This came from a note which is intended as a review of monopoles, dyons, and moduli dynamics thereof in the context of supersymmetric Yang-Mills theories. The draft is very much in preparation and still contain a lot of typos and errors. If you happen to find any such, I would be very happy to hear about it. This file is intended for use by participants of the summer school on strings, gravity, and cosmology at Perimeter Institute, June-July 2005.

I have selected those part which are most relevant to the lectures here, and tried to avoid most of technical details, other than bare minimum necessary for understanding the final result. Nor does it describe relationship to similar problems in the context of string theory, but this I hope to cover in the lectures as much as I can. I hope you would find it a useful guide with many details I skipped in the lectures.

Piljin Yi

Chapter 6

The moduli space of BPS monopoles

Up to this point we have considered monopoles and dyons as classical solitons of Yang-Mills-Higgs theory. While we started with general properties, we saw how supersymmetry introduced many simplifications into the study of solutions. The study of these BPS monopoles and dyons has, in turn, contributed immensely toward our understanding of supersymmetric Yang-Mills field theories, especially in regard to the nonperturbative symmetries of $\mathcal{N} \geq 2$ supersymmetric Yang-Mills (SYM) theories known as dualities.

One important handle for studying the behavior and classification of monopoles and dyons is the low-energy moduli space approximation [12]. In this description, most of the field theoretical degrees of freedom are ignored, with the remaining finite number of bosonic and fermionic variables to be quantized. The bosonic variables are the collective coordinates that encode the positions and phases of the individual monopoles, while the fermionic pieces complete certain low-energy supersymmetries that remain unbroken by the monopoles. Dyons arise in this description as excited states with nonzero momenta conjugate to the phase coordinates.

The moduli space approximation ignores radiative interactions and is relevant only when we ask questions suitable for the low-energy limit [39][40]. For instance, while one can study scattering of monopoles within this framework, the result is only reliable if none of monopoles are moving rapidly or radiating a lot of electromagnetic energy. This can be ensured by restricting to low velocity and by working in the regime with small Yang-Mills coupling constant. This restriction is harmless if we wish to ask what kind of low-energy bound states of monopoles exist, which is one of the main goals when we want to make contact with the nonperturbative aspects of the underlying Yang-Mills theories.

Although supersymmetry, specifically the supersymmetry that is left unbroken by the monopoles, is important for understanding the low-energy dynamics, we will start, in this chapter, with the purely bosonic part. When there is only one adjoint Higgs, we have the notion of fundamental monopoles, which was introduced in Chap. 4. Each fundamental monopole carries four collective coordinates, and thus a $4n$-dimensional moduli space emerges as the natural setting for describing n monopoles interacting with each other. We will presently define, characterize, and find explicit examples of such moduli spaces. It also turns out that, for some special collections of monopoles, a sensible limit of massless monopoles exists and allows a similar low-energy approach to monopoles with non-Abelian magnetic charges. This latter example shows how one can get around the famous "global color problem" in magnetic monopole systems, and we will also explore the moduli spaces corresponding to this limit.

Of course, SYM with extended supersymmetry comes with two or six adjoint Higgs fields in the vector multiplet. With the exception of SU(2) theories, this feature turns out to qualitatively modify the low-energy dynamics and is in fact quite crucial for recovering most of the dyonic states in the theory. However, by taking a suitable limit in which one of the Higgs field takes a dominant role in the symmetry breaking, we can study monopole dynamics in such multi-Higgs vacua with a simple and universal modification of the moduli space dynamics. This modified moduli space dynamics will occupy the second half of this review. For now, we will concentrate on the conventional moduli space dynamics.

We begin, in Sec. 6.1, by describing some general properties of monopole moduli spaces. We then go on describe how the moduli space metric can be determined in several special cases. In Sec. 6.2, we use the interactions between well-separated monopoles to infer the metric for the corresponding asymptotic regions of moduli space. Next, in Sec. 6.3, we show how these asymptotic results, together with the general mathematical constraints on the moduli space, determine the full moduli space for the case of two fundamental monopoles. If the two monopoles are of distinct types, it turns out that the asymptotic form of the metric is actually the exact form for the entire moduli space. This result is extended to the case of an arbitrary number of distinct monopoles in Sec. 6.4. In Sec. 6.5, we consider moduli spaces for systems with massless monopoles and non-Abelian magnetic charges. Finally, in Sec. 6.6, we discuss monopole scattering within the moduli space framework.

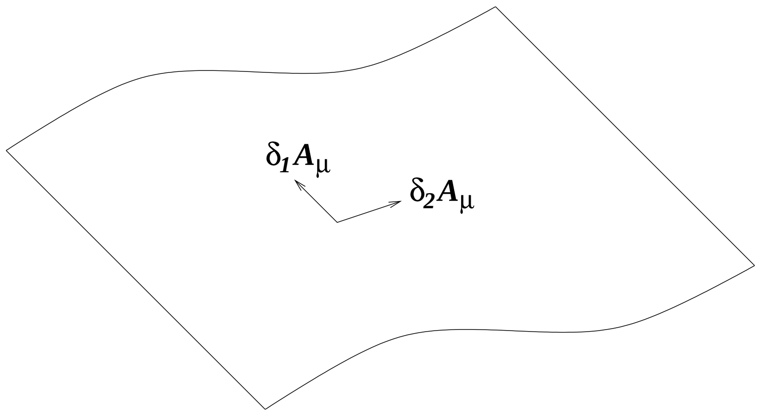

Figure 6.1: Monopole moduli space is a curved manifold whose points correspond to monopole solutions. Thus, tangent vectors at any given point on the moduli space of a BPS monopole encode infinitesimal deformations of the corresponding monopole solution which preserve the BPS condition.

6.1 General properties of monopole moduli spaces

We recall from the discussion in Chap. 2 that in the moduli space approximation the dynamics is described by a Lagrangian of the form

$$L = -(\text{total rest mass of monopoles}) + \frac{1}{2}\, g_{rs}(z)\dot{z}^r \dot{z}^s \tag{6.1.1}$$

where the z_r are the collective coordinates that parameterize the monopole configurations, and the constant first term will usually be omitted in our discussions.

The moduli space is naturally viewed as a curved manifold with metric $g_{mn}(z)$, as illustrated in Fig. 6.1. As was shown in Sec. 2.5, the metric can be obtained from the background gauge zero modes via

$$g_{rs}(z) = 2\int d^3x \; \mathrm{tr}\,\{\delta_s A_i \delta_s A_i + \delta_r \Phi \delta_s \Phi\} = 2\int d^3x \; \mathrm{tr}\,\{\delta_r A_a \delta_s A_a\} \tag{6.1.2}$$

In the last integral we have used the convention, introduced previously, of letting Roman indices from the beginning of the alphabet run from 1 to 4, with $A_4 \equiv \Phi$.

This expression for the metric is anything but accessible. The computation of the metric would seem to require that we know the entire family of BPS monopole solutions, which remain a very difficult task. Historically, moduli space metrics have been found by various indirect methods, by invoking the symmetries of the underlying gauge theory and the moduli space properties that derive from them.

One essential property of a monopole moduli space is its hyper-Kähler structure. In Sec. 4.2.2 we found that at each point on the moduli space there are three complex structures $J^{(r)}$ that map the the tangent space onto itself and that obey the quaternionic algebra

$$J^{(s)}J^{(t)} = -\delta^{st} + \epsilon^{stu}J^{(u)} \tag{6.1.3}$$

Furthermore, it turns out that the manifold is Kähler with respect to each of the $J^{(r)}$, which is equivalent to saying that

$$\nabla J^{(s)} = 0 \tag{6.1.4}$$

with respect to the affine connection of the moduli space metric. The Kähler two-forms $w^{(s)}$ are defined by

$$w^{(s)}_{km} = -g_{kl}(J^{(s)})^l{}_m \tag{6.1.5}$$

and are closed. When a manifold possesses such a triplet of Kähler structures, it is called a hyper-Kähler manifold, and so we learn that moduli space of BPS monopoles is always a hyper-Kähler manifold. This puts a tight algebraic constraint on the curvature tensor and thus provides a differential constraint on the moduli space metric.

The fact that the $J^{(r)}$ and $w^{(r)}$ satisfy the requisite integrability condition is much easier to see when supersymmetry is used to express things in terms of fermion zero modes. For this reason, we will postpone the proof of the hyper-Kähler property to Chap. 8. A more detailed discussion of complex structures, integrability, and Kähler and hyper-Kähler geometry is given in the Appendix.

Another important property of the moduli space is its isometries. These isometries reflect the underlying symmetries of the BPS monopole solutions themselves. For instance, since we are discussing monopoles in an R^3 space with rotational and translational symmetries, the moduli space should possess corresponding isometries. The translation isometry shows up somewhat trivially in the center-of-mass part of the collective coordinates and does not enter the interacting part of the moduli space.

The SO(3) rotational isometry of the moduli space proves to be very useful, since it acts on the relative position vectors of the monopoles. Spatial rotation of a BPS solution always produces another BPS solution. This takes one point on the moduli space to another, and thus induces a mapping of the moduli space onto itself. Because the physics is invariant under such spatial rotations, this mapping preserves the moduli space Lagrangian, and thus the metric. Actually this rotational isometry can, in general, be elevated to an SU(2) isometry.

The infinitesimal generators of the isometries are realized as vector fields on the moduli space. We will denote the three generators of the SU(2) isometry by L^s with

$s = 1, 2, 3$. The statement that the L^s generate isometries is reflected in the fact that they are Killing vector fields, whose components therefore satisfy

$$0 = (\mathcal{L}_{L^s}[g])_{mn} \equiv \nabla_m L^s_n + \nabla_n L^s_m \tag{6.1.6}$$

where $\mathcal{L}_V$ denotes the Lie derivative with respect to the vector field V.

The SU(2) structure of these isometries is in turn encoded in the commutators of these vector fields,

$$[L^s, L^t] = \epsilon^{stu} L^u \tag{6.1.7}$$

where the commutator of two vector fields, X and Y, is defined as

$$[X, Y]^m \equiv X^n \partial_n Y^m - Y^n \partial_n Y^m \tag{6.1.8}$$

This SU(2) isometry does not leave the complex structures, $J^{(s)}$, invariant. Rather, the latter transform as a triplet:

$$\mathcal{L}_{L^s}[J^{(t)}] = \epsilon^{stu} J^{(u)} \tag{6.1.9}$$

Equivalently, the three Kähler forms w^s transform as

$$\mathcal{L}_{L^s}[w^t] = \epsilon^{stu} w^u \tag{6.1.10}$$

This can be easily understood by recalling that the action of $J^{(s)}$ originates from the action of the 't Hooft tensor $\eta^s_{\mu\nu}$ on the zero modes. After carefully sorting through how spatial rotation acts on the $\eta^s_{\mu\nu}$, one finds that the $J^{(s)}$ form an SU(2) triplet.

The unbroken gauge group, $\mathrm{U}(1)^r$, can also be used to rotate a BPS solution, and this generates another set of isometries of the moduli space. There are at most r independent isometries of this sort. The zero modes associated with these gauge isometries take the particularly simple form

$$\delta_a A_s = D_s \Lambda_a, \qquad \delta_a \Phi = ie[\Phi, \Lambda_a] \tag{6.1.11}$$

with $a = 1, 2, \ldots, r$ labelling the r possible gauge rotations. Accordingly, the zero mode equations simplify to a single second-order equation,

$$D^2 \Lambda_a + e^2 [\Phi, [\Phi, \Lambda_a]] = 0 \tag{6.1.12}$$

While the long-range part of the solution commutes with the unbroken gauge group, the monopole cores are transformed. Throughout this review, we will denote the Killing vector fields associated with these U(1) isometries by K^a. Unlike the rotational isometry, these U(1) isometries preserve the complex structures of the moduli space, so

$$\mathcal{L}_{K_a}[J^s] = 0 \tag{6.1.13}$$

These properties of the isometries are quite useful in identifying the exact moduli space geometry.

In the following we will find it useful to have an explicit coordinate system where the gauge isometries acts as translations of the angular coordinates. Generally, we may consider a coordinate system where these Killing vectors are written as

$$K_a = \frac{\partial}{\partial \xi^a} \tag{6.1.14}$$

for some angular coordinates ξ^a. The Lagrangian must then have no explicit dependence on the ξ^a, other than via their velocities, and so may be written most generally as

$$L = \frac{1}{2}\, h_{pq}(y)\dot{y}^p\dot{y}^q + \frac{1}{2}\, k_{ab}(y)\ \left(\dot{\xi}^a + \dot{y}^p w^a_p(y)\right)\left(\dot{\xi}^b + \dot{y}^q w^b_q(y)\right) \tag{6.1.15}$$

where y^p denotes the rest of the coordinates. In other words, the ξ^a are all cyclic coordinates whose conjugate momenta are conserved quantities, just as in the case of SU(2) monopoles. We can identify these conjugate momenta as the electric charges that arise when the monopole cores are excited in such a manner that the monopoles are converted into dyons.

6.2 The moduli space of well separated monopoles

The metric on the moduli space determines the motion of slowly moving dyons. Conversely, the form of the moduli space metric can be inferred from a knowledge of the interactions between the dyons. In general, this is hardly an easy task, since the complete interaction between the dyons is no easier to understand than the complete form of the classical Yang-Mills solitons.

On the other hand, a drastic simplification occurs when we restrict our attention to cases where the monopole cores are separated by large distances. In this limit, the only interactions between the monopoles come about by the exchange of massless fields, which are completely Abelian [15][42][43]. In other words, the interactions involved are simply the Maxwell forces and their scalar analogue. By studying these interactions, then, we will be able to recover those regions of the moduli space where the intermonopole distances are all large. In this section, we will show how to do this.

6.2.1 Asymptotic dyon fields and approximate gauge isometries

Let us imagine that we have a set of N fundamental monopoles, all well separated from each other. We label these by an index j. Because only Abelian interactions are relevant at long distances, the non-Abelian process of electric charge hopping from one monopole core to another is extremely suppressed. Consequently, in this regime we have a larger number of "gauge" isometries than we have a right to expect. Instead of having a conserved electric charge for each unbroken U(1) gauge group, we effectively have a conserved electric charge for each monopole core. The $4N$ moduli of the monopole solution are easily visualized as $3N$ position coordinates $\mathbf{x}_j$ and N angular coordinates, ξ_j, with j labelling the monopole cores. Translation along ξ_j is then an approximate symmetry of the moduli space metric, so we have an approximate gauge isometry associated with each monopole. The effective Lagrangian of this approximate moduli space must be of the form

$$L = \frac{1}{2}\, M_{ij}(\mathbf{x})\mathbf{x}^i \cdot \mathbf{x}^j + \frac{1}{2}\, K_{ij}(\mathbf{x}) \left(\dot{\xi}^i + \mathbf{W}^i_k(\mathbf{x}) \cdot \dot{\mathbf{x}}^k\right) \left(\dot{\xi}^j + \mathbf{W}^j_l(\mathbf{x}) \cdot \dot{\mathbf{x}}^l\right) \qquad (6.2.1)$$

for some functions M_{ij}, K_{ij}, and $\mathbf{W}^i_j$ of the $\mathbf{x}_k$. This Lagrangian is similar in form to that displayed in Eq. (6.1.15), but with the significant difference that there is now a phase angle for every monopole, rather than just one for each unbroken U(1) factor, no matter how many fundamental monopoles of a given species are present.

Let us work in a gauge where the asymptotic Higgs field lies in the Cartan subalgebra. Then, as was described in Sec. 4.1, the jth monopole, located at $\mathbf{x}_j$, gives rise to an asymptotic magnetic field

$$\mathbf{B}^{(j)} = g_j\, (\boldsymbol{\alpha}^*_j \cdot \mathbf{H})\, \frac{(\mathbf{x} - \mathbf{x}_j)}{4\pi\, |\mathbf{x} - \mathbf{x}_j|^3} \qquad (6.2.2)$$

where $\boldsymbol{\alpha}_j$ is one of the fundamental roots and $g_j = 4\pi/e$. Exciting Q_j, the momentum conjugate to ξ_j, gives rise to a long-range electric field

$$\mathbf{E}^{(j)} = Q_j\, (\boldsymbol{\alpha}^*_j \cdot \mathbf{H})\, \frac{(\mathbf{x} - \mathbf{x}_j)}{4\pi\, |\mathbf{x} - \mathbf{x}_j|^3}, \qquad (6.2.3)$$

Because of the appearance of $\boldsymbol{\alpha}^*_j$, instead of $\boldsymbol{\alpha}_j$, the electric charge Q_j is quantized in integer units of $e\boldsymbol{\alpha}^2_j$.

We will also need the long-range effects of these dyons on the Higgs field. Applying a Lorentz transformation to the solution of Eq. (4.1.14), we see that the jth dyon induces a deviation

$$\Delta\Phi^{(j)} = -\frac{(\boldsymbol{\alpha}^*_j \cdot \mathbf{H})}{4\pi|\mathbf{x} - \mathbf{x}_j|} \sqrt{1 - \mathbf{v}^2_j}\sqrt{g^2_j + Q^2_j} + O(r^{-2}) \qquad (6.2.4)$$

from the vacuum value Φ_0.

The interactions among these dyons are most easily described by a Legendre transformation of the original monopole Lagrangian, in which we trade off the ξ_j in favor of their conjugate momenta Q_j/e. The resulting effective Lagrangian is often called the Routhian, and has the form

$$R = L - \frac{Q_j}{e}\,\dot{\xi}^j = \frac{1}{2}\,M_{ij}\,\dot{\mathbf{x}}^i \cdot \dot{\mathbf{x}}^j - \frac{1}{2}\,(K^{-1})^{ij}\,\frac{Q_i}{e}\frac{Q_j}{e} + \frac{Q_i}{e}\mathbf{W}^i_j \cdot \dot{\mathbf{x}}^j \tag{6.2.5}$$

In the following section we will compute this Routhian directly from the long-range interactions of dyons, and extract the asymptotic geometry of the moduli space.

6.2.2 Asymptotic pairwise interactions and the asymptotic metric

We begin by considering a pair of well-separated dyons, and asking for the effect of dyon 2 on the motion of dyon 1. This has two parts — the long-range electromagnetic interaction and the scalar interaction. The former is a straightforward generalization of the interaction between a pair of moving point charges in Maxwell theory. Given two U(1) dyons with electric and magnetic charges Q_j and g_j, the electromagnetic effects of the second on the first are described by the Routhian

$$R^{(1)}_{\text{Maxwell}} = Q_1\left[\mathbf{v}_1 \cdot \mathbf{A}^{(2)}(\mathbf{x}_1) - A^{(2)}_0(\mathbf{x}_1)\right] + g_1\left[\mathbf{v}_1 \cdot \tilde{\mathbf{A}}^{(2)}(\mathbf{x}_1) - \tilde{A}^{(2)}_0(\mathbf{x}_1)\right]. \tag{6.2.6}$$

Here $\mathbf{A}^{(2)}$ and $A^{(2)}_0$ are the ordinary vector and scalar electromagnetic potentials due to charge 2, while $\tilde{\mathbf{A}}^{(2)}$ and $\tilde{A}^{(2)}_0$ are dual potentials defined so that $\mathbf{E} = -\nabla \times \tilde{\mathbf{A}}$ and $\mathbf{B} = -\nabla\tilde{A}_0 - \partial\tilde{\mathbf{A}}/\partial t$.

Using standard methods to obtain these potentials, and keeping only terms of up to second order in Q_j or $\mathbf{v}_j$, we obtain

$$R^{(1)}_{\text{maxwell}} = \frac{g_1 g_2}{4\pi r_{12}}\left[\mathbf{v}_1 \cdot \mathbf{v}_2 - \frac{Q_1 Q_2}{g_1 g_2}\right] - \frac{1}{4\pi}(g_1 Q_2 - g_2 Q_1)(\mathbf{v}_2 - \mathbf{v}_1)\cdot \mathbf{w}_{12}, \tag{6.2.7}$$

where $r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$ and the Dirac monopole potential

$$\mathbf{w}_{12} = \mathbf{w}(\mathbf{x}_1 - \mathbf{x}_2) \tag{6.2.8}$$

obeys

$$\nabla \times \mathbf{w}(\mathbf{r}) = -\frac{\mathbf{r}}{|\mathbf{r}|^3}. \tag{6.2.9}$$

In terms of the usual spherical coordinates for $\mathbf{r}$, we can write

$$\mathbf{w}(\mathbf{r}) \cdot d\mathbf{r} = \cos\theta\, d\phi \tag{6.2.10}$$

locally.

These electromagnetic interactions can all be traced back to the $F_{\mu\nu}^2$ term in the Maxwell Lagrangian. In the Yang-Mills case, the analogous term involves a trace over the group generators. The result is that the right-hand side of Eq. (6.2.7) must be multiplied by a factor of[1]

$$2\mathrm{Tr}\left[(\boldsymbol{\alpha}_1^* \cdot \mathbf{H})(\boldsymbol{\alpha}_2^* \cdot \mathbf{H})\right] = \boldsymbol{\alpha}_1^* \cdot \boldsymbol{\alpha}_2^* \tag{6.2.11}$$

The scalar interaction is manifested as a position-dependent modification of the dyon mass [42]. The effective mass of dyon 1 becomes

$$m_1^{\mathrm{eff}} = 2\sqrt{g_1^2 + Q_1^2}\ \mathrm{Tr}\left[(\boldsymbol{\alpha}_1^* \cdot \mathbf{H})\left(\Phi + \Delta\Phi^{(2)}(\mathbf{x}_1)\right)\right] \tag{6.2.12}$$

and hence

$$\begin{aligned} R^{(1)}_{\mathrm{scalar}} &= m_1^{\mathrm{eff}}\sqrt{1 - \mathbf{v}_1^2} \\ &= m_1\left(1 - \frac{\mathbf{v}_1^2}{2} + \frac{Q_1^2}{g_1^2}\right) - \frac{g_1 g_2\, \boldsymbol{\alpha}_1^* \cdot \boldsymbol{\alpha}_2^*}{8\pi r_{12}}\left(\mathbf{v}_1^2 + \mathbf{v}_2^2 - \frac{Q_1^2}{g_1^2} - \frac{Q_2^2}{g_2^2}\right) \end{aligned} \tag{6.2.13}$$

Adding these contributions, subtracting the rest mass m_1, and keeping terms up to second order in Q_j or $\mathbf{v}_j$, we obtain

$$\begin{aligned} R^{(1)} = &= -m_1\left(1 - \frac{1}{2}\mathbf{v}_1^2 + \frac{Q_1^2}{2g_1^2}\right) \\ &- \frac{g_1 g_2\, \boldsymbol{\alpha}_1^* \cdot \boldsymbol{\alpha}_2^*}{8\pi r_{12}}\left[(\mathbf{v}_1 - \mathbf{v}_2)^2 - \left(\frac{Q_1}{g_1} - \frac{Q_2}{g_2}\right)^2\right] \\ &- \frac{\boldsymbol{\alpha}_1^* \cdot \boldsymbol{\alpha}_2^*}{4\pi}(g_1 Q_2 - g_2 Q_1)(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{w}_{12}, \end{aligned} \tag{6.2.14}$$

By interchanging particles 1 and 2, a similar expression is obtained for $R^{(2)}$, the Routhian describing the effects of particle 1 on particle 2.

The extension to an arbitrary number of well-separated dyons [43] is straightforward. Since we are considering fundamental dyons that all carry unit magnetic charges, we can set all of the g_j equal to $4\pi/e$. The Routhian obtained by adding all the pairwise interactions is of the form of Eq. (6.2.1), with

$$M_{ij} = \begin{cases} m_i - \displaystyle\sum_{k \neq i} \frac{4\pi \boldsymbol{\alpha}_i^* \cdot \boldsymbol{\alpha}_k^*}{e^2 r_{ik}}, & i = j \\ \dfrac{4\pi \boldsymbol{\alpha}_i^* \cdot \boldsymbol{\alpha}_j^*}{e^2 r_{ik}}, & i \neq j \end{cases} \tag{6.2.15}$$

[1] The factor 2 arises because our normalization convention, Eq. (4.1.1), replaces the usual 1/4 of the Maxwell Lagrangian to a 1/2, as in Eq. (2.1.2).

$$\mathbf{W}_i^j = \begin{cases} -\sum_{k\neq i} \boldsymbol{\alpha}_i^* \cdot \boldsymbol{\alpha}_k^* \mathbf{w}_{ik}\,, & i = j \\ \\ = \boldsymbol{\alpha}_i^* \cdot \boldsymbol{\alpha}_k^* \mathbf{w}_{ij}\,, & i \neq j \end{cases} \tag{6.2.16}$$

and

$$K = \frac{(4\pi)^2}{e^4} M^{-1} \tag{6.2.17}$$

The asymptotic moduli space metric is obtained by returning from the Routhian back to the Lagrangian via a Legendre transform. Substituting Eqs. (6.2.15)–(6.2.17) into Eq. (6.2.1), we obtain the desired asymptotic metric [43],

$$\mathcal{G}_{\text{asym}} = M_{ij} d\mathbf{x}_i \cdot d\mathbf{x}_j + \frac{(4\pi)^2}{e^4} (M^{-1})_{ij} (d\xi_i + \mathbf{W}_{ik} \cdot d\mathbf{x}_k)(d\xi_j + \mathbf{W}_{jl} \cdot d\mathbf{x}_l) \tag{6.2.18}$$

6.2.3 Why does the asymptotic treatment break down?

It is easy to see that this asymptotic approximation to the moduli space metric cannot be exact for the case of two identical monopoles. First of all, the M_{jj} vanish and the asymptotic form becomes singular if the intermonopole distance is too small. Second, for the case of two identical monopoles the approximate metric is independent of the relative phase angle $\xi_1 - \xi_2$. If this isometry were exact, it would imply that the two-monopole solutions was axially symmetric, which we know is not the case. Furthermore, such an isometry would correspond to an additional U(1) isometry, but for the SU(2) case there is only one unbroken U(1) gauge group.

