SUSY basics

Lecture "1":

Why SUSY? (≡ symmetry relating bosons & fermions)

(a.) particle theory : * beyond SM physics, "MSSM", GUTs...

* typically weak coupling models, but not always

(b.) strong coupling dynamics : * dynamical (super-)symmetry breaking
* confinement, dualities...

* both in 4d, as well as $d>4$ $(5,6)$ & $d=2,3$ (SUSY critical systems – tricritical Ising model)

(c.) string theory : worldsheet theory w/ fermions is supersymmetric.

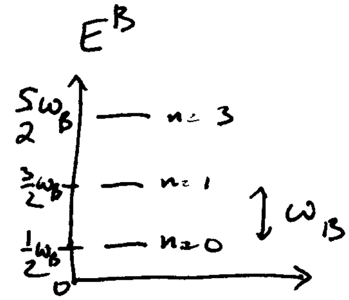
(d.) mathematical results : index theorems, ...

(e.) getting jobs (empirical)

Simplest SUSY system: SUSY oscillator

(a) bosonic operators: b, b^+ : $[b, b^+] = 1$

$$H_B = \omega\left(b^+ b + \frac{1}{2}\right) = \frac{\omega}{2}\{b^+, b\}$$

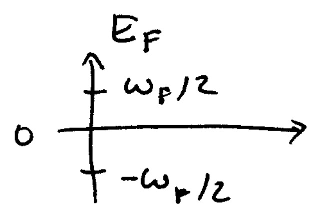


$$E_n^B = \omega_B\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

(b) fermionic operators: f, f^+ : $\{f, f^+\} = 1$, $f^2 = 0$, ${f^+}^2 = 0$

$$H_F = \omega_F\left(f^+ f - \frac{1}{2}\right) = \frac{\omega}{2}[f^+, f]$$

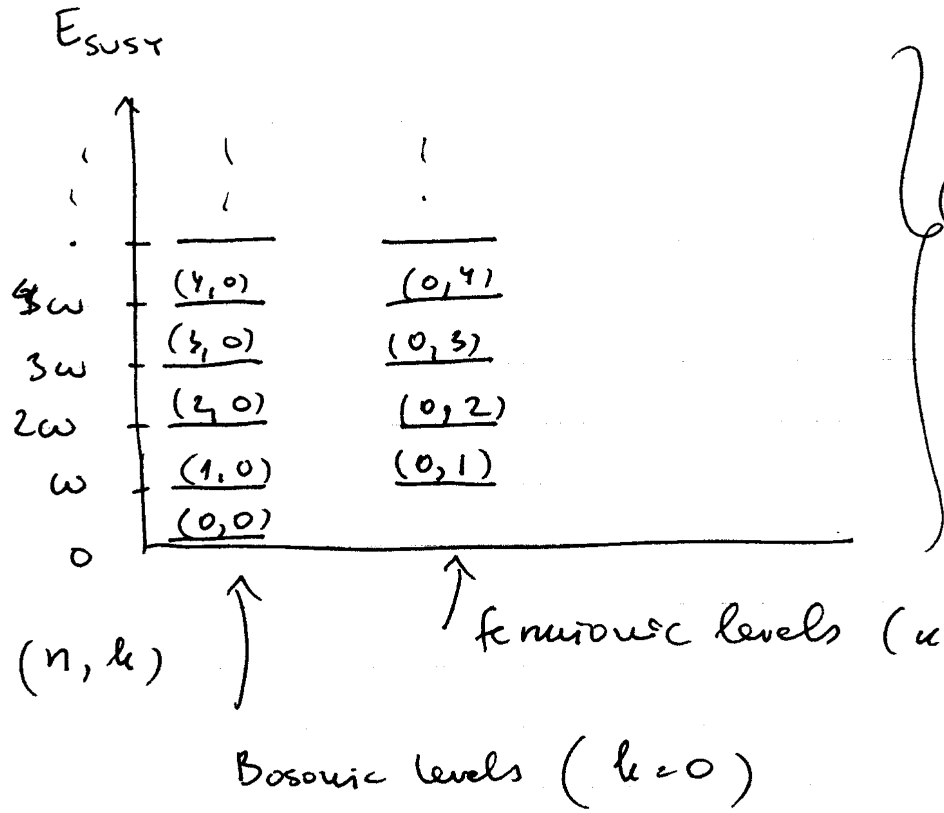
$$E_n^F = \omega_F\left(k - \frac{1}{2}\right), \quad k = 0, 1$$



