Quantum Error Correction: Solution Set #2

It for Qubit
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Problem #1. Error Syndromes and Cosets for Stabilizer Codes

Recall that $N(S) = \{P|\, [P, M] = 0\, \forall M \in S\}$, that S is a normal subgroup of $N(S)$, and that the elements of $N(S)/S$ (i.e., cosets of S in $N(S)$) correspond to logical $\overline{X}$ and $\overline{Z}$ operators.

a) Show that two Pauli errors E and F have the same error syndrome for a stabilizer code S iff they are in the same coset of $N(S)$ in the Pauli group.

Solution: Suppose E and F are in the same coset of $N(S)$. Then $E = FN$, with $N \in N(S)$. Let $M \in S$, so $MN = NM$. Then

$$EM = FNM = FMN = (-1)^f MFN = (-1)^f ME, \tag{1}$$

where $FM = (-1)^f MF$. Therefore E and F have the same commutation or anticommutation relationship with all the elements of the stabilizer and therefore have the same error syndrome.

Conversely, if E and F have the same error syndrome, they have the same commutation/anticommutation relationship with all elements of the stabilizer: Let $N = F^\dagger E$, and for some given $M \in S$, suppose $FM = (-1)^f MF$, so $EM = (-1)^f ME$ also. Then

$$NM = F^\dagger EM = (-1)^f F^\dagger ME = (-1)^{2f} MF^\dagger E = MN. \tag{2}$$

Since M was arbitrary, this means that $N \in N(S)$, and therefore that $E = FN$ is in the same coset of $N(S)$ as F.

b) Suppose that for each coset of $N(S)$ we pick some particular coset representative E and perform E whenever syndrome measurement indicates that coset. Suppose, however, a different error F had actually occurred. Relate the overall action on the codespace to an element of $N(S)/S$.

Solution: Note that when error F occurs and we correct by E, we end up with an error syndrome equal to the sum of the two error syndromes. When E and F are in the same coset, the overall error syndrome is therefore 0, meaning we have returned to the code. However, we may have changed the encoded state. Indeed, we have performed $EF \in N(S)$; this is in some coset in $N(S)/S$, which corresponds to some logical Pauli operation on the encoded state.

c) For the 5-qubit code, we choose the coset representatives to be the single-qubit errors (and the identity for the 0 syndrome), as there is exactly one in each coset. Use the result of part b to find the actions resulting from the errors $X_1 Z_3$ and $Y_2 X_4 Z_5$.

Solution: Recall we have the stabilizer

$$\begin{array}{ccccc} X & Z & Z & X & I \\ I & X & Z & Z & X \\ X & I & X & Z & Z \\ Z & X & I & X & Z \end{array} \tag{3}$$

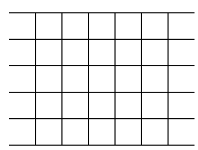
and the logical $\overline{X}$ and $\overline{Z}$ operations can be chosen to be $X \otimes X \otimes X \otimes X \otimes X$ and $Z \otimes Z \otimes Z \otimes Z \otimes Z$.

The error syndrome of $X_1 Z_3$ is 0011, which is the same as the error syndrome of the one-qubit Pauli operation X_5. Therefore, the overall operation is $X_1 Z_3 X_5$. We need to figure out which coset $N(S)/S$ this is in. We could solve this systematically using linear algebraic methods (we want to write the vector corresponding to $X_1 Z_3 X_5$ as a sum of the vectors corresponding to the generators of S, $\overline{X}$, and $\overline{Z}$), but for a problem this size, it is probably just as easy to figure it out just by looking at it. We note that $X_1 Z_3 X_5$ times $\overline{Z}$ has the form $-Y \otimes Z \otimes I \otimes Z \otimes Y$, which does have the same sort of structure as elements of S. Indeed, we can multiply together the first, second, and fourth generators to get $Y \otimes Z \otimes I \otimes Z \otimes Y$. The overall minus sign between the two has no physical significance, so we can conclude that the error $X_1 Z_3$ results in an overall $\overline{Z}$ after correction.

The error syndrome of $Y_2 X_4 Z_5$ is 1111, which is the same as the error syndrome of Y_4. The overall operation is thus $Y_2 Z_4 Z_5$, with an overall phase that has no physical significance. Multiplying this by $\overline{X}$ and $\overline{Z}$, we get $Y \otimes I \otimes Y \otimes X \otimes X$, again with some overall phase. We can just get this by multiplying together the second, third, and fourth generators of S, so the net operation is a $\overline{Y}$ after correction.

Problem #2. Surface Codes with Boundary

For this problem, consider a code with qubits located on the edges of the following graph, extended to a square grid of $L \times L$ vertices:



Note that is this case, we are *not* identifying top and bottom or left and right. For each face or vertex in the interior of the graph, have Z_f or X_v in the stabilizer as for the toric code. On the rough edges, we have a three-qubit stabilizer element $Z_f = Z \otimes Z \otimes Z$ for the three edges around each incomplete face. On the smooth edges, we have a three-qubit stabilizer element $X_v = X \otimes X \otimes X$ for the three edges incident at each vertex on the boundary.

a) How many physical qubits does this code have?

Solution: There are $L+1$ qubits in each horizontal line and L horizontal lines. There are $L-1$ qubits in each vertical line and L vertical lines. Therefore the total number of qubits is $n = (L+1)L + (L-1)L = 2L^2$.

b) How many logical qubits does this code have?

Solution: There are a total of $(L-1)^2$ internal faces plus $2(L-1)$ boundary partial faces. There are a total of $L(L-2)$ internal vertices plus $2L$ boundary vertices on the smooth edges. This is a total of $2L^2 - 1$ X_v and Z_f operators. In this case, the product of all X_v operators does not give the identity, nor does any other product of X_v's. Similarly, no product of Z_f operators gives the identity. Thus, all X_v and Z_f are independent and there is $k = 2L^2 - (2L^2 - 1) = 1$ logical qubit.

c) Consider a path that starts and ends on edges that are part of a rough boundary. Show that the tensor product of Zs along the path is an element of $N(S)$.

Solution: Such a path has either two or four edges incident on any vertex (interior or boundary). Thus the operator commutes with all X_v operators and is in $N(S)$.

d) Consider a dual path starting and ending on smooth boundaries. Show that the tensor product of Xs along the path is an element of $N(S)$.

Solution: Similarly, the path has either two or four edges around each face or partial face. Thus the operator commutes with all Z_f and is in $N(S)$.

e) Characterize the non-trivial logical $\overline{X}$ and $\overline{Z}$ operators. What is the distance of this code?

Solution: As with the toric code, exact cycles and cocycles give elements of S. There are no topologically non-trivial cycles or cocycles for this code, so none of the logical Paulis can come from that source. However, $N(S)$ also contains tensor products of Zs along a path that starts and ends on a rough edge (or products of such paths). If the path starts and ends at the same edge, then it is a product of Z_f operators along the faces inside the path (bordered by the rough edge in question). However, a path that stretches from one rough edge to the other cannot be written in this way. A product of Z's along a path stretching between the two rough edges is a logical $\overline{Z}$. Similarly, X's on the qubits in a dual path stretching from one smooth edge to the other is a logical $\overline{X}$ operator.

The minimal weight of $\overline{X}$ is L. The minimal weight of $\overline{Z}$ is $L+1$. Therefore the distance of the code is $d = L$.