Neither of these problems would be presemt if we were considering a pair of distinct monopoles [44]. As will be shown in detail in next section, the moduli space for a pair of distinct monopoles can be continued to a complete, smooth manifold. Also, for a pair of distinct monopoles there are always two different unbroken U(1) isometries acting on the BPS solutions, so the appearance of an additional U(1) is in fact desired. In next section, we will discuss these two cases in great detail and show how the asymptotic metric is corrected for a pair of identical monopoles, and why the asymptotic metric is in fact exact for a pair of distinct monopoles.

6.3 Exact moduli spaces for two monopoles

For a pair of monopoles, the moduli space is eight-dimensional. Of these eight dimensions, three encode the center-of-mass motion of the two-body system and must remain free, while at least one corresponds to an exact gauge rotation. Thus the nontrivial part of the moduli space is at most four-dimensional. With the various constraints on the moduli space, such as its hyper-Kähler property and the SO(3) isometry from spatial rotations, not much choice is left. In fact, it is via these abstract considerations that Atiyah and Hitchin [] were able to find the exact moduli space for two identical monopoles. In this section, we will consider an arbitrary pair of monopoles, identical or distinct, and find the exact moduli space thereof.

In the process, we will earn an interesting fact. For a pair of distinct monopoles, the asymptotic form of the metric that we found above turns out to extend naturally to all distances and forms a complete and smooth metric that is identical to the exact moduli space metric found via rigorous mathematical considerations.

6.3.1 Geometry of two-monopole moduli spaces

Symmetry considerations tell us a good deal about the form of the two-monopole moduli space $\mathcal{M}$. First of all, there must be three directions, corresponding to overall spatial translations of the two-monopole system, that are free of interaction. In other words, there must be no nontrivial metric components for these directions. Furthermore, the hyper-Kähler structure relates these three free directions to a fourth one, at least locally, so that at least a four-dimensional part of the moduli space comes with a flat metric. This fourth direction must come from gauge rotations that are a mixture of the various U(1) gauge angles associated with the fundamental monopoles. The latter allows, in principle, a discrete mixing between the free part of the gauge angles and the rest, and so we conclude that the space must be of the form

$$\mathcal{M} = R^3 \times \frac{R^1 \times \mathcal{M}_0}{\mathcal{D}} \tag{6.3.1}$$

where $\mathcal{D}$ is a discrete normal subgroup of the isometry group of $R^1 \times \mathcal{M}_0$.

The isometry group of $\mathcal{M}_0$ is also easily determined. Since a spatial rotation of a BPS solution about any fixed point is again a solution, $\mathcal{M}_0$ must possess a three-dimensional rotational isometry. As we noted in Sec. 6.1, this SU(2) isometry does not preserve all the complex structures, but rather mixes them among themselves.

Depending on whether the two monopoles are distinct or identical, we may have one additional U(1) isometry. This arises for a pair of distinct monopoles and so is possible only if the gauge group is rank 2 or higher, with two unbroken U(1) factors. One linear combination of the two unbroken U(1) gauge degrees of freedom generates the translational symmetry, alluded to above, along the overall R^1. The remaining generator must then induce a U(1) isometry acting on $\mathcal{M}_0$. Hence, $\mathcal{M}_0$ must be a four-dimensional manifold which is equipped either with four Killing vector fields that span an $su(2) \times u(1)$ algebra, or with three Killing vectors that span $su(2)$.

Furthermore, we saw in the previous section that the orbits of the rotational isometry on the asymptotic metric were three-dimensional; clearly the exact metric must also possess this property at large r.

For a four-dimensional manifold the fact that the moduli space must be hyper-Kähler implies that the manifold must be a self-dual Einstein manifold. From this, together with the rotational symmetry properties of the manifold, it follows that the metric can be written as

$$ds^2 = f(r)^2\, dr^2 + a(r)^2\sigma_1^2 + b(r)^2\sigma_2^2 + c(r)^2\sigma_3^2, \tag{6.3.2}$$

where the metric functions obey

$$\frac{2bc}{f}\frac{da}{dr} = b^2 + c^2 - a^2 - 2\epsilon bc \tag{6.3.3}$$

(and cyclic permutations thereof) with ϵ either 0 or 1, while the three one-forms σ_k satisfy

$$d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k. \tag{6.3.4}$$

An explicit representation for these one-forms is

$$\begin{aligned}
\sigma_1 &= -\sin\psi d\theta + \cos\psi\sin\theta d\phi,\\
\sigma_2 &= \cos\psi d\theta + \sin\psi\sin\theta d\phi,\\
\sigma_3 &= d\psi + \cos\theta d\phi.
\end{aligned} \tag{6.3.5}$$

The ranges of θ and ϕ are $[0,\pi]$ and $[0,2\pi]$, respectively. The function f depends on the coordinate choice for r, and after suitably fixing this choice the four coordinates may be regarded as the usual spherical coordinate system on R^3 plus a single phase angle. A conventional choice is to assign (r,θ,ϕ) to form the three-vector $\mathbf{r}$ on R^3, and ψ to a U(1) angle. With this choice, it is easy to see that

$$\sigma_3 = d\psi + \mathbf{w}(\mathbf{r})\cdot d\mathbf{r} \tag{6.3.6}$$

where $\mathbf{w}$ is the same Dirac potential as in Eq. (6.2.10). In order that the metric tend to the asymptotic form $\mathcal{G}_{\rm rel}$, the range of ψ must be $[0,4\pi]$.

Recalling the asymptotic pairwise interaction of monopoles, we see that the functions in Eq. (6.3.2) should behave as

$$f^2 \to k; \quad a^2 \to kr^2; \quad b^2 \to kr^2; \quad c^2 \to k' \tag{6.3.7}$$

as $r \to \infty$, with k and k' being constants. We now quote the results of Atiyah and Hitchin [] and list all the solutions to the hyper-Kähler conditions above that correspond to smooth geometries:

- $\epsilon = 0$ produces only one smooth solution with an asymptotic region, the so-called Eguchi-Hanson gravitational instanton [51]. Its asymptotic geometry is R^4/Z_2 and does not have a compact circle corresponding to a gauge U(1) angle.

- $\epsilon = 1$, $a = b = c$ gives

$$f^2 = 1; \qquad a^2 = b^2 = c^2 = r^2 \tag{6.3.8}$$

  which corresponds to flat R^4. This can be divided by Z to make a cylinder, $R^3 \times S^1$, which would be $\mathcal{M}_0$ for a pair of noninteracting monopoles. For an interacting pair, however, this is not acceptable, because it has too much symmetry.

- $\epsilon = 1$ $a = b \neq c$ gives

$$f^2 = 1 + \frac{2l}{r}; \qquad a^2 = b^2 = r^2 f^2; \qquad c^2 = \frac{4l^2}{f^2} \tag{6.3.9}$$

  with $l > 0$. (A possible overall multiplicative constant has been suppressed.) This is the Taub-NUT geometry with an SU(2) rotational isometry [50], which is illustrated in Fig. 6.2. The range of ψ is $[0, 4\pi]$. Since $a = b$, the metric has no dependence on ψ, and a shift of ψ is a symmetry. This generates an additional U(1) isometry, which is also triholomorphic and thus could be associated with an unbroken U(1) gauge symmetry.

- $\epsilon = 1$, $a \neq b \neq c$ yields the Atiyah-Hitchin geometry with an SO(3) rotational isometry and no gauge isometry [13][38]. There are two such smooth manifolds. Their topology and global geometry are a bit involved. We will come back to these in Sec. 6.3.3

Thus, only two of the four cases, namely the Taub-NUT manifold and the Atiyah-Hitchin manifold, can be part of the exact moduli space for a pair of interacting monopoles.

It is worthwhile to note that these two geometries share the same general form for the asymptotic metric,

$$\begin{aligned} ds^2 &= \left(1 + \frac{2l}{r}\right)\left(dr^2 + r^2\sigma_1^2 + r^2\sigma_2^2\right) + \left(\frac{4l^2}{1 + 2l/r}\right)\sigma_3^2 \\ &= \left(1 + \frac{2l}{r}\right) d\mathbf{r}^2 + \left(\frac{4l^2}{1 + 2l/r}\right)(d\psi + \cos\theta\, d\phi)^2 \end{aligned} \tag{6.3.10}$$

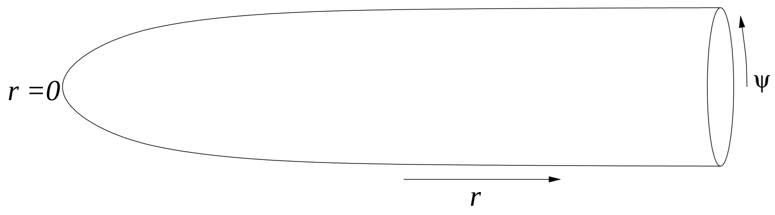

Figure 6.2: The Taub-NUT manifold with two of the three Euler angles suppressed. The origin $r = 0$ is a special point where one circle collapses to a point. Everywhere else, we have a squashed S^3 at each fixed value of $r > 0$.

up to an overall multiplicative constant. The difference between the two is that the parameter l is positive for the Taub-NUT manifold and negative for the Atiyah-Hitchin manifold. With negative l this metric develops an obvious singularity at $r = 2l$, signalling that the Atiyah-Hitchin geometries must behave differently from this asymptotic form as r become comparable to $2l$. On the other hand, with positive l, this asymptotic form is exact for the Taub-NUT geometry. In the subsequent sections, we will find the physical reasons for such different behavior.

6.3.2 Taub-NUT manifold for a pair of distinct monopoles

Let us now specialize the results of Sec. 6.2 to the case of two distinct fundamental monopoles. If the corresponding simple roots are orthogonal (i.e., if they are not connected in the Dynkin diagram), then Eq. (6.2.18) reduces to a flat metric, corresponding to the fact that the monopoles do not interact with each other. The more interesting case is when $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are connected in the Dynkin diagram. These may be roots of equal length; if not, we can, without loss of generality, take $\boldsymbol{\alpha}_2$ to be the shorter root. If we define

$$\lambda = -2\boldsymbol{\alpha}_1^* \cdot \boldsymbol{\alpha}_2^* \tag{6.3.11}$$

then the general properties of Dynkin diagrams imply that $\lambda\boldsymbol{\alpha}_2^2 = 1$ and

$$p = \lambda\boldsymbol{\alpha}_1^2 = \frac{\boldsymbol{\alpha}_1^2}{\boldsymbol{\alpha}_2^2} \tag{6.3.12}$$

is an integer equal to 1, 2, or 3.

The first step is to convert from the original coordinates to center-of-mass and relative variables. For the spatial coordinates we define the usual variables

$$\mathbf{R} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2}, \qquad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2. \tag{6.3.13}$$

To separate the phase variables, we first define a total charge q_χ and a relative charge q_ψ by

$$q_\chi = \frac{(m_1 Q_1 + m_2 Q_2)}{e\,(m_1+m_2)}, \qquad q_\psi = \frac{\lambda(Q_1 - Q_2)}{2e}. \tag{6.3.14}$$

The coordinates conjugate to these charges are

$$\chi = (\xi_1 + \xi_2), \qquad \psi = \frac{2(m_2\xi_1 - m_1\xi_2)}{\lambda\,(m_1+m_2)}. \tag{6.3.15}$$

When expressed in these variables, the metric of Eq. (6.2.18) separates into a sum of two terms

$$\mathcal{G}_{\rm asym} = \mathcal{G}_{\rm cm} + \mathcal{G}_{\rm rel} \tag{6.3.16}$$

where

$$\mathcal{G}_{\rm cm} = (m_1+m_2)\left[d\mathbf{R}^2 + \frac{(4\pi)^2}{e^4(m_1+m_2)^2}d\chi^2\right] \tag{6.3.17}$$

is a flat metric and

$$\mathcal{G}_{\rm rel} = \left(\mu + \frac{2\pi\lambda}{e^2 r}\right) d\mathbf{r}^2 + \left(\frac{2\pi\lambda}{e^2}\right)^2 \left(\mu + \frac{2\pi\lambda}{e^2 r}\right)^{-1} (d\psi + \mathbf{w}(\mathbf{r})\cdot d\mathbf{r})^2 \tag{6.3.18}$$

Here μ is the reduced mass and $\mathbf{w}(\mathbf{r}) = \mathbf{w}_{12}(\mathbf{r})$.

Apart from an overall factor of μ, the relative metric has the same form as the Taub-NUT metric of Eq. (6.3.9), with $l = \pi\lambda/e^2$ [44][45][48]. To verify that the manifold defined by the asymptotic metric is indeed the Taub-NUT space, all that remains is to show that ψ has periodicity 4π, which is required for the manifold to be nonsingular at $r = 0$.

We first recall, from the discussion in Sec. 6.2.1, that Q_j is quantized in units of $e\boldsymbol{\alpha}_j^2$. This implies that ξ_j has period $2\pi/\boldsymbol{\alpha}_j^2$. Hence, a shift of ξ_1 by $2\pi/\boldsymbol{\alpha}_1^2$ implies the identification

$$(\chi,\psi) = (\chi + \frac{2\pi}{\boldsymbol{\alpha}_1^2}, \psi + \frac{4\pi m_2}{\lambda\boldsymbol{\alpha}_1^2(m_1+m_2)}), \tag{6.3.19}$$

while a $-2\pi/\boldsymbol{\alpha}_2^2$ shift of ξ_2 gives

$$(\chi,\psi) = (\chi - \frac{2\pi}{\boldsymbol{\alpha}_2^2}, \psi + \frac{4\pi m_1}{\lambda\boldsymbol{\alpha}_2^2(m_1+m_2)}). \tag{6.3.20}$$

Combining p steps of the first shift and one of the second then gives

$$(\chi,\psi) = (\chi, \psi + 4\pi) \tag{6.3.21}$$

thus showing that ψ has the 4π periodicity that is required to obtain the nonsingular Taub-NUT space. The identification shown in Eq. (6.3.19) then defines the discrete subgroup $\mathcal{D}$ that appears in Eq. (6.3.1)

As a consistency check, note that Eq. (6.3.14) shows that the quantization of Q_j in units of $e\boldsymbol{\alpha}_j^2$ implies that q_ψ has integer or half-integer eigenvalues, as is appropriate for a momentum conjugate to an angle with periodicity 4π. By contrast, χ is not periodic, and q_χ is not quantized, unless the ratio of the monopole masses is a rational number.

Thus, by analytic continuation of asymptotic form of the moduli space metric, we have apparently found a smooth manifold that has all the properties required of the exact moduli space. Not only have we learned that the Taub-NUT manifold is the interacting part of the exact moduli space, but we also learned that the naive asymptotic approximation yields an exact metric for the case of a pair of distinct monopoles [44].

6.3.3 Atiyah-Hitchin geometry for two identical monopoles

This finally brings us to the other choice for the exact moduli space for a pair of interacting monopoles. The decomposition of the full moduli space into a free part and an interacting part should follow from the asymptotic form of the metric. Since two identical monopoles have exactly the same mass and the same magnetic charge, the decomposition of the full moduli space should be

$$\mathcal{M} = R^3 \times \frac{S^1 \times \mathcal{M}_0}{Z_2}. \tag{6.3.22}$$

where again $\mathcal{M}_0$ is a four-dimensional hyper-Kähler space.

As we saw above, only one class of possibilities remains for $\mathcal{M}_0$, namely the Atiyah-Hitchin geometry with $a \neq b \neq c$. In this section, we will characterize this geometry with an emphasis on its topology and its global geometry. Without loss of generality we can rescale[2] the radial coordinate so that $l = -1$.

It is quite clear that the asymptotic form of the metric,

$$ds^2 \simeq \left(1 - \frac{2}{r}\right)\left(dr^2 + r^2\sigma_1^2 + r^2\sigma_2^2\right) + \left(\frac{4}{1-2/r}\right)\sigma_3^2 \tag{6.3.23}$$

will hit a singularity at $r = 2$, which tells us there must be some correction when the separation between the two monopoles is small. Writing the metric more generally, as before,

$$f(r)^2\, dr^2 + a(r)^2\sigma_1^2 + b(r)^2\sigma_2^2 + c(r)^2\sigma_3^2, \tag{6.3.24}$$

[2] In order to restore the correctly normalized metric we need to repeat the exercise carried out above for a pair of distinct monopoles, mindful that we must employ two identical masses $m_1 = m_2$ and that the coupling is modified to $\lambda\boldsymbol{\alpha}_1^2 = \lambda\boldsymbol{\alpha}_2^2 = -2$.

we have a further choice for the radial coordinate. We will fix it by setting

$$f = -\frac{b}{r} \tag{6.3.25}$$

following Gibbons and Manton [], according to whom we have the following implicit solution. Introduce β to parameterize the radial coordinate by

$$r = 2K(\sin(\beta/2)), \qquad K(x) \equiv \int_0^{\pi/2} dt\, \frac{1}{\sqrt{1 - x^2 \sin^2 t}} \tag{6.3.26}$$

so that, as β varies from 0 to π, the range of r is $[\pi, \infty)$. The functions a, b, c are then determined by

$$\begin{aligned} ab &= -(\sin\beta)\frac{r\,dr}{d\beta} + \frac{1}{2}(1 - \cos\beta)r^2 \\ bc &= -(\sin\beta)\frac{r\,dr}{d\beta} - \frac{1}{2}(1 + \cos\beta)r^2 \\ ca &= -(\sin\beta)\frac{r\,dr}{d\beta} \end{aligned} \tag{6.3.27}$$

This metric indeed asymptotes to Eq. (6.3.23) as $r \to \infty$ ($\beta \to \pi$), without further change of the radial coordinate. In order to see whether the singularity at small r is replaced by a regular geometry, we must also understand the metric near $r = \pi$. Again following Gibbons and Manton, we have

$$ds^2 \simeq dr^2 + 4(r - \pi)^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2 \tag{6.3.28}$$

In order that this metric give a smooth manifold near $r = \pi$, the angle associated with σ_1 must have a period π instead of the usual 2π. We can rephrase this by defining a new set of Euler angles by

$$\begin{aligned} \sigma_1 &= d\tilde{\psi} + \cos\tilde{\theta} d\tilde{\phi} \\ \sigma_2 &= -\sin\tilde{\psi} d\tilde{\theta} + \cos\tilde{\psi}\sin\tilde{\theta} d\tilde{\phi}, \\ \sigma_3 &= \cos\tilde{\psi} d\tilde{\theta} + \sin\tilde{\psi}\sin\tilde{\theta} d\tilde{\phi}, \end{aligned} \tag{6.3.29}$$

and imposing the identification

$$I: \qquad \tilde{\psi} \to \tilde{\psi} + \pi \tag{6.3.30}$$

In terms of the original Euler angles, this is

$$I: \qquad \theta \to \pi - \theta, \qquad \phi \to \phi + \pi, \qquad \psi \to -\psi \tag{6.3.31}$$

From the viewpoint of the monopole solutions this identification is quite natural, since it exchanges the positions of the two identical monopoles, and thus maps any two-monopole solution to itself. The manifold that is obtained after making this identification is known as the double-cover of the Atiyah-Hitchin manifold [13]. Near the "origin" at $r = \pi$ its geometry is that of $R^2 \times S^2$.

A second smooth manifold can be obtained by making a further Z_2 division, defined by

$$I' : \qquad \theta \to \theta, \qquad \phi \to \phi, \qquad \psi \to \psi + \pi \tag{6.3.32}$$

This is known as the Atiyah-Hitchin manifold, and is the manifold denoted as M_2^0 in Ref. []. Near $r = \pi$ it has the geometry of $R^2 \times RP^2$.

To decide which is the proper choice of $\mathcal{M}_0$, we need to return to the definition of the center-of-mass and relative phase angles χ and ψ. We proceed as in Sec. 6.3.2, except that when $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 \equiv \boldsymbol{\alpha}$, we have $\lambda\boldsymbol{\alpha}^2 = -2$. The analogues of the identifications in Eqs. (6.3.19) – (6.3.21) tell us that ψ has period 2π, χ has a period $4\pi/\boldsymbol{\alpha}^2$, and that

$$(\chi, \psi) = (\chi + \frac{2\pi}{\boldsymbol{\alpha}^2}, \psi - \pi), \tag{6.3.33}$$

This corresponds to a Z_2 division on the product manifold, thus yielding a manifold

$$\mathcal{M} = R^3 \times \frac{S^1 \times \mathcal{M}_0}{Z_2} \tag{6.3.34}$$

We now remember that the only role of the $R^3 \times S^1$ in monopole-monopole scattering is to supply a conserved momentum and a conserved electric charge that are not affected by the scattering process. If we set these quantities equal to zero, then the scattering is completely described by $\mathcal{M}_0/Z_2$; in order that this be a smooth manifold, we must identify the double-cover of the Atiyah-Hitchin manifold as $\mathcal{M}_0$.

6.4 Exact moduli spaces for arbitrary numbers of distinct monopoles

In the previous section, we saw that the asymptotic form of the moduli space metric for a pair of distinct fundamental monopoles is in fact the exact moduli space metric for all values of the monopole separation. The key to this surprising result lies in the gauge isometry. As we noted at the very beginning of our discussion of the asymptotic interactions between monopoles, the long-range interactions involve only the interchange of photons and their scalar analogues, because in the maximally broken phase all the other particles — the charged vector and scalar mesons — are heavy and cannot propagate over long distances. The interactions mediated by these massive particles fall exponentially with distance. Thus, the asymptotic form of the metric for k monopoles is always equipped with k U(1) isometries.

For a pair of SU(2) monopoles, or for a pair of identical monopoles, the two U(1) isometries cannot both be exact, since there is only one U(1) gauge rotation acting on these monopoles. One might view the short distance corrections in the Atiyah-Hitchin manifold as the removal of the redundant gauge isometry. This is also reflected in the fact that electric charge can hop from one monopole to the other.

For a pair of distinct monopoles, on the other hand, two U(1) gauge isometries are, in fact, required. However small the impact parameter is, the electric charges on the two monopole cores are separately conserved. If there were some short-distance correction to the asymptotic metric, it would have to respect the additional constraint of preserving two U(1) gauge isometries, in addition to all the usual properties that are associated with monopole moduli spaces. In the case of a two-monopole system, this constraint turns out to be sufficiently stringent to fix the metric uniquely to be Taub-NUT.

What really happened here is that the only possible short-distance correction comes from the exchange of heavy charged vector mesons, but this is disallowed by the gauge symmetry combined with the BPS equation. Even with many distinct monopoles, this intuitive picture of why the asymptotic form of the metric is actually the exact metric should still work as long as no two monopoles are identical [43]. In this section, we will show that the asymptotic metric for an arbitrary number of distinct monopoles is in fact the exact moduli space metric. We will start by showing that it is smooth.

6.4.1 The asymptotic metric is smooth everywhere

We consider a system of n fundamental monopoles with charges $\boldsymbol{\alpha}_i^*$, each corresponding to a different simple root of the Lie algebra. This set of simple roots defines a subdiagram of the Dynkin diagram of the algebra. If this subdiagram has several disconnected components, the monopoles belonging to one component will have no interactions with those belonging to others, and the the total moduli space will be a product of moduli spaces for each connected component. It is therefore sufficient to to consider the case where the $\boldsymbol{\alpha}_j$ correspond to a connected subset of simple roots, and thus to the full Dynkin diagram of a (possibly smaller) simple gauge group.