like a spin in magnetic field

(a) + (b) w/ $\omega_B = \omega_F \Longleftrightarrow$ SUSY oscillator

$$H_{SUSY} = H_B + H_F = \omega\left(b^+ b + f^+ f\right)$$



note: free SUSY QFT in any dim'n has a spectrum that looks like this, for every oscillator mode!

* all levels above $E=0$ are doubly degenerate

* all energies are ≥ 0

$\underbrace{\hphantom{\text{all energies are} \geq 0}}_{\text{generic properties!}}$

symmetry!

$$Q = b^{+} f$$

$$Q^{+} = b f^{+} \quad (\text{take } [b, f] = 0)$$

$$\left\{ \begin{array}{l} H = \omega \{Q^{+}, Q\} \\ Q^2 = Q^{+2} = 0 \text{ due to nilpotency of } f, f^{+} \end{array} \right\}$$

moreover clearly $[Q, H] = [Q^+, H] = 0$

so Q & Q^+ are symmetry generators ($\underline{\mathcal{N}=2}$ $\underline{\text{SUSY QM}}$)

since $Q = b^+ f$: annihilates $\underline{f}$, creates $\underline{b}$

when acting on (n, k) state gives

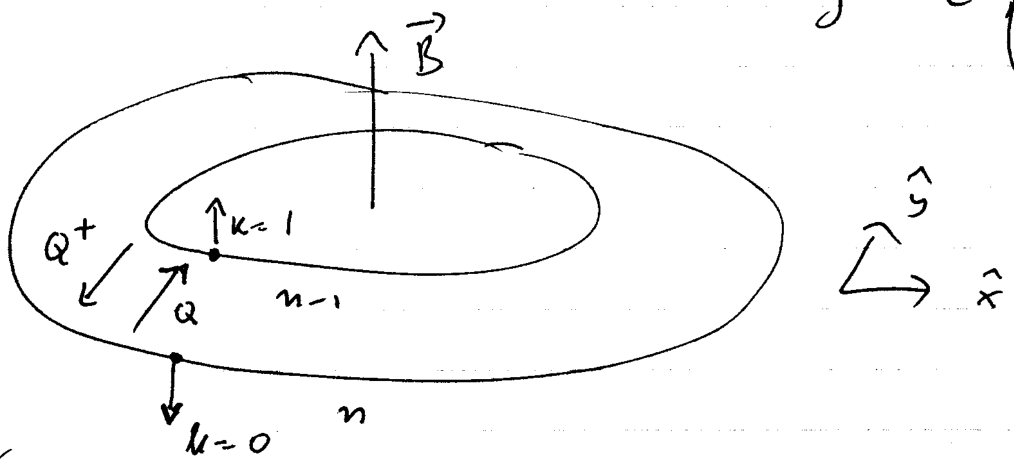
$(n+1, k-1)$ state ;

similarly $Q^+ = b f^+$ on (n, k) state gives

$(n-1, k+1)$ state

(of course, since $k = 0$ or 1 only, will get zero when acting w/ Q on $k=0$ or Q^+ on $k=1$)

[simplest real system: 2d electrons in $\vec{B}$-field along $\hat{z}$ (when $g_{el.} = 2$)]



↳ this system (Landau levels, $g=2$) + ^{4}He monolayer on Kr-plated graphite $\simeq$ tricritical Ising $\simeq (1,1)\ \phi^3$ 2d QFT

-- about only SUSIC !! in nature !!

