Day 1

Problem: entropy for Gaussian states

a) We have for a Gaussian state

$\langle f_1 \cdots f_i f_{i+1} \cdots f_{2n} \rangle = \sum$ possible products of two point functions

$= \langle f_i f_{i+1} \rangle \langle f_1 \cdots f_{i-1} f_{i+2} \cdots f_{2n} \rangle +$ terms that do not contain $\langle f_i f_{i+1} \rangle$

$\langle f_1 \cdots f_{i+1} f_i \cdots f_{2n} \rangle = \langle f_{i+1} f_i \rangle \langle f_1 \cdots f_{i-1} f_{i+2} \cdots f_{2n} \rangle +$ terms that do not contain $\langle f_{i+1} f_i \rangle$

In this two expression the second term in the left hand side is the same.
Hence $\langle f_1 \cdots f_i f_{i+1} \cdots f_{2n} \rangle - \langle f_1 \cdots f_{i+1} f_i \cdots f_{2n} \rangle = \langle f_i f_{i+1} - f_{i+1} f_i \rangle \langle f_1 \cdots f_{i-1} f_{i+2} \cdots f_{2n} \rangle = [f_i, f_{i+1}] \langle f_1 \cdots f_{i-1} f_{i+2} \cdots f_{2n} \rangle$

This follows using the Gaussian law for correlators, but is the correct result since this is $\langle f_1 \cdots (f_i f_{i+1} - f_{i+1} f_i) \cdots \rangle = [f_i, f_{i+1}] \langle f_1 \cdots f_{i-1} f_{i+2} \cdots f_{2n} \rangle$

It is important then that commutator are numerical for the variables in order to have a Gaussian state.

The linearity: if we take one of the variables in the expectation value as a linear combination, $\langle f_1 \cdots (a f_a + b f_b) \cdots f_{2n} \rangle$ this is equal to $a \langle f_1 \cdots f_a \cdots f_{2n} \rangle + b \langle f_1 \cdots f_b \cdots f_{2n} \rangle$. The expression in terms of two point function is also linear in each of the variables, giving the sought equation.

b) $\langle \phi_i \phi_j \rangle^* = \langle \phi_j \phi_i \rangle = \langle \phi_i \phi_j \rangle$ so X_{ij} is real symmetric, and the same holds for P_{ij}. They are positive definite: $\langle 0 | \alpha_i^* \phi_i | \alpha_j \phi_j | 0 \rangle = \| \alpha_j \phi_j | 0 \rangle \|^2 \geq 0$

$\Rightarrow \alpha_i^* X_{ij} \alpha_j \geq 0$

$\langle \phi_i \pi_j \rangle^* = \langle \pi_j \phi_i \rangle = \langle \phi_i \pi_j \rangle - i \delta_{ij}$ then if $\langle \phi_i \pi_j \rangle = \frac{i}{2} \delta_{ij} + D_{ij}$

it follows D is real. Time inversion symmetry is due to an antiunitary operator T / $T \phi T = \phi$, $T \pi T = -\pi$ $\Rightarrow$ $\langle T \phi_i \pi_j \rangle = -\langle \phi_i \pi_j \rangle$

On the other hand, if T leaves the state invariant, being antiunitary

$\langle T\phi_i \pi_j \rangle = \langle \phi_i \pi_j \rangle^* \Rightarrow -\langle \phi_i \pi_j \rangle = \langle \phi_i \pi_j \rangle^* \Rightarrow D = 0$

c) $\rho = K\, e^{-\sum \varepsilon_e a_e^+ a_e}$

$$\left.\begin{aligned} \phi_i &= \alpha_{ij} a_j^+ + \alpha_{ij} a_j \\ \pi_i &= -i\beta_{ij} a_j^+ + i\beta_{ij} a_j \end{aligned}\right\} \text{ hermitians}$$

For each mode a_e there is an independent density matrix. The hilbert space of this mode has a basis on occupation number $|n\rangle$. The density matrix is diagonal on this bases with eigenvalues $e^{-\varepsilon_e n}$. Then the normalization constant is $\left(\sum_{n=0}^{\infty} e^{-\varepsilon_e n}\right)^{-1} = (1 - e^{-\varepsilon_e})$. The normalization constant for ρ is $k = \prod_e (1 - e^{-\varepsilon_e})$

The spectrum of ρ is given by al possible products of eigenvalues of each oscillator mode: $\quad k \cdot \prod_e e^{-\varepsilon_e n_e} = \lambda\{n_1 \dots n_e\}$