There are several ways in which this moduli space could fail to be smooth. First, the $n \times n$ matrix M would not be invertible if $\det M$ vanished. Second, the metric would be degenerate if its determinant vanished; since

$$\det \mathcal{G}_{\rm asym} = \left(\frac{4\pi}{e^2}\right)^{2n} (\det M)^2 \,. \qquad (6.4.1)$$

this possibility is equivalent to the first. Finally, there could be singularities when one or more of the r_{ij} vanish.

We begin by showing that $\det M$ is nonzero whenever the r_{ij} are nonzero. We start by recalling that

$$\begin{aligned} M_{ii} &= m_i + \sum_{j\neq i} c_{ij}, \\ M_{ij} &= -c_{ij}\,, \qquad\qquad \text{if } i \neq j, \end{aligned} \qquad (6.4.2)$$

where the c_{ij} are all nonnegative and the m_i are all positive definite.

It is trivial to see that $\det M > 0$ for $n = 2$. We then proceed inductively. We note that the determinant vanishes if all of the m_i are zero, and that its partial derivative with respect to any one of the masses is the determinant of the $(n-1)\times(n-1)$ matrix obtained by eliminating the row and column corresponding to that mass. The new matrix is of the same type as the first (but with a shifting of the m_j), and so has a positive determinant by the induction hypothesis. If follows that $\det M > 0$.

To study the behavior when some of the r_{ij} vanish, it is more convenient to switch to center-of-mass and relative coordinates. To do this, we observe that the Dynkin diagram contains n links, which we label by an index A. Each of these is associated with a pair of roots $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$ for which $\lambda_A \equiv -2\boldsymbol{\alpha}_i^* \cdot \boldsymbol{\alpha}_j^*$ is nonzero. By analogy with our treatment of the two-monopole case, we define center-of-mass and relative coordinates

$$\mathbf{R} = \frac{\sum m_i \mathbf{x}_i}{\sum m_i}, \qquad \mathbf{r}_A = \mathbf{x}_i - \mathbf{x}_j. \qquad (6.4.3)$$

and charges

$$q_\chi = \frac{\sum m_i Q_i}{e\sum m_i}, \qquad q_A = \frac{\lambda_A}{2e}(Q_i - Q_j), \qquad (6.4.4)$$

As before, the q_A have half-integer eigenvalues and their conjugate angles ψ_A have period 4π.

When rewritten in terms of these variables, the metric splits into the sum of a flat metric for $\mathbf{R}$ and χ and a relative moduli space metric,

$$\begin{aligned} \mathcal{G}_{\rm rel} &= C_{AB}\, d\mathbf{r}_A \cdot d\mathbf{r}_B \\ &+ \frac{(2\pi)^2 \lambda_A \lambda_B}{e^4} (C^{-1})_{AB}\, (d\psi_A + \mathbf{w}(\mathbf{r}_A)\cdot d\mathbf{r}_A)(d\psi_B + \mathbf{w}(\mathbf{r}_B)\cdot d\mathbf{r}_B). \end{aligned} \qquad (6.4.5)$$

where the $(n-1)\times(n-1)$ matrix C_{AB} is

$$C_{AB} = \mu_{AB} + \delta_{AB}\,\frac{2\pi\lambda_A}{e^2 r_A}, \tag{6.4.6}$$

with $r_A = |\mathbf{r}_A|$ and μ_{AB} being a reduced mass matrix.

This relative metric is manifestly invariant under independent constant shifts of the periodic coordinates ψ_A. These isometries, together with the isometry under uniform translation of the global phase χ, correspond to the action of the n independent global U(1) gauge rotations, generated by the $\boldsymbol{\alpha}_j \cdot \mathbf{H}$, of the unbroken gauge group.

The $\boldsymbol{\alpha}_i$'s are connected and distinct. It is easy to see that the sum $\sum \boldsymbol{\alpha}_i^*$ is then equal to $\boldsymbol{\gamma}^*$ for some positive root $\boldsymbol{\gamma}$ of the group G. Embedding of the SU(2) BPS monopole using the subgroup generated by $\boldsymbol{\gamma}$ gives a solution that is both spherically symmetric and invariant under the $n-1$ U(1) gauge rotations orthogonal to $\boldsymbol{\gamma}\cdot\mathbf{H}$. It thus corresponds to a maximally symmetric point on the relative moduli space that is a fixed point both under overall rotation of the n monopoles and under the $(n-1)$ U(1) translations. This fixed point is clearly the origin, $\mathbf{r}_A = 0$ for all A. In the neighborhood of this point, the factors of $1/r_A$ are all sufficiently large that the matrix C_{AB} is effectively diagonal, so that

$$\mathcal{G}_{\rm rel} \simeq \frac{2\pi}{e^2}\sum_A \lambda_A \left(\frac{1}{r_A} d\mathbf{r}_A^2 + r_A\left(d\psi_A + \mathbf{w}(\mathbf{r}_A)\cdot d\mathbf{r}_A\right)^2\right), \tag{6.4.7}$$

with the leading corrections being linear in the r_A. Comparing this with the results of Sec. 6.3.2, we see that the manifold is nonsingular at the origin.

Finally, we consider the points where only some of the r_A's vanish; we us a subscript V to distinguish those that vanish. In inverting C_{AB} to leading order, it suffices to remove all components of μ_{AB} in the rows or the columns labeled by the V's. The matrix C then becomes effectively block-diagonal, and consists of the diagonal entries $2\pi\lambda_V/e^2 r_V$ and a number of smaller square matrices. Looking for the part of metric along the $\mathbf{r}_V$ and ψ_V directions, we find

$$\mathcal{G}_{\rm rel} \simeq \frac{2\pi}{e^2}\sum_V \lambda_V \left(\frac{1}{r_V} d\mathbf{r}_V^2 + r_V\left(d\psi_V + \mathbf{w}(\mathbf{r}_V)\cdot d\mathbf{r}_V\right)^2\right) + \cdots. \tag{6.4.8}$$

The terms shown explicitly give a smooth manifold, as previously. The remaining terms, indicated by the ellipsis, consist of harmless finite terms that are quadratic in the other $d\mathbf{r}_A$ and $d\psi_A$ as well as mixed terms that involve a $d\mathbf{r}_V$ or a $d\psi_V$ multiplied by a $d\mathbf{r}_A$ or a $d\psi_A$. The off-diagonal metric coefficients corresponding to the latter vanish linearly near $\mathbf{r}_V = 0$, and hence cannot introduce any singular behavior at that point. We thus conclude that the relative metric, and thus the total metric, remains smooth as any number of monopoles come close together.

6.4.2 The asymptotic metric is a hyper-Kähler quotient

Actually, a cleaner way of showing that the asymptotic metric is smooth (as well as that it is hyper-Kähler) is to show that it can be obtained by a hyper-Kähler quotient procedure [49]. This alternate derivation is important not only for showing the smoothness, but also for making contact with the moduli space metric derived from the Nahm data, which should give the exact form. For simplicity, we take the case of an SU$(n+2)$ theory broken to U$(1)^{n+1}$, and consider a collection of $n+1$ distinct fundamental monopoles.

The hyper-Kähler quotient procedure is more or less the same as for a symplectic quotient, so let us briefly recall the latter first. For more complete details, we refer readers to the Appendix. Suppose that one is given a symplectic form w (say on a phase space) together with a symmetry coordinate ξ, or equivalently a Killing vector $\partial/\partial\xi$ that not only preserves the metric but also preserves the symplectic form w. A symplectic quotient is a procedure for removing two dimensions associated with such a cyclic coordinate. Formally, one does this by first identifying a "moment map" ν — a function on the manifold — by

$$d\nu = \left\langle \frac{\partial}{\partial\xi}\,, w \right\rangle \tag{6.4.9}$$

The right hand side is an inner product between the Killing vector field and the symplectic 2-form w, and the resulting 1-form is guaranteed to be closed if

$$dw = 0, \qquad \mathcal{L}_K w = 0 \tag{6.4.10}$$

Assuming trivial topology, the moment map ν is well-defined.

The submanifold on which ν takes a particular value, say s, is a manifold $\nu^{-1}(s)$ with one fewer dimensions. One can reduce by one more dimension by dividing $\nu^{-1}(s)$ by the group action G of the Killing vector ∂_ξ. The resulting manifold with two fewer dimensions,

$$\nu^{-1}(s)/G \tag{6.4.11}$$

is the symplectic quotient of the original manifold, and is itself a symplectic manifold. The symplectic quotient takes a more familiar shape if we consider the manifold as the phase space for some Hamiltonian dynamics. There, the quotient effectively corresponds to restricting our attention to motions with a definite conserved momentum, $\nu = s$, along a cyclic coordinate.

A hyper-Kähler manifold is essentially a symplectic manifold with three symplectic forms, namely the three Kähler forms, defined componentwise from the complex structure and the metric by

$$w^{(s)}_{mn} = g_{mk}(J^{(s)})^k{}_n \tag{6.4.12}$$

The hyper-Kähler quotient reduces the dimension by four, since we can now impose three moment maps for each Killing vector field. We define the moment maps by

$$d\nu_a = \left\langle \frac{\partial}{\partial \xi}, w^{(s)} \right\rangle \tag{6.4.13}$$

where $\partial/\partial\xi$ preserves all three Kähler forms, and consider the manifold

$$\left(\nu_1^{-1}(s_1) \cap \nu_2^{-1}(s_2) \cap \nu_3^{-1}(s_3)\right)/G \tag{6.4.14}$$

This new manifold is also a hyper-Kähler manifold. If the initial manifold was smooth the quotient is also smooth, provided that the group action does not have a fixed submanifold, since the metric on the quotient is inherited from the old manifold.

Consider a flat Euclidean space, $H^n \times H^n = R^{4n} \times R^{4n}$, whose $8n$ Cartesian coordinates are grouped into $2n$ quaternions q^A and t^A $(A = 1, 2, \ldots, n)$. We will assume a flat metric of the form

$$ds^2 = \sum dq^A \otimes_s d\widehat{q}^A + \sum \mu_{AB}\, dt^A \otimes_s d\widehat{t}^B \tag{6.4.15}$$

(Here conjugation is denoted by a hat, and acts like Hermitian conjugation

$$\widehat{ab} = \widehat{b}\widehat{a} \tag{6.4.16}$$

because quaternions do not commute.)

The three Kähler forms can be compactly written as the expansion of

$$-\frac{1}{2}\left(\sum dq^A \wedge d\widehat{q}^A + \sum \mu_{AB} dt^A \wedge d\widehat{t}^B\right) = iw^{(1)} + jw^{(2)} + kw^{(3)} \tag{6.4.17}$$

which is necessarily purely imaginary since μ_{AB} is a symmetric matrix. The metric and the Kähler forms are nondegenerate as long as the matrix μ is nondegenerate.

A useful reparametrization of the q^A is obtained by introducing n three-vectors $\mathbf{r}_A$ such that

$$q^A i \widehat{q}^A = ir^1_A + jr^2_A + kr^3_A \tag{6.4.18}$$

and n angular coordinates χ^A defined indirectly by rewriting the first term in the metric as[3]

$$\sum dq^A \otimes_s d\widehat{q}^A = \frac{1}{4}\sum\left[\frac{1}{r_A}\, d\mathbf{r}_A^2 + r_A(d\chi^A + \mathbf{w}(\mathbf{r_A})\cdot d\mathbf{r}_A)^2\right] \tag{6.4.19}$$

[3]For an explicit expression of the coordinate transformation, see [].

A shift of χ_A by $\Delta\chi_A$ is a multiplicative map

$$q^A \to q^A e^{i\Delta\chi_A/2} \tag{6.4.20}$$

The reparametrization we want for t^A is

$$t^A = \sum_B (\mu^{-1})_{AB} y_0^B + i y_1^A + j y_2^A + k y_3^A \tag{6.4.21}$$

from which it follows that the second term in the metric is

$$\sum \mu_{AB}\, dt^A \otimes_s d\hat{t}^B = \sum \left[(\mu^{-1})_{AB} dy_0^A dy_0^B + \mu_{AB}\, d\mathbf{y}^A \cdot d\mathbf{y}^B \right] \tag{6.4.22}$$

In the new coordinates the Kähler forms are the three imaginary parts of

$$\frac{1}{4} \sum_A d\chi^A \wedge (i\, dr_1^A + j\, dr_2^A + k\, dr_3^A) + \sum_A dy_0^A \wedge (i\, dy_1^A + j\, dy_2^A + k\, dy_3^A) + \cdots \tag{6.4.23}$$

where the ellipsis denotes parts involving neither χ_A nor y_0^A.

We wish to start with this flat hyper-Kähler metric and use a hyper-Kähler quotient to obtain a $4n$-dimensional curved hyper-Kähler manifold. To this end, consider the n Killing vectors

$$K_A = 2\frac{\partial}{\partial \chi^A} + \frac{\partial}{\partial y_0^A} \tag{6.4.24}$$

that generate

$$\begin{aligned} q^A &\to q^A e^{i\theta_A} \\ t^A &\to t^A + \sum_B (\mu^{-1})^{AB}\, \theta_B \end{aligned} \tag{6.4.25}$$

The $3n$ moment maps are thus the n purely imaginary triplets in

$$\frac{1}{2}\left(i r_A^1 + j r_A^2 + k r_A^3 \right) + \left(i y_1^A + j y_2^A + k y_3^A \right) = \frac{1}{2}\left[q^A i \hat{q}^A + (t^A - \hat{t}^A) \right] \tag{6.4.26}$$

Setting these $3n$ moment maps to zero, we may remove the $\mathbf{y}^A$ in favor of the $\mathbf{r}^A$,

$$\mathbf{y}^A = -\frac{1}{2}\mathbf{r}^A \tag{6.4.27}$$

This replacement gives us a $4n+n$ dimensional manifold which can be further reduced by the symmetry action of R^n.

The simplest method for doing this last step is to express the metric in the dual basis in terms of some basis vector fields, instead of one-forms, and set the generators of the isometry in Eq. (6.4.24) to zero. We will choose to work with the of coordinates defined by

$$\frac{\partial}{\partial \psi^A} = \frac{\partial}{\partial \chi^A}; \qquad \frac{\partial}{\partial \theta^A} = 2\frac{\partial}{\partial \chi^A} + \frac{\partial}{\partial y_0^A} \tag{6.4.28}$$

and set $\partial/\partial\theta^A$ to zero. With this choice of coordinates, the metric of the quotient manifold

$$\left(\nu_1^{-1}(0)\cap\nu_2^{-1}(0)\cap\nu_3^{-1}(0)\right)/R^n \tag{6.4.29}$$

is

$$\begin{aligned} ds^2 &= \frac{1}{4}\,\mathcal{C}_{AB}\,d\mathbf{r}_A\cdot d\mathbf{r}_B \\ &+ \frac{1}{4}\,(\mathcal{C}^{-1})_{AB}\,(d\psi_A+\mathbf{w}(\mathbf{r}_A)\cdot d\mathbf{r}_A)(d\psi_B+\mathbf{w}(\mathbf{r}_B)\cdot d\mathbf{r}_B) \end{aligned} \tag{6.4.30}$$

where the matrix $\mathcal{C}_{AB}$ is

$$\mathcal{C}_{AB}=\mu_{AB}+\delta_{AB}\,\frac{1}{r_A}, \tag{6.4.31}$$

Up to an overall factor of 1/4 and a coupling constant $g^2/8\pi = 2\pi/e^2$, this is precisely the relative part of the asymptotic metric for a chain of $n+1$ distinct monopoles in SU(n+2) theory. The reduced mass matrix μ_{AB} is a positive definite matrix of rank n, as the construction here assumes. Furthermore, its inverse μ^{-1} is also nondegenerate as long as the monopoles are all of finite mass, and this ensures that there is no fixed point under the R^n action used above. From this, we can conclude that this manifold is free of singularities.

6.4.3 The asymptotic metric is the exact metric

While there is plenty of reason to believe that the asymptotic metric for the case of all distinct monopoles is exact, there is as yet no direct field theoretical proof of this assertion.[4] However, very compelling support can be found from the ADHMN construction. The Nahm data reproduces the complete family of BPS monopoles and, furthermore, has its own intrinsic definition of a moduli space metric. At first encounter, this latter definition appears to have little to do with the field theoretical definition of the moduli space metric, although for the case of an SU(2) gauge group it has been shown mathematically that the two definitions give the same metric.

However, recent progress in string theory has given us a much better understanding of the ADHMN construction in terms of D-branes. In particular, it has become quite clear why the two definitions of the moduli space metric should produce one and the same geometry; we refer readers to Chap. 11 for more details. Using this knowledge, we show here that the asymptotic form of the metric is precisely the same as the exact metric from the Nahm data [46] and thereby prove the main assertion of this section.

[4] An alternate approach to this proof can be found in Ref. [47].

Before invoking the Nahm data, however, it is useful to generalize slightly the hyper-Kähler quotient construction above. Instead of using $H^n \times H^n$ as the starting point, we want to start with $H^n \times H^{n+1}$, where the H^n is to be taken the same as the first factor in the previous construction. We have $2n+1$ quaternionic variables, q^A $(A = 1, 2, \ldots, n)$ and

$$T^i = \frac{1}{m_i}\, x_0^i + i x_1^i + j x_2^i + k x_3^i \,, \qquad i = 0, 1, 2, \ldots, n \tag{6.4.32}$$

We introduce the flat metric

$$ds^2 = \sum_a dq^A \otimes_s d\widehat{q}^A + \sum_i m_i\, dT^i \otimes_s d\widehat{T}^i \tag{6.4.33}$$

As the notation suggests, the m_i will later be identified with the masses of individual monopoles.

Let us take a hyper-Kähler quotient with the action

$$T^i \to T^i + \eta \tag{6.4.34}$$

for any real number η. The three moment maps are

$$\frac{1}{2}\, \sum_i m_i (T^i - \widehat{T}^i) \tag{6.4.35}$$

The subsequent hyper-Kähler quotient reduces the H^{n+1} factor to H^n with the metric,

$$\sum \Big((\mu^{-1})_{AB} dy_0^A dy_0^B + \mu_{AB}\, d\mathbf{y}^A \cdot d\mathbf{y}^B \Big) \tag{6.4.36}$$

where the reduced mass matrix μ is associated with the m_i and the y coordinates are constructed from the x coordinates by writing

$$\mathbf{y}^A = \mathbf{x}^{A-1} - \mathbf{x}^A, \qquad \frac{\partial}{\partial y_0^A} = \frac{\partial}{\partial x_0^{A-1}} - \frac{\partial}{\partial x_0^A} \tag{6.4.37}$$

while setting

$$0 = \sum m_i \mathbf{x}^i, \qquad 0 = \sum m_i \frac{\partial}{\partial x_0^i} \tag{6.4.38}$$

From this, then, we can proceed as before to produce the relative part of the smooth asymptotic metric by a hyper-Kähler quotient. Since the two quotient operations commute, we conclude that our moduli space metric can be thought of as the hyper-Kähler quotient of $H^{n+1} \times H^n$ with respect to the $n+1$ isometries

$$\sum_{i=0}^{n} m_i \, \frac{\partial}{\partial x_0^i} \tag{6.4.39}$$

and

$$\frac{\partial}{\partial x_0^{A-1}} - \frac{\partial}{\partial x_0^{A}} + 2\frac{\partial}{\partial \chi^{A}}, \quad A = 1, 2, \ldots, n \tag{6.4.40}$$

where the χ^A are certain phases of the q^A, as in Eq. (6.4.19). In fact, the role of the first isometry is not difficult to guess. Its associated moment maps are $\sum_i m_i \mathbf{x}^i$, so the quotient due to this simply removes the center-of-mass part of the moduli space. We leave it to interested readers to verify that the quotient of $H^{n+1} \times H^n$ by R^n with only the n isometries

$$\frac{\partial}{\partial x_0^{A-1}} - \frac{\partial}{\partial x_0^{A}} + 2\frac{\partial}{\partial \chi^{A}}, \quad A = 1, 2, \ldots, n \tag{6.4.41}$$

reproduces our asymptotic form for the total moduli space metric, up to a periodic identification of one free angular coordinate. The condition that the $3n$ moment maps vanish can be written more suggestively in terms of the coordinates of $H^n \times H^{n+1}$,

$$\frac{1}{2}\, q^A i \widehat{q}^A = \mathrm{Im}(T^A - T^{A-1}) \tag{6.4.42}$$

where $\mathrm{Im}(T) \equiv (T - \widehat{T})/2$.

There is a very obvious correspondence with the Nahm data for this system, which were discussed in Sec. 4.5.4. Because we are considering a chain of $n+1$ distinct $\mathrm{SU}(n+2)$ monopoles, we need $n+1$ contiguous intervals, of lengths proportional to the m_p. Since there is only one monopole of each type, the Nahm data on the pth interval includes a triplet of functions $T_i^{(p)}(s)$ that, by the Nahm equation, are equal to a constant, x_i^p, on the interval, together with $T_0^{(p)}(s)$, which we are not assuming to have been gauged away. We can identify the former with the imaginary part of a quaternion $\tilde{T}^p$, with the real part being

$$\frac{1}{m_p} x_0^p \equiv \int ds T_0^{(p)}(s) \tag{6.4.43}$$

A natural metric for this part of the Nahm data is then

$$\sum_p \frac{1}{m_p}(dx_0^i)^2 + m_p\,(d\mathbf{x}^p \cdot d\mathbf{x}^p) = \sum_p m_i\, d\tilde{T}^p \otimes_s d\widehat{\tilde{T}}^p \tag{6.4.44}$$

In this trivial example of the ADHMN construction, the only subtle part was obtaining the jumping data at the boundaries. It is not hard to see that the matching condition of Eq. (4.5.37) is equivalent to requiring that there be quaternions $\tilde{q}^A$ such that

$$\frac{1}{2}\, \tilde{q}^A i \widehat{\tilde{q}}^A = \mathrm{Im}(\tilde{T}^A - \tilde{T}^{A-1}) \tag{6.4.45}$$

The natural metric for these is $\tilde{q}^A$ is the canonical one,

$$\sum_A d\tilde{q}^A \otimes_s d\widehat{\tilde{q}}^A \tag{6.4.46}$$

When we studied this example in Sec. 4.5.4, we worked in a gauge where the $T_0^{(0)}$ were identically zero. Had we not done so, we would have found that the gauge action of Eqs. (4.4.10) and (4.4.11) also acts on the jump data, with the effect being that the phase χ^A associated with $\tilde{q}^A$ is shifted by an amount that is determined by the transformations of the $T_0^{(p)}$ in the adjacent intervals. The invariance under this local gauge action is then equivalent to the isometry generated by

$$\frac{\partial}{\partial x_0^{A-1}} - \frac{\partial}{\partial x_0^A} + 2\frac{\partial}{\partial \chi^A}, \quad A = 1, 2, \ldots, n \tag{6.4.47}$$

The correspondence with the moduli space metric is clear. We simply drop the tildes and associate the Nahm data and the jumping data with the H^{n+1} and H^n factors, respectively. The vanishing of the moment maps is the matching condition on the Nahm data, while the division by R^n is the identification due to the gauge action on the Nahm data. With this mapping of variables, the metric derived from the Nahm data is exactly equal to the asymptotic form of the metric that we found by considering only the long-range interactions. This concludes the proof.

6.5 Massless limits and non-Abelian charges

With an exact and explicit form of the metric everywhere, we can now explore different corners of theory space. A particularly interesting limit is when some of the monopole masses vanish. While we demanded in the previous section that the reduced mass matrix μ be nondegenerate, that had to do with the coordinate choice we made in the Nahm data, such as x_0^i's. The actual geometry of the moduli space is perfectly well-behaved even if μ is degenerate.

On the other hand, the massless limit also means that the symmetry-breaking pattern is changed. When there is only one adjoint Higgs, a monopole become massless precisely when an unbroken U(1) is enhanced to an unbroken SU(2) or higher gauge group. This gives us a chance to understand magnetic solitons in theories where the unbroken gauge group is not entirely Abelian. Some examples of such solutions were given earlier, and we will study some of these here in terms of their moduli space geometry and thereby learn more general properties intrinsic to such solutions.

