Day 2

Problem: mass term in EE (1)

a) The euclidean path integral is

$$Z = N^{-1}\int \mathcal{D}\varphi \; e^{-S[\varphi]} \quad \text{with } S[\varphi] = \int d^2x \; \frac{1}{2}\left(\partial_\mu\varphi\partial_\mu\varphi + m^2\varphi^2\right)$$

Then $$\frac{d}{dm^2} Z = N^{-1}\int \mathcal{D}\varphi \; e^{-S(\varphi)} \left(-\frac{1}{2}\right)\int \varphi^2_{(x)}\, d^2x$$

Since $$G(x,y) = \langle \phi(x)\,\phi(y)\rangle = N^{-1}\int \mathcal{D}\varphi \; e^{-S(\varphi)} \phi(x)\,\phi(y)$$

it follows that $\frac{d}{dm^2} Z = -\frac{1}{2}\int d^2x \; G(x,x)$ (this is $-\frac{1}{2}\,(\mathrm{tr}\, G)$)

The action can also be written $\int d^2x \; \frac{1}{2}\; \varphi\,(-\nabla^2 + m^2)\varphi$

It is an elementary property of path integrals that the two point function for quadratic action is given by the inverse of the operator in the quadratic form:

$$G(x,y) = (-\nabla^2 + m^2)^{-1} \qquad (-\nabla^2 + m^2)_x\, G(x,y) = \delta^2(x-y)$$

b) In polar coordinates the Laplacian writes $\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{\partial_\theta^2}{r^2}\right) = \nabla^2$

Then the eigenvector equation is

$$\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{\partial_\theta^2}{r^2}\right]\psi - m^2\psi = -\lambda\psi$$

We use separation of variables, only a derivative on ∂_θ^2 apears, so we can use $\psi = f(r)\cdot e^{ia\theta}$. Periodicity in the angular variable requires. $e^{ia2\pi n} = 1 \Rightarrow \boxed{a = \frac{k}{n}, \; k = \text{integer}}$

Using this the radial equation becomes:

$$\left[\partial_r^2 + \frac{1}{r}\partial_r - \frac{(k/n)^2}{r^2} - m^2 + \lambda\right] f(r) = 0$$

writing $\nu = \sqrt{\lambda - m^2}$, $\boxed{\lambda = \nu^2 + m^2}$, the general solution of this second order diff. equation is:

$$f(r) = c_1 \, J_{|k/\mu|}(\nu r) + c_2 \, Y_{|k/\mu|}(\nu r)$$

in terms of Bessel functions. The second solution $Y_{|k/\mu|}$ is divergent at the origin, so taking the regular solution

$$\boxed{f(r) = c \, J_{|k/\mu|}(\nu r)}$$

A complete set of solutions is then

$$\boxed{\psi_{\nu,k}(r,\theta) = c_{\nu,k} \, e^{i k/\mu \, \theta} \, J_{|k/\mu|}(\nu r)}$$

c) We have to normalize the eigenfunctions. We have

$$\int_0^{2\pi\mu} d\theta \int_0^\infty dr \, . \, r \; e^{-ik/\mu \, \theta} \; e^{ik'/\mu \, \theta} \, . \, J_{|k/\mu|}(\nu r) \, J_{|k'/\mu|}(\nu' r) =$$

$$= \underbrace{\mu \, \delta_{kk'} \, (2\pi)}_{\text{angular integration}} \times \underbrace{\int_0^\infty dr \, r \; J_{|k/\mu|}(\nu r) \, J_{|k/\mu|}(\nu' r)}_{\text{This is known to be } \frac{1}{\nu}\delta(\nu-\nu') \text{ from the theory of Bessel transforms.}} =$$

$$= \frac{2\pi\mu}{\nu} \, \delta_{k,k'} \, \delta(\nu-\nu')$$

Then we normalize $\quad \psi_{\nu,k}(r,\theta) = \sqrt{\frac{\nu}{2\pi\mu}} \, e^{ik/\mu \, \theta} \, J_{|k/\mu|}(\nu r)$

$$\int d^2x \; \psi^*_{\nu,k}(x) \, \psi_{\nu',k'}(x) = \delta_{k,k'} \, \delta(\nu-\nu')$$

Since the Green function is the inverse of the operator we have diagonalized, we can compute it by writing the inverse of the eigenvalues in between the eigenvectors:

$$G(r,\theta,r',\theta') = \sum_{k=-\infty}^{\infty} \int_0^\infty d\nu \; \frac{1}{\nu^2+m^2} \; \psi_{k,\nu}(r,\theta) \, \psi^*_{k,\nu}(r',\theta')$$

d) $$G(r,\theta,r,\theta) = \sum_{k=-\infty}^{\infty} \int_0^\infty d\nu \; \frac{\nu}{\nu^2+m^2} \; \frac{J^2_{|k/n|}(\nu r)}{2\pi n}$$