The entropy is the sum of the entropies of each oscillator. These have density matrics with eigenvalues $(1 - e^{-\varepsilon_e})\, e^{-\varepsilon_e n}$, $n = 0 \dots \infty$

$$S_e = -\sum_n (1 - e^{-\varepsilon_e})\, e^{-\varepsilon_e n} \log\left((1 - e^{-\varepsilon_e})\, e^{-\varepsilon_e n}\right) = -\log(1 - e^{-\varepsilon_e}) + \frac{\varepsilon_e e^{-\varepsilon_e}}{1 - e^{-\varepsilon_e}}$$

$$S(\rho) = \sum_e S_e$$

d) Correlation functions involving different modes will factorize. For a single mode the density matrix is diagonal in occupation number, so only products with the same number of creation and anihilation operators are non vanishing. Then we have only to check the gaussian relation for the correlator

$$\langle a_e^{+k} a_e^k \rangle = \sum_{n=k}^{\infty} (1 - e^{-\varepsilon_e})\, e^{-\varepsilon_e n}\, n(n-1)\cdots(n-k+1) =$$

$$e^{-\varepsilon_e k}\, k! \sum^{\infty} (1 - e^{-\varepsilon_e})\, e^{-\varepsilon_e n} \binom{n+k}{} = k! \,/\, (e^{+\varepsilon} - 1)^k$$

This is exactly what gives the Gaussian law

$$\langle a_\ell^{\dagger k} a_\ell^k \rangle = k! \, \langle a_\ell^\dagger a_\ell \rangle^k = \frac{k!}{(e^{\varepsilon}-1)^k}$$

e) $[\phi_i, \pi_j] = i\,\delta_{ij} = 2\,\alpha_{i\ell}\,[a_\ell^\dagger, a_s]\; i\,\beta_{js} = -2i\,\alpha_{i\ell}\,\beta_{j\ell} = -2i(\alpha\beta^T)_{ij}$

$$\Rightarrow \boxed{\alpha\beta^T = -1/2}$$

f) $\mathrm{tr}\,\rho\,\phi_i\phi_j = \mathrm{tr}\,\rho\,(\alpha_{i\ell}a_\ell^\dagger + \alpha_{i\ell}a_\ell)(\alpha_{js}a_s^\dagger + \alpha_{js}a_s) =$

$$= \alpha_{i\ell}\, n_\ell\, (\alpha^T)_{\ell j} + \alpha_{i\ell}\,(1+n_\ell)(\alpha^T)_{\ell j} \Rightarrow \boxed{X = \alpha(2n+1)\alpha^T}$$

Similarly

$$\langle \pi_i \pi_j \rangle = \langle (-i\beta_{i\ell}a_\ell^\dagger + i\beta_{i\ell}a_\ell)(-i\beta_{js}a_s^\dagger + i\beta_{js}a_s) \rangle =$$

$$= \beta_{i\ell}\, n_\ell\, (\beta)^T_{\ell j} + \beta_{i\ell}\,(n_\ell+1)(\beta^T)_{\ell j} \Rightarrow \boxed{P = \beta(2n+1)\beta^T}$$

g) Multiplying $XP = \alpha(2n+1)\alpha^T\beta(2n+1)\beta^T = \alpha\frac{(2n+1)^2}{4}\alpha^{-1}$

If ν_k are the eigenvalues of $C = \sqrt{XP}$ we have

$$(2n_k+1)^2\frac{1}{4} = \nu_k^2 \Rightarrow \nu_k = \frac{1}{2}\left(2(e^{\varepsilon_k}-1)^{-1}+1\right) = \frac{1}{2}\coth(\varepsilon_k/2)$$

Then it is algebra to prove that the entropy of the mode k:

$$S_k = -\log(1-e^{-\varepsilon_k}) + \varepsilon_k\frac{e^{-\varepsilon_k}}{1-e^{-\varepsilon_k}} = (\nu_k+1/2)\log(\nu_k+1/2) - (\nu_k-1/2)\log(\nu_k-1/2)$$

and then $S(\rho) = \mathrm{tr}\left((C+1/2)\log(C+1/2) - (C-1/2)\log(C-1/2)\right)$

For the renyi entropies we have for each mode:

$$\mathrm{tr}\,\rho_\ell^n = (1-e^{-\varepsilon_\ell})^n \sum_{m=0}^{\infty} e^{-\varepsilon_\ell n m} = \frac{(1-e^{-\varepsilon_\ell})^n}{(1-e^{-\varepsilon_\ell n})}$$

$$S_\ell^n = \frac{1}{\quad}\left(n\log(1-e^{-\varepsilon_\ell}) - \log(1-e^{-\varepsilon_\ell n})\right)$$

Again converting this to the variable ν_e, we get summing over l

$$S(\rho) = \frac{1}{n-1} \operatorname{tr} \log\left((C+1/2)^n - (C-1/2)^n\right)$$

the modular Hamiltonian is $H = \sum \varepsilon_e a_e^+ a_e$.

Inverting the relation

$$\phi_i = \alpha_{ij}\,(a_j^+ + a_j) \quad \Rightarrow \quad a_j^+ + a_j = \alpha^{-1}_{ji}\,\phi_i$$

$$\pi_i = -i\,\beta_{ij}\,(a_j^+ - a_j) \quad \Rightarrow \quad a_j^+ - a_j = i\,\beta^{-1}_{ji}\,\pi_i = -2i\,\alpha^T_{ji}\,\pi_i$$

$$a_j = \left(\alpha^{-1}_{ji}\,\phi_i + 2i\,\alpha^T_{ji}\,\pi_i\right)\frac{1}{2}$$

$$H = \left(\alpha^{-1}_{ji}\phi_i \mp i2\alpha^T_{ji}\pi_i\right)\varepsilon_j\left(\alpha^{-1}_{je}\,\phi_e + i2\alpha^T_{je}\,\pi_e\right)\frac{1}{4}$$

the terms with products of $\phi_i\,\pi_e$ are:

$$\frac{i}{2}\left(\alpha^{-1}_{ji}\,\phi_i\,\varepsilon_j\,\alpha^T_{je}\pi_e - \alpha^T_{ji}\,\pi_i\,\varepsilon_j\,\alpha^{-1}_{je}\,\phi_e\right)$$

Except for a constant from the commutation relation this is zero. We do not include this constant in the modular Hamiltonian since it can always be redefined by an additive constant that goes away with the normalization of ρ.

