f_g(\lambda) = \sum_{i=0}^{\infty} \alpha_{g,i}\, \lambda^i\ . \tag{1.2}$$

where g is the genus of the two dimensional surface on which the Feynman diagram can be drawn. You will find it useful to recall the Gauss-Bonnet theorem, which gives the Euler character χ of a graph (and hence its genus) in terms of the vertices V, edges E and faces F:

$$\chi = 2 - 2\,g = \text{V} - \text{E} + \text{F}\ . \tag{1.3}$$

(iv) Use the result (1.2) to show that in the 't Hooft scaling limit:

$$N \to \infty \quad \text{with} \quad \lambda = \text{fixed}, \tag{1.4}$$

the free energy is given in terms of just the planar diagrams i.e., $g = 0$. Compare this expansion with the string perturbation expansion.

(v) Using this result argue that the large N limit effective classicalizes the theory, viz., $\hbar_{\text{eff}} \sim \frac{1}{N}$.

(vi) Bonus: How would you deal with multi-trace terms, viz., $h\,\text{Tr}\,(\Phi\Phi)\ \text{Tr}\,(\Phi\,\Phi)$?

Q2. AdS_{d+1} can be embedded in $\mathbb{R}^{d,2}$ as an hyperboliod

$$\mathbf{X}_d^2 - X_{-1}^2 - X_0^2 = -\ell_{\mathrm{AdS}}^2 \,, \tag{1.5}$$

with $\mathbf{X}_d = \{X_1, X_2, \cdots X_d\}$ are the standard Cartesian coordinates on $\mathbb{R}^d$. The metric on AdS_{d+1} is induced from the standard flat metric on $\mathbb{R}^{d,2}$ with signature $(d,2)$.

(i) Use this embedding to show that the metric on AdS_{d+1} can be written either in terms of the global coordinates:

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega_{d-1}^2 \,, \qquad f(r) = \left(1 + \frac{r^2}{\ell_{\mathrm{AdS}}^2}\right). \tag{1.6}$$

or in terms of the Poincaré coordinates

$$ds^2 = \frac{\ell_{\mathrm{AdS}}^2}{z^2}\left(-dt^2 + dz^2 + d\mathbf{y}_{d-1}^2\right) \,. \tag{1.7}$$

(ii) Exploit the embedding to find a coordinate transformation between the Poincaré coordinate (t, z, y_i) and the global coordinates (t, r, Ω_i), with Ω_i representing the angles of the $\mathbf{S}^{d-1}$.

(iii) Discuss the region of the global AdS spacetime covered by the Poincaré coordinates and the symmetries manifest in the two coordinate systems.

(iv) Show that the boundary of global AdS_{d+1} is $\mathbb{R} \times \mathbf{S}^{d-1}$ and that of the Poincaré patch is $\mathbb{R}^{1,d-1}$.

(v) **Bonus:** Describe the Penrose diagram of AdS_{d+1} by embedding the spacetime into the Einstein Static Universe (the Lorentzian cylinder)

$$ds^2 = -dt^2 + d\Omega_d^2 \,. \tag{1.8}$$

starting from the global coordinates (1.6).

(vi) **Bonus 2:** Find coordinate charts which foliate the AdS_{d+1} spacetime in slices preserving

(a) $SO(d,1)\times\mathbb{R}$: This give a foliation where the spatial leaves are Euclidean hyperbolic space and the boundary is a copy of the hyperbolic cylinder $\mathbb{H}_{d-1} \times \mathbb{R}$.

(b) $SO(d-1,2)$: This foliates AdS_{d+1} in terms of AdS_d slices.

(c) $SO(d,1)$: This foliates the spacetime in de Sitter_d slices.

Q3. Consider a Klein-Gordon scalar field propagating in the global AdS_{d+1} geometry (1.6) with Lagrangian

$$\mathcal{L} = \int d^{d+1}x\sqrt{-g}\; \frac{1}{2}\left(\partial_A\phi\,\partial^A\phi - m^2\,\phi^2\right) \tag{1.9}$$

Solve the equation of motion for the scalar field and show the eigenfrequencies ω associated with the Killing field $\left(\frac{\partial}{\partial t}\right)^A$ are quantized as in a harmonic oscillator

$$\omega_n R = \lambda_\pm + l + 2\,n\ , \qquad n = 0, 1, 2, \cdots \tag{1.10}$$

$\lambda_\pm$ are related to the dimension d of the spacetime and the mass of the field via

$$\lambda_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\,\ell_{\rm AdS}^2}\ , \tag{1.11}$$

and l labels the spherical harmonics on $\mathbf{S}^{d-1}$.

You should first argue that eigenstates of the scalar Laplacian on $\mathbf{S}^d$ are given by scalar spherical harmonics $Y_l(\Omega_d)$ with eigenvalues $l\ (l+d-1)$ which is a natural generalization of standard spherical harmonics on $\mathbf{S}^2$ and then proceed to separate variables.

Use the result for the eigenfrequencies to argue that scalar field in AdS are allowed to have a negative mass squared $m^2 < 0$ provided $m^2\,\ell_{\rm AdS}^2 \geq -\frac{d^2}{4}$. This bound on particle masses is known as the *Breitenlohner-Freedman* bound and particles violating this bound are tachyonic in AdS$_{d+1}$.

Q4. The AdS/CFT correspondence states that the generating function of connected Green's functions in the gauge theory $W_{\rm CFT}\,[\mathcal{J}]$ is given as

$$\begin{aligned} W_{\rm CFT}\,[\mathcal{J}] &= -\log\langle\,\exp\left(\int\, d^4x\,\mathcal{J}(x)\,\mathcal{O}(x)\right)\rangle_{CFT} \\ &\simeq \text{extremum}\,(\,\mathrm{I}_{\rm grav}\,[\phi])\Big|_{\phi(\mathrm{z}=\epsilon)=\mathcal{J}} \end{aligned} \tag{1.12}$$

where $I_{\rm grav}$ is the supergravity action evaluated on-shell subject to the specified boundary condition on the fields $\phi(z,x)$. Taking the bulk metric to be given by the Euclidean AdS$_5$ metric

$$ds^2 = \frac{\ell_{\rm AdS}^2}{z^2}\left(dz^2 + \sum_{i=1}^4 dx_i\,dx_i\right) \tag{1.13}$$

with an explicit cut-off at $z=\epsilon$, and the bulk field to be a Klein-Gordon scalar of mass m^2, evaluate the on-shell supergravity action

$$S = \mathcal{N}\int d^5x\,\sqrt{g}\,\left(\frac{1}{2}\,(\partial\phi)^2 + \frac{1}{2}\,m^2\,\phi^2\right)\ , \tag{1.14}$$

subject to the aforementioned boundary condition $\phi(z=\epsilon,\mathbf{x}) = \mathcal{J}(\mathbf{x})$ by working in momentum space.

From this show that the two-point function of a CFT operator $\mathcal{O}(x)$ whose source is $\mathcal{J}(x)$ has a two point function

$$\langle\,\mathcal{O}(\mathbf{x})\,\mathcal{O}(\mathbf{y})\,\rangle = \mathcal{N}\,\epsilon^{2\Delta-8}\,\frac{2\Delta-4}{\Delta}\,\frac{\Gamma(\Delta+1)}{\pi^2\,\Gamma(\Delta-2)}\,\frac{1}{|\mathbf{x}-\mathbf{y}|^{2\Delta}} \tag{1.15}$$

with $\Delta = 2 + \sqrt{4+m^2\,\ell_{\rm AdS}^2}$.