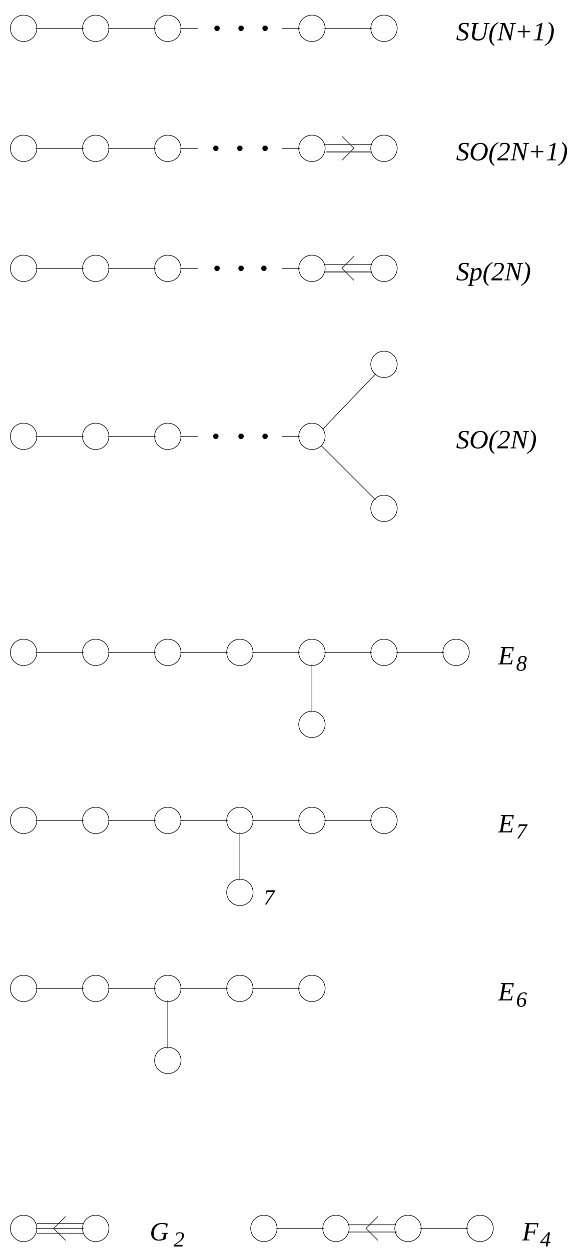

Figure 6.6: Dynkin diagrams for all simple Lie algebras. SU, even-dimensional SO, and E types are simply laced, meaning that all roots are of same lengths.

Chapter 7

Multi-Higgs vacua in SYM and multicenter dyons

Up to now, we have concentrated on the physics of monopoles and dyons when only one adjoint Higgs field acquired a vacuum expectation value. However, this turns out to be overly restrictive when studying dyons in arbitrary vacua of supersymmetric Yang-Mills theories. For the simplest gauge group with monopoles, SU(2), this restriction hardly matters, because one can always use the global R symmetry of supersymmetric Yang-Mills theory to remove all but one of the vevs. This appears to be one reason why the rich new physics of multi-Higgs vacua had been neglected for a long time.

For larger gauge groups and generic dyonic charges, this possibility is no longer available. The reason is simple: the expectation value of an adjoint Higgs field entails r mass scales, corresponding to the generators of the Cartan subalgebra. If there are two adjoint Higgs expectation values, there are $2r$ independent mass scales. On the other hand, the global R symmetry is independent of the rank of the gauge group, and so in general cannot rotate $2r$ masses into r masses. For the classification of generic dyons in a generic vacuum, we can no longer stick to the single-Higgs model.

We have already noted, in Chap. 3, that the supersymmetry condition involving both magnetic and electric charges is a bit involved, and we saw the possibility of 1/4-BPS dyons [62] in $\mathcal{N} = 4$ theories. Yet, all the dyons we have discussed so far have been 1/2-BPS in that context. In this chapter, we discuss how this discrepancy is cured in generic vacua of $\mathcal{N} = 4$ SYM, and explore the nature of 1/4-BPS solitons. In the process, we will learn that the modified BPS equations involve at most two independent adjoint Higgs fields, and are thus directly applicable to the $\mathcal{N} = 2$ case as well; the only difference is the amount of supersymmetry that is preserved. Any given 1/4-BPS soliton solution of $\mathcal{N} = 4$ SYM can be thought of as a solution to

$\mathcal{N} = 2$ SYM with the same gauge group. The supersymmetry properties of the latter are a more subtle issue, to which we will return in later chapters.

7.1 Generalized BPS equations

It turns out that when more than one adjoint Higgs field has a nonzero expectation value, while preserving the same Cartan subgroup, the BPS equations are modified in an essential way. The details of these generalized BPS equations and some examples of exact dyon solutions will be presented in the second half of this chapter. One unexpected and important characteristic of this new class of BPS solutions is that they should be really regarded as composites of two or more solitonic cores balanced against each other by long-range static forces. These static forces can be derived rigorously from the Yang-Mills-Higgs Lagrangian and are a combination of long-range Coulomb forces and forces mediated by scalar particle exchange.

7.1.1 Energy bound

We start by recalling the Lagrangian for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory that was given in Eq. (3.3.1). The correponding energy density is [64]

$$\mathcal{H} = \mathrm{Tr}\left\{E_i^2 + B_i^2 + (D_0\Phi_I)^2 + (D_i\Phi_I)^2 - \frac{e^2}{2}\sum_{I,J}[\Phi_I, \Phi_J]^2\right\} \tag{7.1.1}$$

Next, we choose two arbitrary six-dimensional unit vectors $\hat{m}_I$ and $\hat{n}_I$ that are orthogonal to each other, and decompose the scalar fields as

$$\Phi_I = b\hat{m}_I + a\hat{n}_I + \zeta_I \tag{7.1.2}$$

where ζ_I is orthogonal to both $\hat{m}_I$ and $\hat{n}_I$.

Using this decomposition, we rewrite the energy density as

$$\begin{aligned}
\mathcal{H} &= \mathrm{Tr}\Big\{\Big[B_i^2 + (D_ib)^2\Big] + \Big[E_i^2 + (D_ia)^2\Big] + \Big[(D_0b)^2 + e[a,b]^2\Big] \\
&\qquad + \sum_I \Big[(D_0\zeta_I)^2 + e[a,\zeta_I]^2\Big] + (D_0a)^2 + e^2\sum_I [b,\zeta_I]^2 \\
&\qquad + e^2\sum_{I,J}[\zeta_I,\zeta_J]^2 + \sum_I (D_i\zeta_I)^2\Big\} \\
&= \mathrm{Tr}\Big\{(B_i - D_ib)^2 + (E_i - D_ia)^2 + (D_0b - ie[a,b])^2
\end{aligned}$$

$$+\sum_I (D_0\zeta_I - ie[a,\zeta_I])^2 + (D_0 a)^2 + e^2 \sum_I [b,\zeta_I]^2$$

$$+e^2 \sum_{I,J} [\zeta_I,\zeta_J]^2 + \sum_I (D_i\zeta_I)^2$$

$$+2B_i\, D_i b + 2E_i\, D_i a + 2ie[a,b]\, D_0 b + 2ie\sum_I [a,\zeta_I]\, D_0\zeta_I \Big\} \quad (7.1.3)$$

The last four cross terms can be rewritten with the aid of the Bianchi identity, $D_i B_i = 0$ and Gauss's law,

$$D_i E_i - ie[\phi_I, D_0\phi_I] = 0 \quad (7.1.4)$$

to give

$$\begin{aligned} \mathcal{H} \;=\; \mathrm{Tr}\Big\{ &(B_i - D_i b)^2 + (E_i - D_i a)^2 + (D_0 b - ie[a,b])^2 + \sum_I (D_0\zeta_I - ie[a,\zeta_I])^2 \\ &+(D_0 a)^2 + e^2 \sum_I [b,\zeta_I]^2 + e^2 \sum_{I,J} [\zeta_I,\zeta_J]^2 + \sum_I (D_i\zeta_I)^2 \\ &+2\partial_i(B_i b) + 2\partial_i(E_i a)\Big\} \end{aligned} \quad (7.1.5)$$

Every term is nonnegative, except for the last two, which are total derivatives, so the total energy is bounded by the surface term arising from the latter:

$$\mathcal{E} = \int d^3x\, \mathcal{H} \;\geq\; \hat{n}_I Q_I^E + \hat{m}_I Q_I^M, \quad (7.1.6)$$

with

$$Q_I^E \;=\; 2\int d^3x\, \partial_i\, (\mathrm{Tr}\, \Phi_I E_i), \quad (7.1.7)$$

$$Q_I^M \;=\; 2\int d^3x\, \partial_i\, (\mathrm{Tr}\, \Phi_I B_i). \quad (7.1.8)$$

The most stringent bound is obtained by varying $\hat{n}_I$ and $\hat{m}_I$ so as to maximize the the right-hand side of Eq. (7.1.6). This requires, first of all, that $\hat{m}_I$ and $\hat{n}_I$ both lie on the plane spanned by Q_I^M and Q_I^E. Next, the directions of $\hat{m}_I$ and $\hat{n}_I$ should be chosen so that $\hat{n}_I Q_I^E$ and $\hat{m}_I Q_I^M$ are both positive. Assuming both of these conditions to hold, let θ be the angle between $\hat{m}_I$ and Q_I^M, and α the one between between Q_I^M and Q_I^E. One then finds that the right-hand side of Eq. (7.1.6) is maximized when

$$\tan\theta = \frac{Q^E|\cos\alpha|}{Q^M + Q^E \sin\alpha} \quad (7.1.9)$$

where Q^M and Q^E are the magnitudes of the vectors Q_I^M and Q_I^E. This gives the bound

$$\mathcal{E} \geq \sqrt{(Q^M)^2 + (Q^E)^2 + 2Q^M Q^E \sin\alpha}. \quad (7.1.10)$$

7.1.2 Primary and secondary BPS equations

The lower bound on $\mathcal{E}$ is saturated when the bulk terms in the energy density all vanish. From this we obtain a total of eight sets of equations. The first is the most familiar,

$$B_i = D_i b. \tag{7.1.11}$$

This is the usual Bogomolny equation, which admits magnetic monopole solutions. Note that this magnetic equation can be solved independently of the remaining equations. The other BPS equations influence only the choice of the unit vector $\hat{m}_I$. This fact is of crucial importance when we construct the BPS solution later. For this reason, we call Eq. (7.1.11) the primary BPS equation.

The other BPS equations are to be solved in the background of this purely magnetic BPS solution. They are

$$E_i = D_i a, \tag{7.1.12}$$

$$D_0 b = -ie[b, a] \tag{7.1.13}$$

$$D_0 \zeta_I = -ie[\zeta_I, a] \tag{7.1.14}$$

$$D_0 a = 0 \tag{7.1.15}$$

and

$$[b, \zeta_I] = 0, \tag{7.1.16}$$

$$[\zeta_I, \zeta_J] = 0, \tag{7.1.17}$$

$$D_i \zeta_I = 0, \tag{7.1.18}$$

In addition, we must impose Gauss's law

$$D_i E_i = ie\left([b, D_0 b] + [a, D_0 a] + [\zeta_I, D_0 \zeta_I]\right). \tag{7.1.19}$$

Inserting Eqs. (7.1.12) – (7.1.15) in Gauss's law gives a linear equation for a,

$$D_i D_i\, a = e^2\, [b, [b, a]] + e^2 [\zeta_I, [\zeta_I, a]]. \tag{7.1.20}$$

Matters can be simplified further by writing the solution to the primary equation in a form where the nontrivial fields occupy irreducible blocks, and working in the unitary, or "string", gauge where b is diagonal and time-independent. With this gauge choice, $\dot{A}_i$ is also zero and Eq. (7.1.12) is solved by

$$A_0 = -a \tag{7.1.21}$$

while $D_0\zeta_I - ie\,[a, \zeta_I] = \partial_0\zeta_I = 0$ requires that ζ_I also be time-independent. In the background of a generic monopole solution, the last three equations, (7.1.16), (7.1.17), and (7.1.18), imply that ζ_I is a constant times the identity in each of the irreducible blocks occupied by the monopole solution.[1]

Now Eq. (7.1.20) is a zero-eigenvalue problem for a nonnegative operator acting linearly on a. In order to have the bosonic potential vanish at infinity, $a(\infty)$ must commute with $b(\infty)$ and $\zeta_I(\infty)$. Furthermore, the actual solution can have nontrivial behavior only inside each of the irreducible blocks, defined by b, where the ζ_I are just numbers times the identity matrix. Thus the ζ_I must commute with a everywhere and the last term in Eq. (7.1.20) drops out, yielding [63][64]

$$D_i D_i\, a = e^2\,[b, [b, a]] \tag{7.1.22}$$

which we call the secondary BPS equation.

Finally, recall that in Sec. 3.3 we showed that a 1/4-BPS solution was obtained by requiring that all but two scalar fields vanish and that the remaining two satisfy Eq. (3.3.12). These requirements are identical to[2] Eqs. (7.1.11) – (7.1.18), thus verifying that solutions obeying the primary and secondary BPS equations are indeed 1/4-BPS.

7.1.3 Multicenter dyons are generic

Now that we have generalized the BPS equations, let us characterize the solutions. We saw above that the BPS equations split into two groups, one involving the original Bogomolny equation for the magnetic sector, and the other leading to the second-order Eq. (7.1.22) to be solved in the background of a purely magnetic solution to the first. Because of this, the solutions are parameterized by the same monopole moduli space. The new story is that for any given BPS monopole solution the electric sector is uniquely determined, because the solution to the second-order equation is completely fixed by the Higgs expectation values and the moduli coordinates that characterize the BPS monopole.

A somewhat unexpected consequence of this result is that, if we fix the asymptotic Higgs field and the electric charge, the relative positions of the monopole cores are

[1] In the language of string web, to be discussed in chapter 11, this translates to the requirement that the string web be planar.

[2] We could have obtained, instead, the equivalent of Eq. (3.3.9) if we had made a different choice of sign when completing the squares in Eq. (7.1.3). In this case, we would have found that the most stringent energy bound was obtained by requiring $\hat{n}_I Q_I^E$ to be negative, and so would have been led to the same solutions, but a redefined in such a way that its sign was reversed.

constrained and generically lead to a collection of well-separated dyonic cores [64]. Unlike the case with only one nontrivial Higgs field, these cores cannot be moved freely relative to one another, unless we also change the electric charge or the Higgs vevs. In the second half of this chapter we will study this odd behavior in more detail, and find that there is really nothing mysterious about it; it is simply a result of classical forces generated by the Yang-Mills-Higgs system on these solitonic objects.

To illustrate the general structure of these solutions, it is instructive to consider the secondary BPS equation (7.1.22) when we have a single fundamental monopole. Since the latter is an embedded SU(2) monopole solution, we have

$$D^2_{\mathrm{SU}(2)}\, a = e^2\, [\Phi_{\mathrm{SU}(2)}, [\Phi_{\mathrm{SU}(2)}, a]]. \tag{7.1.23}$$

For this somewhat degenerate case, there is really only one solution for a, which can be written as

$$a = c\, \Phi_{\mathrm{SU}(2)} + \mathrm{constant} \tag{7.1.24}$$

where c is an integration constant and the last term must commute with the magnetic part of the solution everywhere. Thus, we also have

$$E_i = cD_i\Phi_{\mathrm{SU}(2)} = cB_i^{\mathrm{SU}(2)} \tag{7.1.25}$$

Note that the electric field is proportional to the magnetic field.

For a collection of well-separated fundamental monopoles, this form of the solution is a good approximation near each of the monopole cores. Thus, turning on the vacuum expectation value $\langle a\rangle$ endows each core with an electric charge in the corresponding SU(2) subgroup. The amount of electric charge is determined by $\langle a\rangle$ and by the particulars of the magnetic solution. Since the general magnetic solution to the primary BPS equation consists of separated fundamental monopoles, the generic dyonic solution in a supersymmetric Yang-Mills theory looks like a collection of many embedded SU(2) dyons whose relative positions are determined by the balance between various long-range forces. An explicit solution involving two such dyonic cores in SU(3) gauge theory can be found in Ref.[].

7.2 Additional Higgs expectation value as a perturbation

Understanding this new breed of solution becomes a little easier, however, when we approach these solutions from a different perspective. In this second half of the chapter we will try to construct these dyons as classical bound states of monopoles, or equivalently as static orbits in the moduli space of monopoles. This latter viewpoint is possible if the second Higgs expectation value is much smaller than the first, in a sense to be precisely outlined below, so that the perturbation due to the second Higgs generates an attractive bosonic potential energy between the monopole cores. In this language the electric charge behaves as an angular momentum and generates a repulsive angular momentum barrier. The resulting BPS dyons are then obtained via the balance between the potential energy and the angular momentum barrier.

7.2.1 Static forces on monopoles

We start with a Yang-Mills theory with a single adjoint Higgs and solve its Bogomolny equation,

$$B_i = D_i \Phi \tag{7.2.1}$$

We then begin to turn on additional adjoint Higgs fields a_I; i.e., we let these additional Higgs fields acquire their own nonzero expectation values. Because of the quartic commutator term in the Lagrangian, the vacuum condition on the $\langle a_I \rangle$ requires that they commute with the expectation value of Φ. With an SU(2) gauge group, this uniquely fixes the direction of the vev's, which then allows one to use a global R-symmetry to remove all but one vacuum expectation value. This is no longer true for gauge groups of rank ≥ 2.

As we have seen, the monopole solutions $(\bar{A}_a, \bar{\Phi})$ of the magnetic BPS equation are not, in general, solutions to the full field equations when the expectation values $\langle a_I \rangle$ are turned on [65]. As a result, the monopoles exert static forces on each other. For sufficiently small $\langle a_I \rangle$, we should be able to treat these forces as arising from an extra potential energy due to the nontrivial a_I fields in the background of the monopole solution. In other words, when $\langle a_I \rangle \neq 0$, the monopole background induces a nontrivial behavior in the a_I that "dresses" the monopoles and contributes to the energy of the system in a manner that depends on which monopole solution was used for the background.

Let us parameterize the size of the additional Higgs expectation values by a dimensionless number

$$\eta \sim |\langle a_I \rangle| / |\langle \Phi \rangle|. \tag{7.2.2}$$

To find the effect to leading order in η, we imagine a static configuration of monopoles that satisfies the Bogomolny equation. Let us try to dress this configuration with a time-independent a_I field, at the smallest possible cost in energy. The strategy is a two-step process. First, we find the minimum energy due to this additional Higgs vev for a given monopole configuration, and then incorporate it into the low-energy monopole dynamics. Second, we solve this modified dynamics to find out how the monopoles react to the additional Higgs vev.

With some hindsight we will call this new interaction energy $-\mathcal{V}$, for it will prove to be a potential energy term. $\mathcal{V}$ is obtained by using the a_I field equations to minimize [67]

$$\Delta E = \int d^3x \ \mathrm{Tr} \left\{ (\bar{D}_j a_I)^2 - e^2 \left([a_I, \bar{\Phi}\,] \right)^2 \right\}, \tag{7.2.3}$$

(We will ignore higher-order terms, such as $[\, a_I, a_J]^2 \sim \eta^2$.) Thus, we solve

$$\bar{D}_j^2 a_I - e^2[\bar{\Phi}, [\bar{\Phi}, a_I]] = 0. \tag{7.2.4}$$

and insert the result back into ΔE to obtain the minimum energy needed to maintain the monopole configuration in the presence of the $\langle a_I \rangle$.

A crucial point to note here is that the equation for the a_I is identical to the equation for the gauge zero modes [66]. The gauge zero modes are always of the form[3] $\delta A_a = -\bar{D}_a \epsilon$ and must obey the background gauge condition. It follows that the gauge function ϵ must satisfy

$$\bar{D}^2 \epsilon - e^2[\bar{\Phi}, [\bar{\Phi}, \epsilon]] = 0 \tag{7.2.5}$$

We notice that the $\bar{D}_a a_I$ have exactly the same form as the global gauge zero modes, $\delta A_a = D_a \epsilon$, with the gauge functions $\epsilon = a_I$. Thus, it must be true that we can express the $D_a a_I$ as linear combinations of gauge zero modes. Consequently, each a_I picks out a linear combination

$$K_A^r \frac{\partial}{\partial z_r} = \frac{\partial}{\partial \xi_A} \tag{7.2.6}$$

of U(1) Killing vector fields on the moduli space. More precisely, each K_A corresponds to a gauge zero mode

$$K_A^r \delta_r A_\mu, \tag{7.2.7}$$

and each $D_a a_I$ is a linear combination of these,

$$D_a a_I = a_I^A K_A^r \delta_r A_a, \tag{7.2.8}$$

where we have expanded the Cartan-valued vev as

$$\mathbf{a}_I = \sum_A a_I^A \boldsymbol{\lambda}_A \tag{7.2.9}$$

[3]We are using here the four-dimensional Euclidean notation in which a runs from 1 to 4, with $\delta A_4 = \delta\Phi$.

with the $\boldsymbol{\lambda}_A$ being the fundamental weights, which obey $\boldsymbol{\lambda}_A \cdot \boldsymbol{\beta}_B = \delta_{AB}$.

We then express the potential energy $\mathcal{V}$, obtained by minimizing the functional ΔE in Eq. (7.2.3) in the monopole background, in terms of the monopole moduli parameters [68] as

$$\mathcal{V} = \int d^3x \, \mathrm{Tr} \, \left\{ (a_I^A K_A^r \delta_r A_a)(a_I^B K_B^s \delta_s A_a) \right\} = \frac{1}{2} g_{rs} \, a_I^A K_A^r a_I^B K_B^s. \tag{7.2.10}$$

The value of this potential energy depends on the monopole configuration we started with. The low-energy effective Lagrangian, which was purely kinetic when the a_I were absent, picks up a potential energy term that lifts some of the moduli, and takes the form

$$\mathcal{L} = \frac{1}{2} g_{rs} \dot{z}^r \dot{z}^s - \mathcal{V}(z) \tag{7.2.11}$$

In the current approximation, where the additional Higgs fields are treated as perturbations, the mass scale introduced by the potential energy is much smaller than that of the charged vector mesons, and we can still consistently truncate to this moduli space mechanics. The procedure we employed here should be a very familiar one. When we talk about, say, the Coulombic interactions among a set of charged particles, we also fix the charge distribution by hand, and then estimate the potential energy that it costs. Using this potential energy, we then find how the charged particles interact at slow speed.

Of course, there is the possibility of interaction terms involving the moduli velocities as well as the a_I fields, but in the low-energy approximation used here the only relevant such terms would be of order $v\eta$. However, it is clear that neither the back-reaction of the a_I on the magnetic background nor the time-dependence of the a_I can produce such a term. Thus, to the leading quadratic order, the Lagrangian of Eq. (7.2.11) captures all of the bosonic interactions among the monopoles in the presence of the $\mathbf{a}_I$.

A special solution to the a_I equation deserves further attention. Since

$$\nabla^2 \bar{\Phi} - e^2 [\bar{\Phi}, [\bar{\Phi}, \bar{\Phi}]] = \nabla^2 \bar{\Phi} = \nabla \cdot \bar{B} = 0 \tag{7.2.12}$$

one can always separate from a_I the part proportional to $\bar{\Phi}$:

$$a_I = c_I \bar{\Phi} + \Delta a_I \tag{7.2.13}$$

by requiring $\mathrm{Tr} \left(\langle \Delta a_I \rangle \langle \bar{\Phi} \rangle \right) = 0$. The U(1) Killing vector associated with the gauge function $\epsilon = \bar{\Phi}$, on the other hand, is the free U(1) angle that is one of the center-of-mass degrees of freedom. (This will be seen clearly when we discuss the classical dyon solutions in the next section.) The square of this Killing vector is independent of the moduli, and the potential energy term in question simply adds a positive constant to the energy of the system.

This "extra" energy can be easily understood by going back to the field theory and reanalyzing the BPS equation. As was mentioned at the beginning of the section, an a_I vacuum expectation value proportional to that of Φ can be rotated away by a redefinition of Φ. Once this is done, we can make the replacement

$$\begin{aligned} \bar{\Phi} &\rightarrow \left(1+\sum_I c_I^2\right)^{1/2} \bar{\Phi} \\ a_I &\rightarrow \Delta a_I \end{aligned} \tag{7.2.14}$$

Expanding the mass formula in terms of the small c_I, we get back the constant energy terms $\sim c_I^2/2$. Thus, we could have started with these rotated Higgs fields and regarded the Δa_I, instead of the a_I, as the perturbation. The potential energy would then be generated entirely by the Δa_I, and there would be no constant energy shift from the center-of-mass part of the moduli space. For this reason, the part of the a_I proportional to $\bar{\Phi}$ will be ignored for most of this review.

7.2.2 Dyonic bound states as classical orbits

In the classical moduli space approximation bound dyons appear as closed, stationary orbits along U(1) phase angles. Let us consider now the effect that adding a potential energy term[4] $\mathcal{V}$ has on the existence of such closed orbits [69]. It is immediately clear that one will generically find many more closed orbits in the presence of $\mathcal{V}$ than otherwise. For example, if one considers the case of n distinct monopoles, it can be shown rigorously that no closed orbits are possible in the absence of such a potential energy. The existence of a potential energy will, understandably, change this completely.