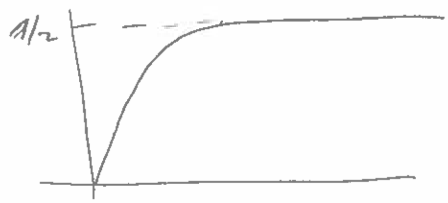
$$\int_0^{2\pi n} d\theta \; G(r,\theta,r,\theta) = \sum_{k=-\infty}^{\infty} \int_0^\infty d\nu \; \frac{\nu}{\nu^2+m^2} \; J^2_{|k/n|}(\nu r) =$$

$$= \sum_{k=-\infty}^{\infty} I_{|k/n|}(rm)\, K_{|k/n|}(rm)$$

To compute $\int d^2x \, G(x,x)$ we still need to integrate in r:

$$\int_0^{2\pi n} d\theta \int_0^\infty dr \; r \; G(r,\theta,r,\theta) = \sum_{k=-\infty}^{\infty} \int_0^\infty dr \; r \; I_{|k/n|}(rm)\, K_{|k/n|}(rm)$$

the integrand is a function $r\, I_a(x)\, K_a(x)$ that has the generic form:



for any $a > 0$. Then the integral in r diverges. If we sum and subtract $1/2$ we get:

$$\frac{1}{m^2}\int_0^\infty d(rm)\left((rm)\, I_{|k/n|}(rm)\, K_{|k/n|}(rm) - 1/2\right) + \underbrace{\frac{1}{m^2}\int_0^\infty d(rm)\; 1/2}_{-\frac{2U}{m^2}}$$

(we call this constant $-\frac{2U}{m^2}$!)

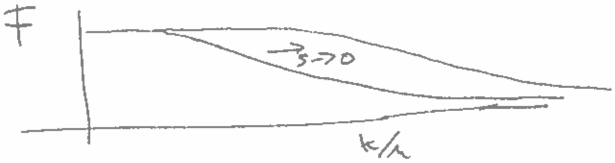
$$= -\frac{|k|}{2m^2 n} - \frac{2U}{m^2}$$

U is divergent. This should disappear from the final result.

We have

$$\frac{d}{dm^2}\log Z = -\frac{1}{2}\int d^2x \, G(x,x) = \sum_{-\infty}^{\infty}\left(\frac{|k|}{4m^2 n} + \frac{U}{m^2}\right) = \boxed{\left(\sum_1^\infty\left(\frac{k}{2m^2 n} + \frac{2U}{m^2}\right)\right) + \frac{U}{m^2}}$$

e) Now we evaluate $\frac{d}{dm^2}\left(\log Z(n) - n \log Z(1)\right)$. In order to have cancelation between divergences in the two terms we impose a cutoff function in the angular variable, a short angle cutoff $\delta\theta$. This is reflected in a cutoff for large $\left(\frac{k}{n}\right)$, that is we can choose a function $F_s(k/n)$ that goes to one for any $|\frac{k}{n}|$ as the parameter $s \to 0$ (the limit that removes the cutoff) but it is always $F_s(k/n) \xrightarrow[k/n\to\infty]{} 0$ for any finite s. For example we can take $F_s(k/n) = e^{-sk/n}$



Then we have:

$$\frac{d}{dm^2}\left(\log Z(n) - n\log Z(1)\right) = \frac{1}{m^2}\lim_{s\to 0}\left[\left(\sum_{k=1}^{\infty}\left(\frac{k}{2n}+2U\right)e^{-\frac{sk}{n}}\right)+U - n\left(\left(\sum_{k=1}^{\infty}\left(\frac{k}{2}+2U\right)e^{-sk}\right)+U\right)\right] = \boxed{\frac{(n^2-1)}{24\, n\, m^2}}$$

As expected, the divergences are eliminated. Then, integrating in m^2, and multiplying by $(1-n)^{-1}$ to get the Renyi entropies

$$S_n = (1-n)^{-1}\left(\log Z(n) - n\log Z(1)\right)$$

we have

$$S_n = -\frac{(n+1)}{12\, n}\log(m\,\varepsilon)$$

The short distance cutoff ε is needed to compensate dimensions. Notice no other scale is present since the size of the interval is ∞, and only entanglement of wavelength $\lesssim m$ and $\gtrsim \varepsilon$ contributes.