Hence: $\quad H = \frac{1}{2}\sum_{ij} \phi_i\,M_{ij}\,\phi_j + \pi_i\,N_{ij}\,\pi_j$

$$M = \frac{1}{2}(\alpha^{-1})^T\,\varepsilon\,\alpha^{-1}$$

$$N = 2\,\alpha\,\varepsilon\,\alpha^T$$

Using $\quad C = \alpha\,\frac{2n+1}{2}\,\alpha^{-1}$, $\quad f(C) = \alpha\,f\!\left(\frac{2n+1}{2}\right)\alpha^{-1}$

and $\quad f(C)\,X = \alpha\,f\!\left(\frac{2n+1}{2}\right)(2n+1)\,\alpha^T$

Choosing f such that $f\!\left(\frac{2n+1}{2}\right)(2n+1) = 2\varepsilon$ we obtain

$$N = \frac{1}{C}\log\left(\frac{C+1/2}{C-1/2}\right)X$$

In a similar way it follows

$$M = P \frac{1}{C} \log\left(\frac{C+1/2}{C-1/2}\right)$$

h) If the state is pure $C = 1/2$, and $\boxed{X \cdot P = 1/4}$ such that the entropy vanishes.

The eigenvalues of XP are greater than $\frac{1}{4}$ because of uncertainty relation

To prove it we make:

$$\langle (v_i^* \phi_i + u_i^* \pi_i)(v_i \phi_i + u_i \pi_i) \rangle \geq 0$$

$$v^+ X v + u^+ P u + \frac{i}{2}(v^+ u - u^+ v) \geq 0$$

Minimizing with respect to u:

$$P u - \frac{i}{2} v = 0 \qquad u = \frac{+i}{2} P^{-1} v$$

Then $\quad v^+ X v + \frac{1}{4} v^+ P^{-1} v - \frac{1}{4} v^+ P^{-1} v - \frac{1}{4} v^+ P^{-1} v =$

$$= v^+ \left(X - \frac{1}{4} P^{-1}\right) v \geq 0$$ for all v. Using $v = \sqrt{P}\, w$

$$w^+ \left(\sqrt{P} X \sqrt{P} - 1/4\right) w^+ \geq 0$$ for all w. Then the eigenvalues of $\sqrt{P} X \sqrt{P}$ are greater than $1/4$. But $\mathrm{tr}\,(\sqrt{P} X \sqrt{P})^q = \mathrm{tr}\,(XP)^q$ for all integer $q \geq 0$ and then the matrices XP and $\sqrt{P} X \sqrt{P}$ have the same eigenvalues.

i) First we choose new variables $\tilde{\phi}_i = O_{ij} \phi_j$ with O orthogonal, and let $\tilde{\pi}_i = O_{ij} \pi_j$ be the conjugate momentum. Choosing

$$O^T K O = \begin{pmatrix} w_1^2 & & \\ & \ddots & \\ & & w_N^2 \end{pmatrix}$$ to be diagonal, the new hamiltonian

is a sum of harmonic oscillators:

$$H = \frac{1}{2} \sum \left(\tilde{\pi}_i^2 + \tilde{\phi}_i^2 w_i^2\right)$$

the fundamental state has $\langle 0 | \tilde{\phi}_i^2 | 0 \rangle = \frac{1}{2 w_i}$, $\langle 0 | \tilde{\pi}_i^2 | 0 \rangle = \frac{1}{2} w_i$

This is a standard computation, but a short-path to show this is to note for the fundamental state $\langle 0_i | H_i | 0_i \rangle = \frac{\omega_i}{2}$ (zero point energy), and that kinetic and potential energies are equidistributed:

$$\left\langle \frac{1}{2} \tilde{\phi}_i^2 \omega_i^2 \right\rangle = \left\langle \frac{1}{2} \tilde{\pi}_i^2 \right\rangle = \frac{\omega_i}{4} = \frac{1}{2} \langle H \rangle$$

Then $\langle \phi_i \phi_j \rangle = (O^T)_{il} \langle \tilde{\phi}_l (O^T)_{js} \tilde{\phi}_s \rangle = (O^T)_{il} \frac{1}{2\omega_l} O_{lj}$

$$\Rightarrow X = \frac{1}{2\sqrt{K}}$$

Similarly $P = \frac{\sqrt{K}}{2}$

The fundamental state then has $XP = \frac{1}{4}$, and the entropy is 0.