It from Qubit: Entanglement Theory Tuesday Problems

Measures of entanglement

(a) The *Schmidt number* of a bi-partite pure state is defined as the number of non-zero coefficients in the state's Schmidt decomposition. Prove that for transformations among pure states, the Schmidt number is an entanglement monotone, i.e., that it is non-increasing under local operations and classical communication (LOCC).

(b) The *global robustness of entanglement* of a multi-partite state ρ is defined as the smallest probabilistic weight s with which one can mix the state ρ with some state τ such that the mixture $\frac{1}{1+s}\rho + \frac{s}{1+s}\tau$ is a separable state. Formally, it is

$$R_G(\rho) = \arg\inf_s \{\frac{1}{1+s}\rho + \frac{s}{1+s}\tau \in \text{Separable}\}. \tag{1}$$

Show that R_G is an entanglement monotone, i.e., show that if $\rho \to \sigma$ under LOCC, then $R_G(\sigma) \leq R_G(\rho)$.

Hint: The proof makes use of the following two facts about an LOCC operation: (1) by virtue of being LOCC, when acting on a separable state, it yields another separable state, and (2) by virtue of being a linear map, it can be distributed over a mixture.

Bound entanglement

(a) Prove that the following five states of a pair of qubits,

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle) \tag{2}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle \tag{3}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle) \tag{4}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle \tag{5}$$

$$|\psi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle) \tag{6}$$

form an *unextendable product basis,* which is to say that the subspace orthogonal to the span of these states contains no product states.

(b) Prove that the mixed state corresponding to the normalized projection onto the subspace orthogonal to the span of the elements of the unextendable product basis is a bound entangled state.