As a special case, let us take a pair of distinct monopoles in a theory with SU(3) broken to U(1) × U(1). Before turning on the additional Higgs fields, the purely kinetic interaction Lagrangian of the pair can be distilled down to

$$L_0 = \frac{1}{2}\left(1+\frac{1}{r}\right)\dot{\mathbf{r}}^2 + \frac{1}{2}\left(1+\frac{1}{r}\right)^{-1}\left(\dot{\psi}+\mathbf{w}(\mathbf{r})\cdot\dot{\mathbf{r}}\right)^2 \tag{7.2.15}$$

[4]As will become clear in Chap. 9, dyons such as these classical monopole bound states can become BPS only if only one such bosonic potential energy is turned on; i.e., only one of the a_I can be excited (up to an orthogonal transformation among the a_I). This corresponds to having only two adjoint Higgs fields participating in the low-energy dynamics and, in the language of the classical BPS equations of Sec. 7.1, corresponds to the decoupling of the ζ_I. This motivates removing all but one of the a_I.

where, for the sake of simplicity, we have taken the Taub-NUT metric of Eq. (6.3.18) and transformed to dimensionless variables defined by the rescalings

$$\mathbf{r} \to \frac{2\pi \mathbf{r}}{e^2\mu}, \qquad t \to \frac{(2\pi)^2 t}{e^4\mu} \tag{7.2.16}$$

A dyonic state with an integer-quantized electric charge q would be governed by the Routhian

$$R_0 = \frac{1}{2}\left(1+\frac{1}{r}\right)\dot{\mathbf{r}}^2 - \frac{q^2}{2}\left(1+\frac{1}{r}\right) + q\mathbf{w}(\mathbf{r})\cdot\dot{\mathbf{r}} \tag{7.2.17}$$

This has three interaction terms: one that modifies the inertia as a function of the separation r; a repulsive potential energy; and a velocity-dependent coupling that generates a Lorentz force, due to a unit monopole sitting at the origin $\mathbf{r} = 0$, on a particle of charge q. Despite the various interaction terms, the conserved energy takes the simple form

$$E = \frac{1}{2}\left(1+\frac{1}{r}\right)\left(\mathbf{v}^2 + q^2\right) \tag{7.2.18}$$

From the form of the effective potential energy, which is monotonically decreasing toward $r = \infty$, it is fairly clear that no bound orbit is possible with this classical dynamics.

A more complete characterization of the classical trajectories is possible if we utilize an additional conserved quantity. The conserved angular momenta has the familiar form

$$\mathbf{J} = \left(1+\frac{1}{r}\right)\mathbf{r}\times\mathbf{v} + q\hat{\mathbf{r}} \tag{7.2.19}$$

with the last term being characteristic of charged particles in a monopole background. This severely restricts the possible trajectories because

$$\mathbf{J}\cdot\hat{\mathbf{r}} = q \tag{7.2.20}$$

is also a conserved quantity. This says that the trajectories lie along a cone going through the origin $\mathbf{r} = 0$, with an opening angle $\cos^{-1}(q/J)$ around $\mathbf{J}$. Also note the inequality

$$J^2 - q^2 \geq 0 \tag{7.2.21}$$

which is saturated only when the cone collapse to a line.

One more conserved vector is known to exist. It is of the Runge-Lenz type [69],

$$\mathbf{K} = \left(1+\frac{1}{r}\right)\mathbf{v}\times\mathbf{J} - (E - q^2)\hat{\mathbf{r}} \tag{7.2.22}$$

and gives us another conserved inner product,

$$[q\mathbf{K} + (E - q^2)\mathbf{J}]\cdot\mathbf{r} = q(J^2 - q^2) \tag{7.2.23}$$

because the linear combination

$$\mathbf{N} \equiv q\mathbf{K} + (E - q^2)\mathbf{J} \tag{7.2.24}$$

of the two conserved vectors is also conserved. Thus, the trajectories also must lie on a plane which is orthogonal to $\mathbf{N}$ and displaced from the origin by

$$\Delta\mathbf{r} = \frac{q(J^2 - q^2)}{N^2}\mathbf{N}. \tag{7.2.25}$$

Combined with the previous result, this shows that the trajectories are always conic sections.

Now let us consider what happens when we turn on a second Higgs field as a perturbation. The only U(1) Killing vector in the Taub-NUT manifold is ∂_ψ, and the effect of turning on a small, second Higgs expectation value a should show up as a potential energy term. Since we are considering a pair of distinct monopoles, the relevant part of the unbroken gauge group is U(1) × U(1), with one factor acting on the center-of-mass part. Because of this, there is only one independent component in the expectation value $\langle \Delta a_1 \rangle$ that generates a nontrivial potential energy term. We denote this value by a. If we introduce the dimensionless combination $\tilde{a}$

$$\tilde{a} \equiv \frac{(2\pi)^2 a}{e^3 \mu} \tag{7.2.26}$$

the additional Higgs expectation value leads to a potential energy term

$$\mathcal{V} = \frac{1}{2}\tilde{a}^2 \left\langle \frac{\partial}{\partial\psi}, \frac{\partial}{\partial\psi} \right\rangle = \frac{1}{2}\frac{\tilde{a}^2}{1 + 1/r} \tag{7.2.27}$$

A remarkable fact is that the dynamics with this potential,

$$L = \frac{1}{2}\left(1 + \frac{1}{r}\right)\dot{\mathbf{r}}^2 + \frac{1}{2}\left(1 + \frac{1}{r}\right)^{-1}\left(\dot{\psi} + \mathbf{w}(\mathbf{r}) \cdot \dot{\mathbf{r}}\right)^2 - \mathcal{V} \tag{7.2.28}$$

admits exactly the same forms for the conserved vectors $\mathbf{J}$ and $\mathbf{K}$ (and thus also for their linear combination $\mathbf{N}$), provided that we keep in mind that the conserved energy

$$E = \frac{1}{2}\left(1 + \frac{1}{r}\right)\left(\mathbf{v}^2 + q^2\right) + \frac{1}{2}\left(1 + \frac{1}{r}\right)^{-1}\tilde{a}^2 \tag{7.2.29}$$

gets an additional contribution from the potential energy. Thus, after we take into account the additional Higgs field, all trajectories are still conic sections.

Of the five kinds of conic sections, only circles and ellipses correspond to bound trajectories. The condition for a closed trajectory is then expressible in terms of the angle between **N** and **J** in the following manner.[5] Given the angular momenta **J**, the cone encloses **J** with an opening angle $0 \le \alpha = \cos^{-1} q/J \le \pi$. Let β be the angle between **J** and **N**. From the explicit form of the conserved vectors, it is a matter of straightforward computation to show that

$$\cos\beta = \frac{\sqrt{J^2 - q^2}}{J} \times \frac{E - q^2}{\sqrt{E^2 - \tilde{a}^2 q^2}} \tag{7.2.30}$$

while

$$\cos(\pi/2 - \alpha) = \sin\alpha = \frac{\sqrt{J^2 - q^2}}{J} \tag{7.2.31}$$

In addition to the inequality $J^2 \ge q^2$, the fact that $\mathbf{N}^2 \ge 0$ gives another constraint,

$$E \ge |\tilde{a}q| \tag{7.2.32}$$

For the sake of simplicity, we will assume that $q \ge 0$ so that $\alpha < \pi/2$. Then, the trajectory will be an ellipse (or a circle) if $\alpha + \beta$ is smaller than $\pi/2$, a parabola if $\alpha + \beta = \pi/2$, and a hyperbola if $\alpha + \beta$ is larger than $\pi/2$. Hence, the trajectory is bound and closed if and only if the ratio

$$\frac{\cos\beta}{\cos(\pi/2 - \alpha)} = \frac{E - q^2}{\sqrt{E^2 - \tilde{a}^2 q^2}} \tag{7.2.33}$$

is strictly larger than 1. The behavior of this ratio in the physical region $E \ge \tilde{a}q$ is qualitatively different depending on whether $a > q$ or $a < q$, since, as a function of E, this has a single extremum at $E = a^2$. Defining a critical energy as

$$E_c \equiv \frac{\tilde{a}^2 + q^2}{2} \tag{7.2.34}$$

we may classify the orbits for positive q as follows:

- $q < |\tilde{a}|$ and $E < E_c$: The trajectory is an ellipse or a circle and corresponds to a bound and closed orbit.
- $q \le |\tilde{a}|$ and $E = E_c$: The trajectory is a parabola.
- Otherwise : The trajectory is a hyperbola.

For negative q, the same statements hold with q replaced by $|q|$.

One simple corollary is that if the potential energy term $\sim \tilde{a}^2$ is absent, no bound orbit at all is possible in this two-body problem. This last statement also holds in the many-body problem with all distinct monopoles, as was shown by Gibbons. Without the potential energy, all classical orbits are hyperbolic.

[5] We thank Choonkyu Lee for useful conversations on this classical dynamics.

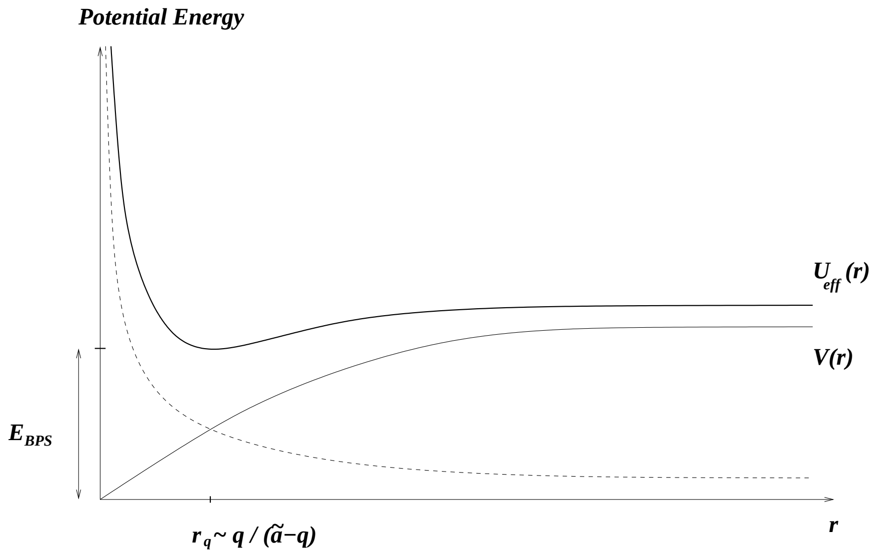

Figure 7.1: Potential energies between a pair of distinct monopoles as a function of separation. The solid line is the potential energy $\mathcal{V}$ between a pair of bare monopoles, while the dotted line is an angular momentum barrier generated by assigning a relative electric charge q. The thick line is the effective potential energy $U_{\rm eff}$ between such a pair of dyons, which has a minimum at a separation $r \sim q/(\tilde{a}-q)$, with the excitation energy saturating the BPS bound.

7.2.3 Static multicenter dyons and balance of forces

An interesting limiting case of a bound orbit is found when the cone collapse to a line, so that $\alpha = 0$ with positive q, in which case the entire angular momentum come from the $q\hat{r}$ piece. In this case the energy must saturate its lower bound, $E = aq$, and the "orbit" is simply a stationary point at a fixed distance. With the two monopoles as above, this static configuration is easy to understand. The effective potential energy in the charge q sector is

$$\frac{q^2}{2}\left(1+\frac{1}{r}\right)+\mathcal{V} = \frac{q^2}{2}\left(1+\frac{1}{r}\right)+\frac{\tilde{a}^2}{2}\left(1+\frac{1}{r}\right)^{-1} \tag{7.2.35}$$

which, for $\tilde{a} > q$, has a global minimum at

$$r = \frac{q}{\tilde{a}-q} \tag{7.2.36}$$

with the minimum energy being $E = \tilde{a}q$. The contribution from the charge q to the effective potential energy behaves exactly like an angular momentum barrier that balances against the attractive potential energy $\mathcal{V}$.

Restoring the physical units is easy; we only need to reverse the rescaling performed above, so that Eq. (7.2.36) becomes

$$\frac{e^2}{2\pi} r = \frac{q}{4\pi^2 a/e^3 - \mu q} \tag{7.2.37}$$

Because the time must be also rescaled back, the physical energy receives an additional multiplicative factor and becomes

$$E = \frac{e^4 \mu}{(2\pi)^2} \tilde{a} q = eaq \tag{7.2.38}$$

For a larger collection of distinct monopoles, the general form of the effective potential energy in physical units is

$$U_{\text{eff}} = \frac{1}{2} \left(\frac{4\pi^2}{e^2} \sum_{A,B} (C^{-1})_{AB}\, \lambda_A a^A\, \lambda_B a^B + \frac{e^4}{4\pi^2} \sum_{A,B} C_{AB}\, \frac{q_A}{\lambda_A} \frac{q_B}{\lambda_B} \right) \tag{7.2.39}$$

with a_A defined as in Eq. (7.2.9). C_{AB} is the matrix in Eq. (6.4.5) that characterized the relative moduli space metric, while $\lambda_A = -2\boldsymbol{\beta}^*_A \cdot \boldsymbol{\beta}^*_{A+1}$ is the dimensionless number that encodes the strength of the interaction between adjacent pair of monopoles. [Recall that $\lambda_A = 1$ for a collection of distinct SU(n) monopoles.] The static, minimum energy configuration is found when

$$\frac{e^3}{4\pi^2} C_{AB} \frac{q_B}{\lambda_B} = \lambda_A a^A \tag{7.2.40}$$

The solution is

$$\frac{e^2}{2\pi} r_A = \frac{q_A}{4\pi^2 \lambda_A a^A/e^3 - \sum_B \mu_{AB}\, q_B/\lambda_B} \tag{7.2.41}$$

We thus find a static dyon solution involving interacting cores separated by finite distances. In particular, the distance r_A becomes infinite as a_A approach the critical value

$$a^c_A = \frac{e^3 \mu}{4\pi^2} \sum_B \frac{\mu_{AB}}{\lambda_A \lambda_B}\, q_B \tag{7.2.42}$$

Note that, although the distances are thus fixed, there are some moduli that remain massless. For instance, the distance between the first and second dyon cores is fixed, as is the distance between the second and third, but the distance between the first and third is not. When a finite size bound state of this type exists, its energy is

$$E = e \sum_A a^A\, q_A \tag{7.2.43}$$

regardless of the λ_A.

Such a dyonic configuration, with more than one soliton core balanced against each other at fixed separations, is quite typical of dyons that preserve at most four supersymmetries. In Chap. 10, we will study the quantum counterparts of these dyons by realizing dyons as quantum bound states of monopoles. To do this, we must first derive the most general monopole moduli space dynamics, which will be the subject of the next chapter. As we saw above, the existence of more than one adjoint Higgs field changes the traditional moduli space dynamics by adding a potential energy term. The main objective of next chapter is to determine how this modifies the complete moduli space dynamics, with fermionic contributions included.

Chapter 10

BPS dyons as quantum bound states

The low-energy dynamics has proven quite useful in exploring the supersymmetric spectrum of charged particles. The earliest such effort can be found with $\mathcal{N} = 4$ SU(2) theory broken to U(1). The celebrated Montonen-Olive duality [84] of $\mathcal{N} = 4$ Yang-Mills theories requires a BPS spectrum that does not prefer the electric or the magnetic charges, while its $\mathrm{SL}(2, Z)$ extension [86] demands that dyons with p units of magnetic charge and q units of electric charge should always exist as a single vector supermultiplet whenever p and q are a pair of co-prime integers.

While $\mathcal{N} = 4$ supersymmetry tends to make the spectrum relatively simple and easy to determine, explicit constructions of such states are rarely trivial, and become cumbersome or even effectively impossible with larger charges. For $\mathcal{N} = 4$ with its maximal supersymmetry, there are further tools we can use. Some such theories can be constructed easily as theory of D3-branes in type IIB string theory. In this approach, the $\mathrm{SL}(2, Z)$ of $\mathcal{N} = 4$ SYM follows from the $\mathrm{SL}(2, Z)$ duality of the type IIB theory, so once the latter is accepted as a fact an $\mathrm{SL}(2, Z)$-invariant spectrum is automatic. Perhaps a more conservative way to view this is that one can use the $\mathrm{SL}(2, Z)$ invariance found in $\mathcal{N} = 4$ SYM as strong supporting evidence for the $\mathrm{SL}(2, Z)$ invariance of type IIB theory. Either way, the point is that the stringy construction allows an easy generalization to a large class of gauge groups and provides an easy pictorial hints to novel BPS states.

A case in point is the 1/4-BPS dyons[64], whose existence was first realized by a construction in type IIB theory [62]. There, these states are constructed by having a supersymmetric web of fundamental strings and D-strings [136][137], with ends on three or more D3-branes. The conventional low-energy dynamics of monopoles is based on models with a single adjoint Higgs, and this necessarily excludes all such-

1/4 BPS states. This in part explains why these objects were ignored for a relatively long period. Only if we begin to turn on the second adjoint Higgs, the field theory can possess such dyons. Thus, the complete low-energy dynamics with potential terms that we presented above is an essential tool for understanding these BPS states from the field theory side.

But at the same time, this is not to say that type IIB theory is more powerful in counting and isolating precise BPS spectra. It is important to remember that the correspondence between the two is not at classical level but at quantum level. A string web of type IIB will correspond to a 1/4 BPS dyons at quantum level. Just we we must quantize moduli space dynamics on the field theory side, the string web must also be quantized. In particular, the moduli space of the latter has little to do with that of the former, and in fact more difficult to quantize. Thus, for some of more precise and specific question on spectrum, the field theory side could give us better control. In the later part of this chapter we will present existence proof and degeneracy counting of some of simpler 1/4 BPS dyons.

In the realm of $\mathcal{N}=2$ Yang-Mills theories, both sides tend to be far more difficult to handle. From string theory side, the construction of the gauge theories are diverse but in all cases finding the corresponding BPS spectrum is quite nontrivial. In the elegant formulation of Seiberg-Witten theory [98][99] as a theory of wrapped M5-branes in M-theory [101], we know how to realize BPS dyons as open membranes yet establishing existence of a given dyon is all but impossible except at particular points of the moduli space [102][103]. From the field theory side also, the constraints [94][**?**] coming from Seiberg-Witten description and the S-duality are difficult to analyze beyond the simple rank 1 case of SU(2) theories [96]. The main culprit is the extremely interesting phenomena that BPS spectrum may change as we change the vacuum of the theory along the Coulomb phase [99][93]. Understanding the spectrum in this approach requires understanding the latter phenomena everywhere on the Seiberg-Witten vacuum moduli space.

However, even if one managed to understand the structure of marginal stability domain walls completely and explicitly, this would be only a beginning of the problem. The reason is that this approach is basically a bootstrap where one tries to find a solution to a set of consistency condition, which often requires additional input such as complete spectra at some limit, for instance the weak coupling limit. A good news is that the semiclassical approach involving moduli space description remains more or less manageable [86][91][92][44], and does not get worse significantly compared to $\mathcal{N}=4$ cases.

In the following, we start with some generalities on BPS spectrum of $\mathcal{N} = 4$ theories, and the associated moduli space problems. But for the most part, we will consider problems of finding and counting BPS states that preserves 4 supercharges. These are 1/4 BPS dyons in $\mathcal{N} = 4$ theories, and generic BPS dyons in $\mathcal{N} = 2$ dyons. For $\mathcal{N} = 4$ theories, counting of 1/2 BPS dyons will be implicitly included here, as they will prove to be subset of the bound states in question with more symmetry than usual.

10.1 $\mathcal{N} = 4$ SYM and electromagnetic duality

Montonen and Olive [84] offered a bold conjecture in early 80's that there may be no fundamental distinction between magnetic monopoles and elementary massive charged particles in spontaneously broken SU(2) Yang-Mills field theories: that one may be able to reformulate the gauge theory in question starting with unbroken U(1) gauge field and the magnetically charged particles and produced the electrically charged particles as solitons. This idea was further sharpened by Osborn [85] who showed such a 1-1 map might be possible at least in the context of $\mathcal{N} = 4$ supersymmetry.

When a theory can be formulated in terms of different sets of fundamental degrees of freedom, we say that there is a duality. Through study of string theory and supersymmetric Yang-Mills theories, physicists have encountered various form of dualities. In case of $\mathcal{N} = 4$ SYM, the duality takes a particularly simple form in that, regardless of which degrees of freedom are chosen to formulate the theory, the general form of the theory remains unchanged. It is always in the form of $\mathcal{N} = 4$ SYM. In section **Erick's part**, we have seen a Z_2 duality proposed by Montonen and Olive. In recent years, this has been shown to be part of much larger duality associated with the infinite discrete group $\mathrm{SL}(2, Z)$. In this section, we will review this $\mathrm{SL}(2, Z)$ duality of $\mathcal{N} = 4$ Yang-Mills theories with emphasis on the BPS spectra.

10.1.1 Supermultiplet structure of a BPS monopole

Let us take the simplest case of $\mathcal{N} = 4$ SU(2) theory. The theory is equipped with three vector multiplets spanning the adjoint representation of SU(2). When massless, each vector multiplet can be regarded as a collection of one massless vector field, a pair of Dirac spinors, and six real scalars. So the actual degrees of freedom associated with each vector multiplet is thus 16. We are studying this theory when an adjoint Higgs field acquire a vacuum expectation value, breaking the gauge symmetry to the

Cartan subgroup U(1). Then of three vector multiplets, only one remain massless while the other two form a massive complex vector multiplet. With respect to the unbroken U(1), the latter is charged with unit electric charge.

This massive and charged vector multiplet is then composed of the following set of representations under the spatial rotation group $SU(2)_L$; a single spin 1 multiplet, four spin 1/2 multiplet, and five spin 0 states. One scalar degrees of freedom is eaten up by the massless vector, by which the latter become massive. The total number of degrees of freedom remain unchanged by the Higgs mechanism of course. Such vector multiplet that became massive and electrically charged with respect to the unbroken U(1) are sometimes referred to as "charged vector meson."

Osborn's simple observation [85] was that a unit magnetic monopole comes with supermultiplet structure as this massive charged vector multiplet, only in $\mathcal{N}=4$ supersymmetric theories. In this section, we will review this fact, for it entails one of most basic and important idea in low-energy dynamics of solitons. For a single monopole, the moduli space consists of those coming from Goldstone modes. That is, on the bosonic side, three translational modes come from that fact that soliton itself breaks the translational symmetry, while a single gauge zero mode comes from the fact that monopole core actually rotates under U(1). On the fermionic side, one started with 8 complex supersymmetry generators, half of which is broken by the soliton.

This set of zero modes are universal features of the moduli space approximation, since they arise as a consequence of global symmetries, namely translation along R^3, global gauge rotation of the unbroken gauge group, and supersymmetries. The soliton breaks these symmetries spontaneously, and as a result massless Goldstone bosons appear as a degree of freedom living on the soliton. For this reason, we sometimes refer to these 4 coordinates $\mathbf{x}$ and ξ and their superpartners the Goldstone zero modes. The Lagrangian involving these basic moduli was written down in section **Erick's**

$$L=\frac{1}{2}M\dot{\mathbf{x}}^2+\frac{1}{2}\frac{(4\pi)^2}{e^4M}\dot{\xi}^2+i\psi^*_\alpha\dot{\psi}_\alpha \tag{10.1.1}$$

with the complex Goldstone fermions labelled by $\alpha=1,2,3,4$. M is the mass of the monopole. The Hamiltonian is then

$$H=\frac{\mathbf{p}^2}{2M}+\frac{e^4M}{2(4\pi)^2}q^2 \tag{10.1.2}$$

with the physical momenta $\mathbf{p}$ of the monopole and the integer-quantized electric charge q. The important point is that the free fermions never enter the Hamiltonian and thus do not contribute to the energy.

This is not to say that the fermions does not affect the physics. The canonical commutation relation says that

$$\{\psi^*_\alpha, \psi_\beta\} = \delta_{\alpha\beta} \tag{10.1.3}$$

so that each complex fermion appears as harmonic oscillator operators. Each fermionic harmonic oscillator generates only two states, "up" state $|\uparrow\rangle$ and "down" state $|\downarrow\rangle$, such that

$$\begin{aligned} \psi\,|\uparrow\rangle &= |\downarrow\rangle \\ \psi^*|\downarrow\rangle &= |\uparrow\rangle \end{aligned} \tag{10.1.4}$$

and

$$\begin{aligned} \psi\,|\downarrow\rangle &= 0 \\ \psi^*|\uparrow\rangle &= 0 \end{aligned} \tag{10.1.5}$$

Thus, the Hilbert space is split into $2^4 = 16$ different sectors, labelled by which of these oscillators in the "up" state. More specifically, there are exactly 16 states with conserved momenta $\mathbf{p}$ and electric charge q, from the "lowest" state with $|\mathbf{p}; q; \downarrow\downarrow\downarrow\downarrow\rangle$ to the "highest" sector with $|\mathbf{p}; q; \uparrow\uparrow\uparrow\uparrow\rangle$, all of which are of exactly the same energy and charge.

On the other hand, complex Goldstone fermions are in two spin 1/2 multiplets, so the raising by ψ^*_α either raises or lowers S_z by 1/2 unit. The unit magnetic monopole carries no angular momentum of its own, so we may assign spin zero to $|\downarrow\downarrow\downarrow\downarrow\rangle$ states. The successive operations by ψ^*_α then either raises and lower S_z by 1/2. It is clear that there is one state with the highest possible value of $S_z = +1$, and also one state with the lowest possible value $S_z = -1$. Two each for $S_z = \pm 1/2$, while the remaining 6 states are of $S_z = 0$. Since these have to be organized into angular momentum multiplets, there is a unique choice: one spin 1, two spin 1/2's, and five spinless, forming the same supermultiplet as that of the charged vector mesons.

As an aside, note that a monopole in $\mathcal{N} = 2$ super Yang-Mills theory will have two complex Goldstone fermions in a single spin 1/2 multiplet, so the supermultiplet of a monopole would be $2^2 = 4$ states with maximal spin 1/2. Since the massive vector meson comes in the so-called vector multiplet with the highest spin 1, a duality between monopole and charged vector mesons cannot exist in $\mathcal{N} = 2$ theories. Duality, if there is one, has to relate monopoles to quarks. Even this latter relation may seem ill justified, since the degeneracy 4 is not quite enough to put together a hypermultiplet. A hypermultiplet carries 8 degrees of freedom instead. It turns out that what happens is that the monopole field must be combined with conjugate of anti-monopole fields in order to fill a complete hypermultiplet, and it is this combination which become dual to quark fields, when such a relation does exist in some $\mathcal{N}=2$ SYM.

This appearance of degeneracy and appropriate angular momentum representation due to the Goldstone fermions, is one of most prevalent phenomenon in study of solitons in supersymmetric theories. In much of what follows, we will not explicitly repeat this part of low-energy dynamics and mainly study interaction that are responsible for formation of bound states of two or more monopoles. However, we shall always keep in mind that one must quantize the free part of the dynamics, and count the degeneracy accordingly.

10.1.2 $\mathrm{SL}(2,Z)$ electromagnetic duality

These similarities between charged vector mesons and magnetic monopoles in $\mathcal{N}=4$ Yang-Mills theories have far-reaching implications. It is conjectured that $\mathcal{N}=4$ supersymmetric Yang-Mills theories have an infinite number of equivalent formulations that are mapped one to another by a discrete duality symmetry $\mathrm{SL}(2,Z)$ that generalizes the Z_2 duality of Montonen and Olive. The simplest such map exchanges charged vector mesons and monopoles. Since this replaces electrically charged particles by magnetically charged ones, we cannot keep the same U(1) gauge field. Instead, we define a new one-form gauge field $\tilde{A}$ such that

$$d\tilde{A} \sim *dA \tag{10.1.6}$$

This relates A to the new gauge field via a complicated nonlocal field redefinition involving the Hodge operator, $*$.

If we ignore the charged particles for the moment, we can derive this duality more precisely, as follows. Consider a quadratic action

$$\int \frac{1}{8\pi}\left(\frac{4\pi}{e^2}\, dA\wedge *dA - \frac{\theta}{2\pi}\, dA\wedge dA\right) \tag{10.1.7}$$

Here we have included the topological θ term that enters crucially into the duality. Eventually, we will want to consider A as the unbroken U(1) part of an SU(2) gauge field

$$A_{SU(2)} = \frac{1}{e}\sum_{a=1,2,3}\frac{1}{2}\tau^a A^{(a)} \tag{10.1.8}$$

with the U(1) field above to be identified with $A^{(3)}$ here. With respect to $A = A^{(3)}$, the magnetic monopole has a charge 4π while the massive vector meson has a charge e^2; i.e,

$$\oint dA = 4\pi\,, \qquad \oint *dA = e^2 \tag{10.1.9}$$

with the integrals being over asymptotic two-spheres surrounding the monopole and the vector meson, respectively.

The usual trick for dualizing such a quadratic action is to resort to a first-order formulation, in which we replace the two-form field strength F by a generic two-form F, and then impose the Bianchi identity $dF = 0$. To do this we write the action in a new form,

$$\int \frac{1}{8\pi}\left(\frac{4\pi}{e^2}\,F\wedge *F - \frac{\theta}{2\pi}\,F\wedge F\right) - \int \frac{1}{4\pi}\,F\wedge d\tilde{A} \tag{10.1.10}$$

Integrating over $\tilde{A}$ imposes the Bianchi identity, forcing $F = dA$ for some one-form field A and yielding the original action. One the other hand, we could opt to solve for F in favor of $d\tilde{A}$,

$$\frac{4\pi}{e^2} * F - \frac{\theta}{2\pi}\,F = d\tilde{A} \tag{10.1.11}$$

Since $*^2 = -1$ on 2-forms in 3+1 dimensions,[1] we have

$$F = \frac{1}{(4\pi/e^2)^2 + \theta^2/(2\pi)^2}\left(-\frac{\theta}{2\pi}\,d\tilde{A} - \frac{4\pi}{e^2} * d\tilde{A}\right) \tag{10.1.13}$$

and

$$*F = \frac{1}{(4\pi/e^2)^2 + \theta^2/(2\pi)^2}\left(\frac{4\pi}{e^2}\,d\tilde{A} - \frac{\theta}{2\pi} * d\tilde{A}\right) \tag{10.1.14}$$

so the action becomes

$$\int \frac{1}{8\pi}\,\frac{1}{(4\pi/e^2)^2 + \theta^2/(2\pi)^2}\left(\frac{4\pi}{e^2}\,d\tilde{A}\wedge *d\tilde{A} + \frac{\theta}{2\pi}\,d\tilde{A}\wedge d\tilde{A}\right) \tag{10.1.15}$$

Note that the general form of the action remains unchanged, but that the coupling constants have changed to

$$\tilde{e}^2 \equiv \left[\frac{(4\pi)^2}{e^4} + \frac{\theta^2}{(2\pi)^2}\right] e^2; \qquad \tilde{\theta} = -\theta/\left[\frac{(4\pi)^2}{e^4} + \frac{\theta^2}{(2\pi)^2}\right] \tag{10.1.16}$$

Repeating the transformation with $F = dA$ replaced by $G = d\tilde{A}$ bring us back to the original Lagrangian in terms of dA, so this is a Z_2 transformation. In the special case

[1] Recall that

$$(*G)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G_{\alpha'\beta'}\eta^{\alpha\alpha'}\eta^{\beta\beta'} \tag{10.1.12}$$

with $\theta = 0$, this transformation reduces to

$$\tilde{e} = \frac{4\pi}{e} \tag{10.1.17}$$

with $\tilde{\theta} = 0$, which is the Montonen-Olive duality map.

SL$(2, Z)$ electromagnetic duality asserts that this Z_2 transformation can be extended to a quantum SU(2) theory broken to U(1); i.e., that there are two equivalent SU(2) gauge theories such that the unbroken U(1) parts of the two gauge fields are A and $\tilde{A}$ respectively and the coupling constants are e^2, θ and $\tilde{e}^2, \tilde{\theta}$ respectively. In order for this to work, the spectra of the quantized theories must be compatible with this statement. The first and foremost constraint is from the topological quantization of magnetic monopoles in these SU(2) theories. We would have

$$\oint dA = 4\pi m\,, \qquad \oint d\tilde{A} = 4\pi n \tag{10.1.18}$$

for integers m and n, assuming that the normalizations for A and $\tilde{A}$ are the right ones. To show that this is indeed consistent with the quantization of electromagnetic charge, we must show that the electric charges are properly quantized as well. With respect to A, we find from Eq. (10.1.11) that

$$\oint *dA = \oint *F = \frac{e^2}{4\pi}\left(\oint d\tilde{A} + \frac{\theta}{2\pi}\oint dA\right) = e^2\left(n + \frac{\theta}{2\pi}\,m\right) \tag{10.1.19}$$

This is exactly the right electric charge quantization condition for dyons in an SU(2) theory with a θ term. [Recall that we are using a normalization convention for the gauge field in which the non-Abelian field strength is $\sim dA + A\wedge A$. The independent integer n is sometimes referred to as integer-quantized electric charge. This gives us an indication that our choice of normalization is the natural one for embedding in the full SU(2) structure.]

Similarly, the electric charges with respect to $\tilde{A}$ can be computed

$$\oint *d\tilde{A} = -\frac{\theta}{2\pi}\oint *F - \frac{4\pi}{e^2}\oint F = -\left[\frac{(4\pi)^2}{e^4} + \frac{\theta^2}{(2\pi)^2}\right] e^2 m - \frac{e^2\theta}{2\pi}\,n \tag{10.1.20}$$

The above transformation of the U(1) action lets us rewrite this as

$$\oint *d\tilde{A} = \tilde{e}^2\left(-m + \frac{\tilde{\theta}}{2\pi}\,n\right) \tag{10.1.21}$$

This shows that the dyonic spectrum of the other SU(2) theory would be also properly quantized, apart from the fact that we are now calling n the magnetic charge and $-m$ the (integer-quantized) electric charge.

These correspondences are conveniently summarized in terms of an SL$(2, Z)$ transformation in which a central role is played by the complex coupling constant

$$\tau \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{e^2} \qquad (10.1.22)$$

The above transformation effectively replaces

$$\tau \to \tilde{\tau} = -\frac{1}{\tau} \qquad (10.1.23)$$

and at the same time maps the electromagnetic charge vector (n, m) to $(-m, n)$. The two actions may be encoded in a single SL$(2, Z)$ matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (10.1.24)$$

S^{-1} acts linearly on the column vector $(n, m)^t$, while S acts as a modular transformation on the complex number τ. Furthermore, we recall that in the original SU(2) theory, θ labeled the so-called theta-vacua and came with a natural period 2π. The requirement that the spectrum should not change under such a shift generates another transformation,

$$\tau \to \tau + 1\,, \qquad (n, m) \to (n - m, m) \qquad (10.1.25)$$

that can be represented by an SL$(2, Z)$ matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad (10.1.26)$$

These two actions, when repeated in an arbitrary sequence, generate the full SL$(2, Z)$ action on the theory, which can be summarized by

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \qquad \begin{pmatrix} n \\ m \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} n \\ m \end{pmatrix} \qquad (10.1.27)$$

with arbitrary integers a, b, c, d satisfying the constraint

$$ad - bc = 1 \qquad (10.1.28)$$

This is the SL$(2, Z)$ duality of the $\mathcal{N} = 4$ SU(2) gauge theory.

This duality generalizes straightforwardly to theories with other simple gauge groups, with the exception of the symplectic groups and odd-dimensional orthogonal groups. From inspection of the expected BPS spectra, it can be seen that these latter two cases are actually dual to each other under S transformations, while they are individually invariant under T.

This electromagnetic duality has a simple consequence for the nature of the BPS spectrum. While the maps generically relate weakly coupled gauge theories to strongly coupled ones, certain BPS spectra are believed to be continuous under changes of τ. If this is taken at face-value, an infinite tower of dyonic states is predicted at any given coupling constants. For SU(2), starting with the massive vector meson with charge $(1,0)$, the SL$(2,Z)$ generates an infinite towers of states of charge (q,p) for all co-prime integer pairs q and p [86].

To see this, we start with the observation that SL$(2,Z)$ always maps a pair of integers of the form (kq,kp) to another such pair (kq',kp'). Because SL$(2,Z)$ is a group, this means that a co-prime pair is always mapped to another co-prime pair. Furthermore, $(1,0)$ is related to $(2,1)$ by the action of

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \tag{10.1.29}$$

so the SL$(2,Z)$ image of the charged vector mesons is composed of co-prime (q,p) dyons. To see that all such co-prime pairs are reachable from $(1,0)$ by an SL$(2,Z)$ transformation, we need to show that, given a pair (q,p), there are always two more integers (r,s) such that

$$qs - rp = 1 \tag{10.1.30}$$

This can only be solved if q and p are co-prime, for otherwise this combination would be always a multiple of the common divisor, no matter what the choice of r and s. Conversely, suppose the smallest positive value for $qs-rp$ that can be obtained after trying all pairs (r,s) is $m>1$. Then $qs-rp=m$ implies that

$$qs = (n+1)m + h; \qquad rp = nm + h \tag{10.1.31}$$

for some integers n and $0 \le h < m$. If $h \neq 0$, we could have chosen $s' = -sn$ and $r' = -(1+n)r$ to yields $qs' - r'p = h$. Since h is positive and smaller than m, this contradicts the initial assumption, so m has to be 1.

On the other hand, under the above assumptions if $h=0$, one of the two pairs, (q,r) or (s,p), must have a common factor m. Without loss of generality, let us say that m is a common divisor of r and q. Furthermore, if n has δ factors of m, all of these factors in $nm=rp$ have to come from r. Then define $\tilde{r} = r/m^{(1+\delta)}$, which is an integer not divisible by m. Letting $s' = ls$ and $r' = lr - \tilde{r}$, we have $qs' - r'p = lm + \tilde{r}p$. Since $\tilde{r}p$ is not divisible by m, the right-hand side can be made to be smaller than

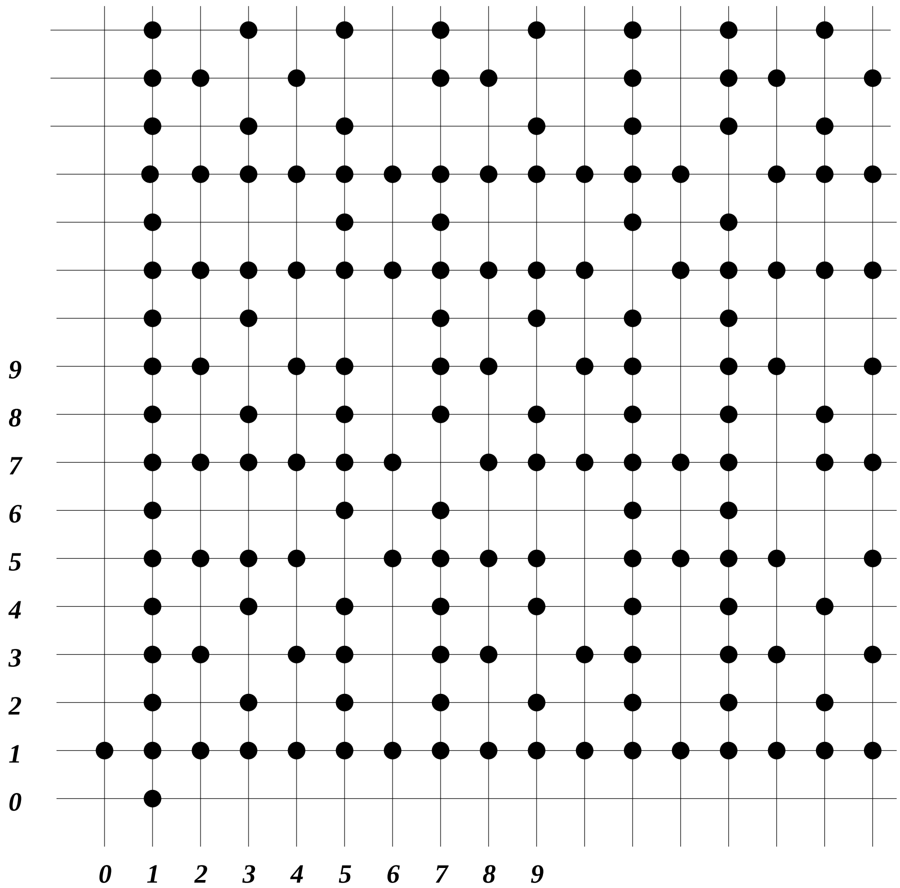

Figure 10.1: Co-prime integer pairs (q, p) are denoted by a solid circle. For each solid dot, there should be an $\mathcal{N} = 4$ vector multiplet in the spectrum.

m, but still positive, by a suitable choice of l. This again contradicts the initial assumption. Therefore, the converse statement is also true, and the SL$(2, Z)$ orbit is composed of (q, p) dyons with all possible co-prime pair of integers.

10.1.3 Counting BPS states in the SU(2) theory

Thus, the SL$(2, Z)$ duality states that for every pair of co-prime integers q and p there should be exactly one massive vector multiplet, with 16 degenerate states, having charges (q, p). The normalization of the charges is such that these states can be considered as bound states of p BPS monopoles and q charged vector mesons. For $(1, 0)$, this is one of the fundamental degrees of freedom of the theory, the charged vector multiplet that becomes massive by the Higgs mechanism.

On the other hand, Osborn's observation tells us that the charge $(0, 1)$ unit monopole also comes in a massive vector multiplet. In fact, for any integer q this observation is trivially extended to all $(q, 1)$ states, which can be thought of bound states of one monopole and n charged vector meson. This is most easily seen from the moduli space description, since such dyons are made by turning on integer quantized momentum along the free U(1) angle ξ. As we described above, the degeneracy of 16 and the spin content of the multiplet simply follow from the quantization of the Goldstone fermions, which completely decouple from the rest of the dynamics. Thus, the supermultiplet structure of charge $(q, 1)$ particles should be identical to that of a pure monopole or of a massive vector.

The first nontrivial test is for states with charge $(q, 2)$ [86]. SL$(2, Z)$ demands that such states should lie in a vector multiplet for any odd integer q. To verify this, we must solve the Schrödinger equation on the moduli space $\mathcal{M}$ of two identical monopoles. Recall that

$$\mathcal{M} = R^3 \times \frac{S^1 \times \mathcal{M}_0}{Z_2}, \tag{10.1.32}$$

where $\mathcal{M}_0$ is the double-cover of the Atiyah-Hitchin manifold. Since the Goldstone fermions, which are the superpartners of the zero modes of the R^3 and S^1 parts, will continue to give the vector multiplet structure, we must find a unique bound state in the supersymmetric dynamics of the sigma model on $\mathcal{M}_0$ for every odd momenta along S^1 Furthermore, since the local form of the metric factorizes as above, the Hamiltonian operator also factorizes, and so we must first solve for the wave functions on $\mathcal{M}_0$.

A further simplification comes from the fact that all of the additional energy due to the electric charge arises from the momentum along the center of mass U(1) angle. Hence, the wave function on $\mathcal{M}_0$ should cost no energy, and we are therefore looking for a zero-energy wave function. In other words, we must find a state on $\mathcal{M}_0$ which is square-normalizable and annihilated by the Hamiltonian. A single bound state on $\mathcal{M}_0$ should then generate an entire tower of dyons with arbitrary odd electric charge, to which we will come back later.

Wave functions of a supersymmetric sigma model with complex supersymmetry can be mapped one-to-one to differential forms on the moduli space, as we saw when we quantized the moduli space dynamics. In the absence of a potential, the superalgebra translates to a set of differential operators on the space of differential forms. A square-normalizable ground state at threshold then translates to a square-normalizable harmonic form. Let us define a one-form basis ω^m

$$\omega^0 = f(r)\, dr,$$

$$\begin{aligned} \omega^1 &= a(r)\,\sigma_1, \\ \omega^2 &= b(r)\,\sigma_2, \\ \omega^3 &= c(r)\,\sigma_3, \end{aligned} \tag{10.1.33}$$

on $\mathcal{M}_0$, so that the metric is

$$ds^2 = \sum_m \omega^m \otimes \omega^m \tag{10.1.34}$$

A unique L^2 harmonic form has to be either self-dual or anti-self-dual. Otherwise, another L^2 harmonic form could be generated by a Hodge dual transform. Also since the Hamiltonian is really a Laplace operator and so does not mix forms of different degree, the wave functions would correspond to forms of definite degrees. This, combined with the uniqueness, allows only middle-dimensional forms (i.e., two-forms) as possible candidate ground states. Furthermore, the uniqueness also requires that it be a singlet under the SO(3) isometry, so we discover that such a wave function must be one of the six possibilities,

$$\Omega_\pm^{(s)} = N_\pm^{(s)}(r)\left(\omega^0 \wedge \omega^s \pm \frac{1}{2}\,\epsilon_{stu}\,\omega^t \wedge \omega^u\right) \tag{10.1.35}$$

where $s = 1$, 2, or 3, and the summation is only over t and u.[2] Harmonicity follows if the two-form is closed, $d\Omega_\pm^{(s)} = 0$. The latter condition gives a first-order equation for the $N_\pm^{(s)}$, which is solved by

$$\begin{aligned} N_\pm^{(1)} &= \frac{1}{bc}\exp\left[\mp\int dr\,\frac{fa}{bc}\right] \\ N_\pm^{(2)} &= \frac{1}{ca}\exp\left[\mp\int dr\,\frac{fb}{ca}\right] \\ N_\pm^{(3)} &= \frac{1}{ab}\exp\left[\mp\int dr\,\frac{fc}{ab}\right] \end{aligned} \tag{10.1.36}$$

Substituting the form of Atiyah-Hitchin manifold, detailed in Sec. 6.3.3, we find that only one of these six possibilities gives a wave function that is normalizable and yet nonsingular at the origin (that is, at the bolt of the manifold), namely $N_+^{(1)}$ with the signs of f, a, b, and c all taken to be positive and a^2 being the function that vanishes at the bolt. The only possible ground state is therefore

$$\Omega_+^{(1)} = N_+^{(1)}(r)\left(\omega^0 \wedge \omega^1 + \omega^2 \wedge \omega^3\right) \tag{10.1.37}$$

[2] The fact that the wave function should be a middle-form, rather than a linear combination of an n-form and a $(4-n)$-form, follows from the fact that the Schrödinger operator in question preserves the degree of the form. This is no longer true when we consider moduli dynamics with potentials.

The physical wave function on the entire moduli space can be constructed from this. Recall that the angle ψ (with range 2π) of the approximate U(1) on $\mathcal{M}_0$ must be twisted with the angle χ (with period 4π) of the exact U(1) angle χ of S^1 by the Z_2 action $\tilde{I}'$

$$(\chi, \psi) \sim (\chi + 2\pi, \psi + \pi) \tag{10.1.38}$$

Since this flips the signs of

$$\begin{aligned}
\sigma_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \\
\sigma_2 \wedge \sigma_3 &= d\sigma_1
\end{aligned} \tag{10.1.39}$$

and thus of $\Omega_+^{(1)}$, a total wave function of the form

$$\Omega_+^{(1)} \otimes \text{constant} \tag{10.1.40}$$

would not be single-valued. If such a wave function existed, it would corresponds to a $(0,2)$ state. The only way to compensate for the double-valuedness is to cancel the sign by introducing a similarly double-valued wave function on the center-of-mass part. This is achieved with

$$\Omega_+^{(1)} \otimes e^{i(k+1/2)\chi} \otimes [R^3 \text{ part}] \tag{10.1.41}$$

for any integer k, which gives an electric charge.

It takes more care to show that the value $k + 1/2$ of the conjugate momentum translates to an electric charge $n = 2k + 1$. The key point is to recall that in all two-body cases, whether the monopoles are identical or distinct, the momentum conjugate to χ is always related to the (approximately) conserved electric charges Q_1 and Q_2 on the individual monopoles by

$$q_\chi = \frac{m_1 Q_1 + m_2 Q_2}{m_1 + m_2} \tag{10.1.42}$$

whereas the U(1) electric charge of a pair of identical monopoles would be a simple sum, $Q_1 + Q_2$. It follows that the total charge of a pair of identical monopoles $(m_1 = m_2)$ is always twice that of q_χ;

$$Q_1 + Q_2 = 2q_\chi \tag{10.1.43}$$

Thus, we have discovered a single tower of supersymmetric bound state dyons with charges $(2k+1, 2)$ for arbitrary integer k, as dictated by the SL$(2, Z)$ electromagnetic duality.

In fact, we can extend the computation a little further and establish a vanishing theorem stating that no other ground state exists on $\mathcal{M}_0$. This reveals something that we might not have known a priori just from the SL$(2, Z)$ invariance. Not only would it show that the requisite $(2k+1, 2)$ BPS states exist with the right supermultiplet structure, but it would also be a strong indication that the spurious states with charges $(2q, 2p)$ are all absent.[3]

10.2 Beyond SU(2): BPS states from binding distinct monopoles

When we try to extend the above computation to higher rank gauge groups, we encounter some qualitative changes. One is that the moduli space dynamics begins to involve the potential energy terms and that typical dyonic states preserve only four supersymmetries. This should be contrasted to the SU(2) case above where all dyons preserve eight supercharges, half of those in $\mathcal{N} = 4$ SYM. The analogues of these more symmetric dyons, as well as purely magnetic bound states, can be found as special limits of generic dyons by allowing only certain combinations of electric charges or by turning them off altogether. For the rest of the chapter we will consider the construction and counting of generic BPS dyons in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM.

Another difference is that even the most basic test of electromagnetic duality becomes an issue. Recall that the basic building blocks for the BPS states are fundamental monopoles. The number of types of these is equal to the rank of the gauge group, as we have seen, yet electromagnetic duality demands that the number of magnetic BPS states be equal to the dimension of the gauge group. For $\mathcal{N} = 4$ SU(n) gauge theory broken maximally to U$(1)^{n-1}$, the S generator of SL$(2, Z)$ must map the $n(n-1)/2$ massive vector mesons to an equal number of magnetic one-particle states. However, we have seen that only $n-1$ of these find their counterparts as fundamental monopoles, which are also the basic units from the topological perspective.

To complete the Z_2 duality, we must find $(n-1)(n-2)/2$ species of purely magnetic bound states built out of these fundamental monopoles. One might think that this would be a trivial consequence of having an attractive potential energy between certain distinct monopoles. However, the existence of the right bound states is rather nontrivial, since we are asking for not just any bound state, but rather for bound states that saturate specific energy bounds. As we will see in this chapter, an analogous bound state question can be asked in the context of pure $\mathcal{N} = 2$

[3]The existence of all the (q, p) towers of BPS states was shown in Ref. [87].

SYM, although there no duality requires a Z_2 symmetric spectra. Even though the quantum dynamics differ, the attractive potential energy between distinct monopoles in the latter example is the same as in the $\mathcal{N} = 4$ SYM case. We will show that no such purely magnetic bound state exists in pure $\mathcal{N} = 2$ SU(n) theories, showing that the existence of the BPS states is much more subtle.

In this section we will explore the moduli space dynamics of a collection of many distinct monopoles in an SU(n) theory. The main objective here will be to establish a dictionary between the conserved quantities in the moduli space description and the electromagnetic charge in the field theory. The actual construction and counting of states will follow in the two subsequent sections.

10.2.1 Electric charges and effective potential

As we have seen earlier, moduli space dynamics of monopoles decompose into the interacting relative part and the non-interacting "center of mass" part. The latter corresponds to a 4-dimensional flat metric of the form,

$$g_{cm} = A\, d\vec{X}^2 + B\, d\chi^2, \tag{10.2.1}$$

where $\vec{X}$ is a three-vector. Since we are interested in establishing existence of bound states, this part of the dynamics will be ignored for the most part. The free center-of-mass sector generates two kinds of quantum numbers, nevertheless. One is the overall, conserved U(1) charge, and the other is a supermultiplet structure generated by the fermionic partners of $\vec{X}$ and χ. The resulting degeneracies, 4 and 16, correspond to the smallest possible BPS multiplet of the underlying $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SUSY Yang-Mills field theories, respectively.

A simple case of this dynamics involves a collection of distinct monopoles in SU(n) gauge theories. The interacting part of the moduli space metric is a simple generalization of four-dimensional Taub-NUT metric. Without loss of generality, consider a collection of $k+1$ distinct monopoles, whose magnetic charges are given by an irreducible (sub)set of simple roots, $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{k+1}$. The simple roots satisfy relations $\boldsymbol{\beta}_a^2 = 1$, $\boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_{a+1} = -1/2$, and $\boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_{a+b} = 0$ for $b > 1$. The relative part of the corresponding metric is

$$g = C_{AB}\, d\vec{r}_A \cdot d\vec{r}_B + \frac{4\pi^2}{e^4}(C^{-1})_{AB}(d\psi_A + \cos\theta_A d\phi_A)(d\psi_B + \cos\theta_B d\phi_B) \tag{10.2.2}$$

where the matrix C for the relative moduli space is

$$C_{AB} = \mu_{AB} + \frac{2\pi}{e^2}\delta_{AB}\,\frac{1}{r_A} \tag{10.2.3}$$

The 3-vectors $\vec{r}_A$ is the relative position vector between a^{th} and $(a+1)^{th}$ monopoles,

$$\vec{r}_A = \vec{x}_{A+1} - \vec{x}_A \qquad (10.2.4)$$

while the angles ψ_A of period 4π are related to the U(1) phases of each monopole, ξ's (of period 2π), as

$$2\frac{\partial}{\partial\psi_A} = \frac{\partial}{\partial\xi_{A+1}} - \frac{\partial}{\partial\xi_A} \qquad (10.2.5)$$

For generic reduced mass matrix μ, the triholomorphic Killing vector fields of this geometry are

$$K_A = \frac{\partial}{\partial\psi_A} \qquad (10.2.6)$$

so the vector fields G and G_I's are linear combinations of K_A's with constant coefficients;

$$\begin{aligned} G &= e\sum_A a_A K_A \\ G^I &= e\sum_A a_A^I K_A \end{aligned} \qquad (10.2.7)$$

The electric charges are measured by the charge operators,

$$-i\mathcal{L}_{K_A} \qquad (10.2.8)$$

whose half-integral eigenvalues we will denote by q_A. In terms of the simple roots β_a, the electric charge of a dyonic state with charge q_A's is

$$\begin{aligned} &e(+q_1+q_2+q_3+\cdots+q_k+n/2)\boldsymbol{\beta}_1+ \\ &e(-q_1+q_2+q_3+\cdots+q_k+n/2)\boldsymbol{\beta}_2+ \\ &e(-q_1-q_2+q_3+\cdots+q_k+n/2)\boldsymbol{\beta}_3+ \\ &e(-q_1-q_2-q_3+\cdots+q_k+n/2)\boldsymbol{\beta}_4+ \\ &\vdots \\ &e(-q_1-q_2-q_3-\cdots-q_k+n/2)\boldsymbol{\beta}_{k+1} \end{aligned} \qquad (10.2.9)$$

where the integer n comes from quantization of an overall U(1) angle and should be even or odd when $2\sum_A q_A$ is even or odd, respectively.

Once we go to such a superselection sector, the soliton cores experience an effective potential which is a combination of the potential energy and angular momentum barrier due to electric charges. Of this, energy due to the overall electric charge n is a constant contribution to the BPS energy while moduli-dependent part of the effective potential has the following closed form,

$$U_{\text{eff}} = \frac{1}{2}\left(\frac{4\pi^2}{e^2}\sum_{A,B}(C^{-1})_{AB}\,a_A\,a_B + \frac{e^4}{4\pi^2}\sum_{A,B}C_{AB}\,q_A\,q_B\right) \qquad (10.2.10)$$

10.2.2 Constituent monopoles

Before solving for the bound states with higher rank gauge groups, which is qualitatively different from the case of SU(2) because of the presence of a potential energy term in the moduli space dynamics, let us stop for a moment and discuss the validity of the low-energy moduli space approximation and point out a subtlety that is easily missed. Exactly when and for what purpose can we use the moduli space dynamics for computing physical quantities? At least two conditions must be met.

The first is that the semiclassical approximation be justified, so that it makes sense to talk about classical solitons. The second is that the nonrelativistic approximation be valid. The former tells us that one must start with a weakly coupled field theory, while the latter puts further constraints on what kind of processes we may look at. The conventional wisdom is that the moduli space approximation is valid for slowly moving monopoles in the Coulomb phase of a weakly coupled Yang-Mills theory.

When the low-energy dynamics involves static potentials, the issue becomes slightly more complicated. As is clear from the derivation of the bosonic potential energy term from field theory, we have imposed the additional condition that one of the two ($\mathcal{N} = 2$) or one of the six ($\mathcal{N} = 4$) Higgs expectation values be much larger than the others. We started with BPS monopoles constructed with this special Higgs field (denoted by b throughout this review), and required the contribution from the remaining Higgs fields to be small. Thus, we can only trust the low-energy dynamics if the Higgs expectation values are such that

$$|\langle a \rangle| \ll |\langle b \rangle| \tag{10.2.11}$$

This condition is also necessary to ensure that the massive charged vector mesons, whose role in mediating interactions we have completely ignored, are much more massive than any of the moduli space degrees of freedom.

There is a potential source of confusion here, since in the derivation of the bosonic potential we required that the extra energy associated with the additional Higgs field a be small compared to the rest masses of the monopoles, and at most comparable to the kinetic energy. The confusion arises from the fact that the requirement of small electric energy is not necessarily equivalent to the above condition on the Higgs expectation values, since the electric part of the energy scales with q and can be made arbitrarily small, whether or not $\langle a \rangle$ is small. This happens because the states with low electric charge are those sitting at the bottom of the potential and are insensitive to the asymptotic behavior of the potential energy, whose size is dictated by the ratio $|\langle a \rangle|/|\langle b \rangle|$. On the other hand, when $|\langle a \rangle|$ is not small, exactly the same kind of low-charge states exist as very tightly bound states of monopole cores with binding energies comparable to their rest masses, hardly a state that can be studied in a nonrelativistic approximation.

The resolution of this confusion lies in the fact that the particles whose interactions are described by the moduli space dynamics are not really combinations of individual monopoles. The monopole cores that enter the moduli space dynamics as constituent particles solve the primary BPS equation,

$$B = Db \tag{10.2.12}$$

where the definition of b as a linear combination of adjoint Higgs fields depends on the total magnetic charge, and thus is affected by the presence of other monopoles, however far away they are.

When we talk about individual monopoles, however, we think of isolated BPS solitons that solve such a BPS equation for a slightly difference choice of b'

$$B = Db' \tag{10.2.13}$$

Here the choice of b' differ from monopole to monopole and depends only on the magnetic charge of the individual monopole, rather than on the total magnetic charge. It is not difficult to convince oneself that the sum of the masses of these isolated individual monopoles is always larger than or equal to the magnetic energy of the solution to the primary BPS equation. The difference between the two is precisely the asymptotic height of the bosonic potential energy.

Thus, a bound state of physical monopoles with a net binding energy $E_{\rm binding}$ is viewed in a very different way from the low-energy dynamics. From the latter viewpoint a bound state is an excited states of constituent monopoles with a nonnegative excitation energy

$$E_{\rm excitation} = \Delta m - E_{\rm binding} > 0, \tag{10.2.14}$$

where $\Delta m \geq 0$ is the difference between the sum of the BPS masses of the individual monopoles and the actual BPS mass of the combined magnetic state. Under the assumption that the supersymmetry is not spontaneously broken, the binding energy $E_{\rm binding}$ approaches Δm in the limit of small e with a finite and fixed value of the quantized electric charge, on the other hand. Therefore, a tightly bound state of physical monopoles is really a low lying excitation state of the moduli space dynamics. Also, the wave function would be concentrated near the zeros of the bosonic potential energy, where it sees only the part of the potential energy that is much smaller than the monopole rest mass. This funny feature of the moduli space dynamics is an inevitable consequence of the low-energy supersymmetry that requires that the ground state energy always be nonnegative.

10.3 Two-body bound states

A special case occurs when the dynamics involves a pair of distinct fundamental monopoles, with masses m_1 and m_2, that are associated with the simple roots $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. Thanks to the simplicity of the moduli space, we can solve for the wave functions explicitly. As we saw in Sec. 6.3.2, the relative moduli space is the Taub-NUT manifold with rotational SU(2) isometry and a triholomorphic U(1) isometry. Its metric is

$$\mathcal{G}_{\rm rel} = \left(\mu + \frac{2\pi}{e^2 r}\right) d\mathbf{r}^2 + \frac{4\pi^2}{e^4}\left(\mu + \frac{2\pi}{e^2 r}\right)^{-1} (d\psi + \cos\theta d\phi)^2 \tag{10.3.1}$$

where μ is the reduced mass. The only triholomorphic Killing vector field is $\partial/\partial\psi$, which is indeed related to the global gauge U(1), so the triholomorphic Killing vector that enters the potential energy is

$$G = ea\frac{\partial}{\partial\psi} \tag{10.3.2}$$

The metric for the free center-of-mass part is

$$\mathcal{G}_{\rm cm} = (m_1+m_2)d\mathbf{R}^2 + \frac{(4\pi)^2}{e^4(m_1+m_2)}d\chi^2, \tag{10.3.3}$$

Locally, the full moduli space is a product of these two manifolds. However, the periodicity of the U(1) phase angles ξ_i imposes certain identifications on χ and ψ, so that the total moduli space is obtained only after a division by a discrete group. These identifications are

$$(\chi,\psi) = (\chi + 2\pi,\ \psi + 4\pi\delta_2)\,, \tag{10.3.4}$$

$$(\chi,\psi) = (\chi - 2\pi,\ \psi + 4\pi\delta_1)\,. \tag{10.3.5}$$

with $\delta_i \equiv m_i/(m_1+m_2)$. The momenta conjugate to ψ and χ, which we denote by q' and q, respectively, are such that

$$Q_1/e \equiv q' + 2\delta_2 q, \qquad Q_2/e \equiv q' - 2\delta_1 q \tag{10.3.6}$$

are integers. In fact, from Eq. (6.3.14), we see that these are the integer-quantized electric charges, conjugate to the ξ_i, on the individual fundamental monopole cores.

Since $Q_1 - Q_2 = 2e(\delta_1 + \delta_2)q = 2eq$, we may write the electric charge of such excitations as

$$\mathbf{q} = e(n/2 + q)\boldsymbol{\beta}_1 + e(n/2 - q)\boldsymbol{\beta}_2 \tag{10.3.7}$$

where $n \equiv 2q' + 2(\delta_2 - \delta_1)q$ is also an integer. By construction n is odd (even) whenever $2q$ is odd (even).

There are no further constraints on the overall electric charge n, other than the requirement, needed for the validity of the moduli space approximation, that total electric energy be kept very small compared to the rest masses of the monopoles. For the case of q, however, the dynamics sets an upper bound if we are interested in bound state dyons. As we saw earlier, q induces an angular momentum barrier that repels the two monopoles from each other. The effective potential energy for the charge q sector is of the form

$$\frac{1}{2}\left[\frac{4\pi^2 a^2}{e^2}\left(\mu + \frac{2\pi}{e^2 r}\right)^{-1} + \frac{q^2 e^4}{4\pi^2}\left(\mu + \frac{2\pi}{e^2 r}\right)\right] \tag{10.3.8}$$

This effective potential energy has a minimum at a finite value of r if and only if

$$q^2 < \left(\frac{4\pi^2 a}{e^3 \mu}\right)^2 \tag{10.3.9}$$

Otherwise, the minimum moves out to infinity, implying that such a dyonic state cannot form. In the following, we will write down the supersymmetric ground state for such "allowed" electric charges q. Once this is done, an infinite tower of dyons follows by exciting the other electric charge, n.

10.3.1 Bound state of a pair of distinct $\mathcal{N} = 4$ monopoles

Finding a supersymmetric (bound) state of two such monopoles boils down to solving a Dirac equation of the form

$$\mathcal{D}_\pm \Omega = (\sqrt{i}\varphi^E \pm \sqrt{-i}\varphi^{*E})(-i\nabla_E \mp G^5_E)\Omega = 0 \tag{10.3.10}$$

where $(\sqrt{i}\varphi^E \pm \sqrt{-i}\varphi^{*E})$ has a Clifford action on Ω that is a differential form on the moduli space; the sign is determined by the sign of the central charge. A supersymmetric state is possible when the square of this operator takes the form

$$\text{(energy)} - \text{(positive central charge)} \tag{10.3.11}$$

As we have done previously, we can remove almost all constants from the problem with the rescalings

$$\begin{aligned} r &\rightarrow \frac{2\pi r}{e^2\mu} \\ t &\rightarrow \frac{4\pi^2 t}{e^4\mu} \end{aligned} \tag{10.3.12}$$

Also as before, we define $\tilde{a} \equiv 4\pi^2 a/e^3\mu$. The problem then reduces to one with a kinetic term derived from the metric

$$\mathcal{G}_{\rm rel} = \left(1+\frac{1}{r}\right) d\mathbf{r}^2 + \left(1+\frac{1}{r}\right)^{-1} (d\psi + \cos\theta d\phi)^2 \tag{10.3.13}$$

and a potential energy obtained from

$$\tilde{G}_5 = \tilde{a}\frac{\partial}{\partial\psi}. \tag{10.3.14}$$

As before, we have defined $\tilde{a} \equiv 4\pi^2 a/e^3\mu$.

Let us assume that $\tilde{a} \geq 0$ and search for BPS bound states obeying

$$\mathcal{D}_+\Omega_q = 0. \tag{10.3.15}$$

For these states, the charge q has to be a nonnegative number in the range $0 \leq q \leq \tilde{a}$. Once the Ω_q are known, the BPS bound states annihilated by $\mathcal{D}_-$, say Ω'_{-q}, are given automatically by

$$\Omega'_{-q} = \Omega^*_q \tag{10.3.16}$$

This is because $\mathcal{D}_+$ turns into $i\mathcal{D}_-$ under a complex conjugation that keeps operators, such as φ and ∇, untouched. The bound states for $\tilde{a} < 0$ can also be constructed from those with $\tilde{a} > 0$ by using the identity

$$\star\mathcal{D}_-(\tilde{a}) = i\mathcal{D}_+(-\tilde{a})\star \tag{10.3.17}$$

where

$$\star \equiv \prod_E (\varphi^E - \varphi^{*E}) \tag{10.3.18}$$

These wave functions are related to those for $\tilde{a} \geq 0$ by

$$\Omega_{-q}(-\tilde{a}) = \star\,\Omega'_{-q}(\tilde{a}), \qquad \Omega'_q(-\tilde{a}) = \star\,\Omega_q(\tilde{a}) \tag{10.3.19}$$

For the $\mathcal{N} = 4$ Yang-Mills case, Ω can be written as a differential form. In this representation, the operation $*$ is a kind of Hodge star operation, just as that of $\tau_\pm$, up to an additional phase dependence on the degree of the form.

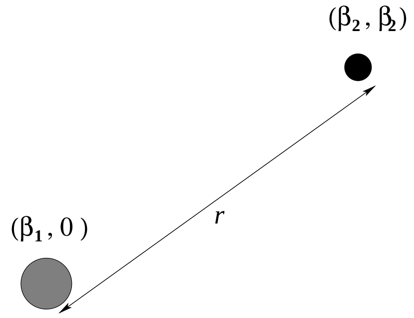

Figure 10.2: The simplest 1/4-BPS dyon is made of a monopole and a dyon of unit electric charge. This configuration corresponds to $q = -1/2$ and $n = 1$. This 1/4-BPS dyon comes in a BPS multiplet with highest spin 3/2 and degeneracy 64.

A bound state wave function in the relative moduli space carries three conserved quantum numbers: the relative electric charge, q, the total angular momentum, j, and the third component of the angular momentum, m. All of these are quantized to be an integer or half-integer. We will denote the BPS wave functions with these quantum numbers as $\Omega^j_{m;q}$. Let us recall some basic properties of the spherical harmonics on S^3, which are usually denoted by D^j_{mk}. A unit S^3 has $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ isometry. The spherical harmonics, D^j_{mk}, have the same quadratic Casimir, $j(j+1)$, for the two SU(2)'s but independent values m and k for the third component eigenvalues for $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$. However, our S^3 is a deformed one on which great circles along one direction are usually much smaller than those along the other two, so that only $\mathrm{SU}(2)_L \times \mathrm{U}(1)_R$ survives the deformation. Because of this, our multiplets have a definite eigenvalue k, which is to be identified with the electric charge contribution to q. In other words, in a given multiplet m ranges over $-j, -j+1, \ldots, j$, while k takes a fixed value in that range. Note how the U(1) charge is shared between the coefficient functions and the two-form parts.

A useful basis for the differential forms is the vielbein

$$\begin{aligned}
\omega^0 &= \sqrt{1+1/r}\, dr, \\
\omega^1 &= \sqrt{r^2+r}\, \sigma_1, \\
\omega^2 &= \sqrt{r^2+r}\, \sigma_2, \\
\omega^3 &= \sqrt{\frac{r}{1+r}}\, \sigma_3. \qquad (10.3.20)
\end{aligned}$$

where, as usual, the σ_j are rotationally invariant one-forms obeying

$$d\sigma_i = \frac{1}{2}\epsilon_{ijk}\,\sigma_j \wedge \sigma_k \tag{10.3.21}$$

Under the U(1) gauge isometry, $\omega^2 + i\omega^3$ transforms as a unit charge state.

When $q \geq 1$, the bound states come in four distinct angular momentum multiplets, of total angular momenta $j = q, q-1/2, q-1/2, q-1$, giving a total of $8q$ wave functions. Each of these $8q$ states acquires an additional factor of 16 degeneracy in the form of an $\mathcal{N}=4$ vector multiplet, thanks to the quantization of the free center-of-mass fermions. Taken all together, these $16 \times 8q$ degenerate states form a single 1/4-BPS supermultiplet with highest angular momentum, $q+1$. When $q = 1/2$, the same set of wave functions works, except that the first three angular momentum multiplets are sufficient. These three by themselves give the degeneracy $8q = 4$. Again, with the additional factor of 16 from the center-of-mass part, a supermultiplet of degeneracy $16 \times 8q = 64$, with highest spin $q+1 = 3/2$ emerges. The $q = 0$ case is special, and will be considered presently.

After some trial and error, one finds that the simplest angular momentum multiplet, with $j = q-1$, takes the form, [4]

$$\Omega^{q-1}_{m;q} = r^{q-1}e^{-(\tilde{a}-q)r}\,(\omega^0 + i\omega^3)\wedge(\omega^1 + i\omega^2)D^{q-1}_{m(q-1)}. \tag{10.3.22}$$

with m taking values $-q+1, -q+2, \ldots, q-1$, while the largest multiplet, with $j = q$, is formed out of

$$\begin{aligned}\Omega^{q}_{m;q} &= \frac{r^q e^{-(\tilde{a}-q)r}}{1+r}\times \\ &\Bigg[\Big(\,\tilde{a} + (\tilde{a} + \frac{1}{1+r})(\omega^0\wedge\omega^3 + \omega^1\wedge\omega^2) + \tilde{a}\,\omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3)\,\Big)D^q_{mq} \\ &-\sqrt{q/2}\,(\omega^0 + i\omega^3)\wedge(\omega^1 + i\omega^2)D^q_{m(q-1)}\Bigg].\end{aligned} \tag{10.3.23}$$

with m taking values $-q, -q+1, \ldots, q$. These two multiplets give a total of $4q$ wave functions.

The remaining $4q$ wave functions with angular momentum $q - 1/2$ can be found most easily by acting with $\mathcal{D}_-$ on the $\Omega^q_{m;q}$ and $\Omega^{q-1}_{m;q}$ found above. From $\Omega^{q-1}_{m;q}$, we find $2q-1$ states

$$\frac{r^q e^{-(a-q)r}}{\sqrt{r+r^2}}\,(\omega^1 + i\omega^2)\wedge(1 + \omega^0\wedge\omega^3)D^{q-1}_{m(q-1)}, \tag{10.3.24}$$

[4]For complete details, see Ref. [76]. The physical angular momentum of these wave functions, and the angular momentum multiplet thereof, has a subtlety which is too involved for this review.

while the action of $\mathcal{D}_-$ on $\Omega^q_{m;q}$ produces $2q+1$ states

$$\frac{r^q e^{-(a-q)r}}{\sqrt{r+r^2}}\Big[(\omega^0+i\omega^3)\wedge(1+\omega^1\wedge\omega^2)\sqrt{2q}D^q_{mq} \\ +i(\omega^1+i\omega^2)\wedge(1+\omega^0\wedge\omega^3)D^q_{m(q-1)}\Big]. \tag{10.3.25}$$

Note that these $4q$ wave functions are not organized into a pair of $j = q - 1/2$ angular momentum multiplets. For the latter, it is important to recognize that the SU(2) rotational isometry we relied on is not quite the physical angular momentum. Owing to the triplet of complex structures, it turns out that a spin contribution must be added to j to give the actual angular momentum. We refer the readers to Ref. [76] for complete details.

We note that these $q > 0$ wave functions are only normalizable if $q < \tilde{a}$.

A special case occurs when $q = 0$; i.e., when the bound state carries no relative electric charges. Such a state generates a purely magnetic bound state (or else a dyonic bound state with only center-of-mass electric charge). The unique wave function is an angular momentum singlet and has the simple form

$$\Omega^0_{0;0} = \frac{e^{-\tilde{a}r}}{1+r}\left[\tilde{a}+(\tilde{a}+\frac{1}{1+r})(\omega^0\wedge\omega^3+\omega^1\wedge\omega^2)+\tilde{a}\,\omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3)\right]. \tag{10.3.26}$$

This BPS state actually preserves all the supercharges of the low-energy dynamics. In the limit of aligned vacua ($a = 0$), this state becomes a threshold bound state of two monopoles with the drastically simpler form [44][45],

$$\Omega^0_{0;0} = d\left(\frac{\sigma_3}{1+1/r}\right) = d\left(\frac{\omega^3}{\sqrt{1+1/r}}\right) \tag{10.3.27}$$

This state acquires a degeneracy of 16 from the center-of-mass fermions, and forms a 1/2-BPS vector supermultiplet of the $\mathcal{N} = 4$ theory. With the exception of this, we can see that a bound state exists only when a strict inequality $q < \tilde{a}$ holds.

10.3.2 Bound states of a pair of distinct $\mathcal{N} = 2$ monopoles

For monopoles in $\mathcal{N} = 2$ Yang-Mills theory, the main difference is that the wave function Ω is now represented by a Dirac spinor on the moduli space. In addition, there is only one Dirac equation to solve,

$$\lambda^E(-i\nabla_E - G_E)\Omega = 0 \tag{10.3.28}$$

where again $\sqrt{2}\lambda^E$ has the Clifford action on spinor Ω. This has something to do with the facts that the low-energy supersymmetry is real and that there is only one real central charge. With a spinorial Ω, writing down the explicit form of the wave function is more cumbersome. One way is to adopt, for example, the Penrose-Newman formalism or to find a simple, if any, complex coordinates where spinors can be transcribed into holomorphic forms. Here, instead of writing down the wave function, we will summarize the result [77].

In the relative moduli space, the bound state wave functions exist only if $1/2 \leq q < \tilde{a}$ or $\tilde{a} < q \leq -1/2$. The wave functions in the relative part of the moduli space are organize into a single angular momentum multiplet with angular momentum $j = |q| - 1/2$, which, combined with the half-hypermultiplet structure from the center-of-mass fermions, produces a single BPS multiplet with highest spin $|q|$ and total degeneracy $4 \times 2|q|$. Note that the dyons with large $|q|$ are in multiplets with large highest spin.[5]

A counterintuitive aspect of $\mathcal{N} = 2$ is that the states with $q = 0$ are nowhere to be found. The lowest lying state of this tower is nothing but a pure monopole state with magnetic charge $\boldsymbol{\beta}_1^* + \boldsymbol{\beta}_2^*$. This is in a stark contrast to the case of $\mathcal{N} = 4$ Yang-Mills theory, where the purely magnetic bound state is actually required by the electromagnetic $\mathrm{SL}(2, Z)$ duality. Although $\mathcal{N} = 2$ SYM lacks such a simple constraint on the spectrum, this absence of a purely magnetic state is still somewhat surprising. In fact, a classical BPS monopole solution of a charge $\boldsymbol{\beta}_1^* + \boldsymbol{\beta}_2^*$ can always be found: all we need to do is to embed the usual SU(2) monopole solution using the root whose dual is $\boldsymbol{\beta}_1^* + \boldsymbol{\beta}_2^*$. Such a root always exists; this is most easily seen in $\mathrm{SU}(n)$ gauge theory, where we can choose the normalization sothat $(\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)^* = \boldsymbol{\beta}_1^* + \boldsymbol{\beta}_2^*$. Analysis of the moduli dynamics shows that no quantum bound state with such a charge can saturate the BPS bound.

10.4 Many-body bound states and index problems

We need more a systematic approach to the problem to generalize the bound state counting to the many-body case, since the explicit construction of bound states becomes quite unlikely beyond the two-body case. For this, we will consider various index problems, instead of direct construction of bound states. The index problems can also be quite involved, given that the quantum mechanics involves many degrees

[5] Such high-spin dyons remain massive everywhere on the vacuum moduli space, and do not enter the Seiberg-Witten description of $\mathcal{N} = 2$ theories in any crucial way. In particular, the states with $|q| > 1$, and possibly those with $|q| = 1$, would be completely missed if we were to use a bootstrap argument to generate dyons by acting with monodromies on simple elementary particles or fundamental monopoles. In order to understand the complete BPS spectrum, one must at least start with the above weak coupling spectrum as an input to the bootstrap.

of freedom with complicated interaction terms. However, we will later see that it is precisely the potential energy terms that simplify the index problems enormously. Let start with some generalities.

The index computations count differences in the number of ± 1 eigenvector ground states with respect to some Z_2 involution τ,

$$\mathcal{I} = \lim_{\beta\to\infty} = \mathrm{Tr}\,\tau e^{-\beta\mathcal{D}^2} = n_+ - n_- \tag{10.4.1}$$

where $\mathcal{D}$ is a Hermitian supercharge that should annihilate the BPS state in question. The Z_2 grading τ must anticommute with $\mathcal{D}$, and $n_\pm$ are the respective number of ground states with respect to $\mathcal{D}^2$. To be more precise, the index we wish to compute is the equivariant index. That is, we count the index in each quantized charge sector. Recall that the low-energy superalgebra is such that we typically have

$$\mathcal{D}^2 = \mathcal{H} - \mathcal{Z} \tag{10.4.2}$$

up to overall numerical factors. In our problems the eigenvalue of $\mathcal{Z}$ is completely determined by the electric charges.

Possible choices for τ and $\mathcal{D}$ have been listed in previous chapters, with the definitions of relevant indices given in Secs 9.2.5 and 9.3.3. In the case of monopole dynamics originating from $\mathcal{N} = 4$ Yang-Mills theories, the supersymmetry is complex and wave functions are represented by differential forms. τ_4 leads to the Euler Index, $\mathcal{I}_4$, while $\tau_\pm$ leads to the signature index, $\mathcal{I}_\pm$. For monopoles from $\mathcal{N} = 2$ theory, the supersymmetry is real and wave functions are represented by Dirac spinors on the moduli space. For this case, the only involution available is the chirality of the Dirac spinor, given by τ_2, which leads to $\mathcal{I}_2$.

We are actually interested in the sum $n_+ + n_-$, for which one needs a more refined understanding of the dynamics, such as a vanishing theorem.[6] We will assume that

[6] A vanishing theorem asserts that certain types of eigenfunctions cannot exist. The simplest example is the argument showing that there is no normalizable solution to

$$\nabla^2\Phi = 0 \tag{10.4.3}$$

in R^n. The absence of square-normalizable solutions can be argued by computing

$$0 = -\int \Phi\nabla^2\Phi = \int (\nabla\Phi)^2 - \oint \Phi\hat{n}\cdot\nabla\Phi \tag{10.4.4}$$

A solution can asymptote to $1/r^{n-2}$ at most, so that the latter boundary term vanishes for $n \geq 3$. This leaves the condition $\nabla\Phi = 0$, which leaves only $\Phi = \mathit{constant}$ as a possible solution, contradicting the assumption. More refined vanishing theorems proceed in more or less the same manner. For instance, for $\mathcal{D}_\pm$, self-dual wave functions and anti-self-dual wave functions couple differently to a self-dual curvature tensor in such a way that only one chirality can evade a vanishing theorem.

such a vanishing theorem does exist, so that $n_+ = 0$ or n_-0, and assume that the absolute value of the index equals the number of ground states annihilated by $\mathcal{D}$.

10.4.1 Bound states of many distinct $\mathcal{N} = 4$ monopoles

When we consider many distinct monopoles, and consider the case where only one combination of the triholomorphic Killing vector fields, $G = e\sum_A a^A K_A$, is involved, the condition for a classical ground state is quite simple;

$$|q_A| < |\tilde{a}_A| \tag{10.4.5}$$

where

$$\tilde{a}_A \equiv \frac{4\pi^2}{e^3}\sum_B (\mu^{-1})_{AB}\, a_B \tag{10.4.6}$$

This condition also guarantees a massgap of the system, and allows us to compute the index using the index theorem [82]. Otherwise, there is a net repulsive force between some of the monopoles, and there cannot be any bound state, classical or quantum. The marginal case of $|q_A| = |\tilde{a}_A|$ is more subtle; we will ignore this case except for some special limits.

A standard index theorem asserts that a Dirac operator $\mathcal{D}$ can be deformed continuously without changing the index, as long as the deformation does not destroy an existing mass gap. Thus, as long as we start with a case that has a mass gap as above, we can safely multiply G by a large number T to find another Dirac operator with an even larger mass gap, but with the same index. On the other hand, a larger coefficient of G means that the potential energy gets stiffer and the low-energy motion gets confined to near the zeros of G, or equivalently near the fixed points of G. In this special set of examples, the one and only fixed point of G is the origin, $\vec{r}_A = 0$, so it suffices to solve a local index problem near the origin. Furthermore, the finite curvature at the origin is overwhelmed by the ever-increasing scale associated with the rescaled Killing vector $T\,G$.

Thus, at sufficiently large T, the problem reduces to one where the geometry is a flat R^{4k}, and G is a linear combination of certain rotational vectors from each R^4 factor. The problem then decomposes into many R^4 problems. On the other hand, we may use the same kind of deformation of the two-monopole problem to reduce it to a flat R^4 problem as well. One difference is now that the two-monopole problems have been solved explicitly, and we already know the value of the index for the R^4 problem. Then, since the multimonopole index problem factorizes into many R^4 problem, all we need to do to recover the value of the index for the multimonopole case is to take a product of the known two-monopole indices for each interacting pair of monopoles within the group.

Thus, when we consider a bound state of $\boldsymbol{\beta}_1$-, $\boldsymbol{\beta}_2$-, ..., $\boldsymbol{\beta}_{k+1}$-monopoles, with relative charges $q_1, q_2, \ldots, q_k$, we can count the number of states by considering successive pairs $(\boldsymbol{\beta}_A\boldsymbol{\beta}_{A+1})$ with relative charges q_A. Counting the degeneracy d_A of each pair as if no other monopoles were present, the degeneracy of the bound state wave function involving all $k+1$ monopoles would be simply the product of all the d_A.[7] In the next two subsections, we will write out the resulting index formulae explicitly, and make some contact with physics.

Supersymmetric quantum mechanics with four complex supercharges describes the dynamics of monopoles in $\mathcal{N}=4$ Yang-Mills theories. There are three possibilities for dyonic bound states:

- The state is 1/2-BPS in the Yang-Mills field theory. Such states would be annihilated by all of the supercharges of the low-energy monopole dynamics, which is possible only if the central charges in the relative part of the dynamics all vanish. This is guaranteed when all the relative electric charges q_a vanish. In particular, this case includes purely magnetic bound states.

- The state is 1/4-BPS in the Yang-Mills field theory. These states would be annihilated by half of the supercharges of the low-energy monopole dynamics and not by the other half. This is possible only if at least one central charge is nonzero.

- The state is non-BPS.

Of the three indices, only $\mathcal{I}_4$ is robust against turning on more than one of the G^I. On the other hand, turning on an additional G^I always increases the mass gap, and is a Fredholm deformation that preserves $\mathcal{I}_4$. The index computation yields

$$\mathcal{I}_4 = \left(\prod_A \left\{ \begin{array}{ll} 1 & \quad q_A = 0 \\ 0 & \quad q_A \neq 0 \end{array} \right\} \right)$$

Since the central charge of the state that contributes to the index is zero, the state must be annihilated by all supercharges of the quantum mechanics and be 1/2-BPS in $\mathcal{N}=4$ Yang-Mills theory. This is consistent with the existence of a unique magnetic 1/2-BPS bound state of monopoles in a generic Coulomb vacuum, as is expected

[7] To get the true degeneracy one wants to multiply by either 16 or 4 from the center-of-mass part, at the end of day, of course.

from the SL$(2,Z)$ electromagnetic duality. One of the generators of the SL$(2,Z)$ maps the massive charged vector multiplets to purely magnetic bound states in a one-to-one fashion. After taking into account the automatic degeneracy of 16 from the free center-of-mass fermions, the total degeneracy of these bound states is always 16, which fits the $\mathcal{N}=4$ vector multiplet nicely. This purely magnetic bound state was previously constructed by Gibbons in special vacua where all the G^I vanish.[8]

The existence of 1/4-BPS states are more sensitive to the vacuum choice and the electric charges. In particular, as can be seen from the BPS equation, 1/4-BPS dyons can exist only if the relevant part of Higgs expectation values is such that only two are effectively involved, which is equivalent to the condition that only one linearly independent G^I is present. This allows us to assume, without loss of generality, that only G^5 is turned on, as far as counting 1/4 BPS states are concerned. While only $\mathcal{I}_4$ is a well-defined index in generic vacua with more than all five G_I are turned on, the Dirac operator $iQ \pm Q^\dagger$ would anticommute with the alternate involution $\tau_\pm$ if only one of the G^I were turned on. This gives us an additional handle on the spectrum: whenever we can discuss 1/4-BPS states, $\mathcal{I}_s^\pm$ become available. Furthermore, as we saw in the ground state wave functions of the two-body problem, all of the ground states in question are of the same chirality with respect to one of $\tau_s^\pm$. Thus, at least in two-body problems, this index actually counts the number of 1/4-BPS dyons.

Secondly, the effective potential energy in the charge-eigensector must be attractive or at least marginal along all asymptotic directions for a bound state to exist. This condition takes the simple form

$$|q_A| \leq |\tilde{a}_A|. \tag{10.4.7}$$

For computation of the index, however, it is often necessary to assume a mass gap. For this reason, we will actually perform the computation with $|q_A| < |\tilde{a}_A|$.

Given a mass gap, the index $\mathcal{I}_s^\pm$ was computed and the result [82] is

$$\mathcal{I}_s^\pm = \left(\prod_A \left\{ \begin{array}{ll} 8|q_A| & \pm\tilde{a}_A q_A > 0 \\ 1 & \tilde{a}_A q_A = 0 \\ 0 & \pm\tilde{a}_A q_A < 0 \end{array} \right\} \right).$$

Note that the index is nonvanishing only if each of $\pm\tilde{a}_A q_A$ (no summation) are non-

[8]One might think that the existence of this bound state is obvious, since the potential energies are all attractive and also there exists a classical BPS monopole with the same magnetic charge. However, none of these guarantees the existence of a BPS bound state at the quantum level. In fact, the same set of facts are true for a pair of distinct monopoles in $\mathcal{N}=2$ SU(3) Yang-Mills theory, but we know that such a purely magnetic bound state does not exist as a BPS state.

negative. This is in addition to the usual requirement

$$\pm e \sum_A \tilde{a}_A q_A > 0 \tag{10.4.8}$$

which is necessary for the states to be annihilated by $\mathcal{H} \mp \mathcal{Z}$ with $\mathcal{Z} = \mathcal{Z}_5 = \sum_A \tilde{a}_A q_A$ being the central charge. The index indicates that the degeneracy of such a 1/4-BPS state is

$$16 \times \prod_A \mathrm{Max}\,\{8|q_A|, 1\}. \tag{10.4.9}$$

The factor of 16 arises from the free center-of-mass fermions.

In the two-monopole bound states, the number $8|q|$ is accounted for by four angular momentum multiplets with $j = |q|, |q| - 1/2, |q| - 1/2$, and $|q| - 1$, except for $|q| = 1/2$ where the first three suffice. The top angular momentum $|q|$ in the relative part of the wave functions has a well-known classical origin: when an electrically charged particle moves around a magnetic object, the conserved angular momentum is shifted by a factor of $eg/4\pi$. While fermions can and do contribute, the number of fermions scales with the number of monopoles, and not with the charge q_A. In fact, the top angular momentum of such a dyonic bound state wave functions is

$$j_{top} = \sum_A |q_A|, \tag{10.4.10}$$

for large charges, so that the highest spin of the dyon would be

$$1 + j_{top} = 1 + \sum_A |q_A|, \tag{10.4.11}$$

after taking into account the universal vector multiplet structure from the free center-of-mass part. The actual multiplet structure is not difficult to derive, and we find

$$V_4 \otimes \Big(\otimes_A \{[|q_A|] \oplus [|q_A| - 1/2] \oplus [|q_A| - 1/2] \oplus [|q_A| - 1]\} \Big) \tag{10.4.12}$$

Here V_4 denotes the vector multiplet of $\mathcal{N} = 4$ superalgebra, and $[j]$ denotes an angular momentum multiplet with the highest spin j.

The largest supermultiplet contain in this has the highest spin $j_{top} + 1$, whose degeneracy is

$$16 \times 8 \sum_A |q_A|, \tag{10.4.13}$$

This is much less than the number of states we found above unless all but one q_A vanishes. Thus, this implies that there are many 1/4-BPS, and thus degenerate, supermultiplets of dyons for a given set of electromagnetic charges. For large electric charges q_A, thus, the number of dyon supermultiplets scales as,

$$\left(\prod_A 8|q_A|\right) / \left(\sum_A 8|q_A|\right) . \tag{10.4.14}$$

While one would expect to find degenerate states within a supermultiplet, there is no natural symmetry that accounts for the existence of many supermultiplets of the same electromagnetic charges and of the same energy.

In the regime where $|q_A| \geq |\tilde{a}_A|$ for some q_A, we cannot rely on the current index computation. On the other hand, since even a single repulsive direction, i.e., $|q_A| > |\tilde{a}_A|$ for some A, prohibits a bound state (supersymmetric or not), the unresolved question boils down to the marginal case, where $|q_A|$ equals $|\tilde{a}_A|$ for some A but $|q_A| < |\tilde{a}_A|$ for others The explicit construction of two-monopole bound states in Ref. [] seem to indicate that no *dyonic* bound state may form along such marginal directions, but this remains to be shown for multimonopole cases.

10.4.2 Bound states of many distinct $\mathcal{N} = 2$ monopoles

In $\mathcal{N} = 2$ Yang-Mills theories, a state can be either BPS or non-BPS. There is no such thing as a 1/4-BPS state. Dyons that would have been 1/4-BPS when embedded in an $\mathcal{N} = 4$ theory are realized as either 1/2-BPS or non-BPS, depending on the sign of the electric charges.

Whenever there is a mass gap, the index $\mathcal{I}_2$ is

$$\mathcal{I}_2 = \left(\prod_A \left\{ \begin{array}{ll} 2|q_A| & \tilde{a}_A q_A > 0 \\ 0 & \tilde{a}_A q_A \leq 0 \end{array} \right\}\right) .$$

which gives us a possible criterion for BPS dyons to exist. This condition is similar to the condition for BPS dyons or monopoles to exist in $\mathcal{N} = 4$ Yang-Mills theories, but differs in two aspects. The first is that given a set of a_A, all of which are positive (negative), the electric charges q_A must be all positive (negative). The overall sign of the electric charge matters.

The second difference from the $\mathcal{N} = 4$ case is that a purely magnetic bound state of monopoles does not seem to exist as a BPS state, even though there exists a classical BPS solution with such a charge. This feature was noted previously in Ref. [], where the bound states of a pair of distinct monopoles were counted explicitly. In fact, the index indicates that all relative q_A must be nonvanishing for a BPS state to exist. Assuming the vanishing theorem, the number of BPS dyonic bound state under the above condition is [82]

$$4 \times \prod_A 2|q_A|, \tag{10.4.15}$$

The overall factor of 4 is from the quantization of the free center-of-mass fermions. The actual multiplet structure is

$$C_2 \otimes \Big(\otimes_A \left[|q_A| - 1/2 \right] \Big) \tag{10.4.16}$$

where C_2 denotes the half-hypermultiplet of $\mathcal{N} = 2$ superalgebra, and $[j]$ denotes an angular momentum multiplet of the highest spin j.

For large electric charges we again observe the proliferation of supermultiplets. The top angular momentum, and thus the size of the largest supermultiplet, can only grow linearly with $\sum |q_A|$. This means that the number of supermultiplets with the same electric charges scales at least as

$$\left(\prod_A 2|q_A| \right) / \left(\sum_A 2|q_A| \right) \tag{10.4.17}$$

for large q_A.

In the regime where $|q_A| \geq |\tilde{a}_A|$ for some q_A, again we cannot rely on the current index computation. For the same reason as in $\mathcal{N} = 4$ Yang-Mills theory, no bound state can exist if even a single repulsive direction ($|q_A| > |\tilde{a}_A|$ for some A) exists, so the unresolved question again boils down to the marginal case, where $|q_A|$ equals $|\tilde{a}_A|$ for some A while others satisfy $|q_A| < |\tilde{a}_A|$. Extrapolating from the explicit construction of two-monopole bound states in Ref. [], we suspect that no bound state can form along such marginal directions.

10.5 Stability of BPS states with four supercharges

In much of the latter part of this chapter, we counted dyonic states that are either 1/4-BPS in $\mathcal{N} = 4$ theories or 1/2-BPS in $\mathcal{N} = 2$ theories. In either case, the BPS states in question preserve 4 supercharges, and because of this a lot of common traits show up. We succeeded in counting dyons made out of a chain of distinct monopoles, and also found that their actual presence depends sensitively on the choice of the vacuum and the coupling constant.

Just as we found in classical analysis earlier, these dyons are typically loose bound states of more than one charged particles and the size of the wave functions grows indefinitely when we increase certain electric charges beyond critical values. Equivalently the bound state become marginal as we move the vacuum expectation values toward the corresponding marginal stability domain wall, and in fact the state disappears at the domain wall and beyond. While the marginal stability domain walls in question are specific to this weakly coupling corner of the moduli space, the basic decay mechanism, involving multi-centered nature of generic dyons, may as well be universal. In fact, in the context of Seiberg-Witten theory coupled to supergravity, dyonic black holes with several horizons have been shown to exist in a manner quite similar to our picture [107][108]. In the latter system again, the dyonic states disintegrate as a marginal stability domain wall is approached. Similar phenomena were found to persist in the strongly coupled regime as well in a series of papers where dyons are realized as (split-)trajectories in Seiberg-Witten moduli space [111][112][113][114][115][116][117].

Counting this type of objects with 4 unbroken supercharges tends to be a little more difficult for $\mathcal{N} = 2$ SYM. In particular, unlike $\mathcal{N} = 4$, existence of classical BPS solution is no guarantee for existence of its quantum counterpart even when the low-energy effective theory has a massgap. The latter is seen most clearly in case of $\mathcal{N} = 2$ pure SU(3) theory in a corner of vacuum moduli space where one monopole can be regarded as a bound state of the other two monopoles with small nonnegative binding energy; in this regime there is a light but massive modes that corresponds to separating the two fundamental monopoles which bound to form the third monopole. The flip side of this aspect is that a massgap exists and stabilize the third, heaviest monopole against splitting into the two fundamental monopoles. Normally, under such circumstances, one would expect all three exist as quantum BPS states. Yet, the spectrum we found is such that the heaviest monopole does not exist as a supersymmetric state. It goes without saying that this absence persists when the mass gap is removed as well.

Overall, counting states with 4 unbroken supercharges turn out to be a rather difficult task in all of Yang-Mills theories and in superstring theories as well. The problem manifest itself in various different mathematical forms. If one realizes $\mathcal{N} = 2$ theories as dynamics of a wrapped M5 brane [101], the BPS states correspond to supersymmetric open membranes with boundary circling some specific combinations of topological cycles on the wrapped M5. Supersymmetry condition for the open membrane is far more restrictive than just minimizing its area, and existence proof of a given BPS state is rather involved, and has been done in very special cases only [102][103]. If one realizes these theories by Calabi-Yau compactification of type II string theories [100], the BPS states are D-branes completely wrapped on some supersymmetric cycles of the Calabi-Yau manifold. Generally speaking, establishing existence of and classifying such cycles is a rather difficult task. Some of more successful examples where computation of BPS spectrum is found at special region of the Coulomb branch of $\mathcal{N} = 2$ include Refs. [105][106][109][110].

Furthermore, the problem typically get worse when the topology of the open membrane or the supersymmetric cycle become complicated, which tends to happen when we consider a highly charged states with large (expected) degeneracy. Complicated topology is accompanied by many internal moduli, namely classical supersymmetric deformations of the shape and the size of the membranes or the D-branes. Quantization of these internal moduli, which is an independent problem of their own unrelated to our moduli space dynamics, is necessary for the counting or even for the existence proof, yet this is typically very difficult task also. These internal moduli space are typically incomplete manifolds with boundaries.[9]

Counting of BPS states in this chapter represents a small corner in the entire landscape of this class of problems, but the one where computation can be performed explicitly and with a well-defined approximation procedure. The hope is that one can eventually find ways to connect the findings or even the methods here usefully to other regimes.

[9]One case that illustrates this point well is the string web picture for 1/4 BPS dyons of $\mathcal{N} = 4$ theories. While this approach is both very intuitive and powerful in understanding marginal stability domain walls, it is as helpless as any other approach when it comes to counting degeneracy or supermultiplet structures. For larger electric and magnetic charges, the corresponding string web has many internal moduli which essentially are sizes of internal loops made out of fundamental strings and D1-branes. The sizes of these loops are bounded from above in a very complicated way because they are interrelated and also because the end-points on D3-branes restricts the size of the web. See next chapter more details on string web construction of 1/4 BPS states.