Algebraic structure generalizes to more complicated systems (than oscillator) ~~see p. 4'~~

For now study algebra abstractly (since algebraic structure holds):

for oscillator

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Q^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$Q^2 = Q^{+\,2} = 0$$

$$H = \{Q^+, Q\} \quad ; \quad (Q)^+ = Q^+$$

((1)) can write H as total square of hermitean op'r

$H = (Q + Q^+)^2$, so H is a ≥ 0 operator

hence <u>all energy eigenvalues are positive semidefinite</u>

$H|E\rangle = |E\rangle E$, so

$$\underbrace{E \langle E|E\rangle}_{=1} = \langle E|(Q+Q^+)(Q+Q^+)|E\rangle = \| \langle E|(Q+Q^+) \|^2 \geq 0$$

& $E = 0$ <u>iff</u> $(Q + Q^+)|E\rangle = 0$

((1')) similarly $H = (i(Q-Q^+))^2$ allows us to conclude $E = 0$ <u>iff</u> $(Q-Q^+)|E\rangle = 0$

((1+1'))

$\Rightarrow E = 0 \Leftrightarrow Q|E\rangle = 0, \; Q^+|E\rangle = 0 \overset{\text{def}}{\Longleftrightarrow}$ SUSY ground states

$\Rightarrow$ in a system w/ unbroken SUSY
the ground state vacuum energy is $\equiv 0$!

(early days hopeful about c.c. ...)

this is called $\mathcal{N}=2$ QM, since redefining

$$\begin{cases} Q_1 = Q + Q^\dagger \\ Q_2 = i(Q - Q^\dagger) \\ H = Q_1^2 = Q_2^2 \\ \{Q_1, Q_2\} = 0 \end{cases}$$

we have an equivalent form

$$\underbrace{\{Q_i, Q_j\} = \delta_{ij} H}, \quad i,j = 1,2.$$

an SO(2) global symmetry of the algebra.

Before continuing, let's look @ an explicit realization of this algebra in a QM system w/ potential different than harmonic...

in an x, p ($[p, x] = -i$) rep., easy to generalize to more complicated systems, e.g.:

$$H = \frac{1}{2} p^2 + \frac{1}{2}(W'(x))^2 + \frac{1}{2}\sigma_3 \; W''(x)$$

acting on $\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ square-integrable functions on the line: particle w/ "spin ± 1" moving on a line w/ potential $V(x) \sim (W'(x))^2$ & "spin-orbit" $\sim W''(x)$; where hermitean Q_i:

$$Q_1 = \frac{1}{2}\sigma_1 \, p + \frac{1}{2}\sigma_2 W'(x)$$

$$Q_2 = \frac{1}{2}\sigma_2 \, p - \frac{1}{2}\sigma_1 W'(x) \quad \text{obey } \{Q_i, Q_j\} = \delta_{ij} H$$

$\Rightarrow$ general structure, indeed. [SHOW!]

(to recover oscillator, take $W(x) = \frac{\omega}{2} x^2$.)

Moreover, call states w/ $\psi_2 = 0$ "Fermionic" and those w/ $\psi_1 = 0$ "BOSONIC" (in analogy w/ oscilator)

Formally: $F \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1 + \sigma_3)$; clearly $[F, H] = 0$

[and also [check!] $\underbrace{[F, Q] = Q \; , [F, \bar{Q}] = -\bar{Q}}$]

(up to interchange!)

def: $\Rightarrow$ (F = fermion #) $\Leftarrow$

now ready to discuss another general property:

((2)) <u>all energy levels are doubly degenerate</u>

define $(-1)^F$ operator : $\equiv +1$ on bosonic states
$\equiv -1$ on fermion states

by $(-1)^F H = H(-1)^F$
$(-1)^F Q_i = -Q_i(-1)^F$ } e.g. H is bosonic, acting on a state does not change f #
while Q_i change the f # by ±1

$(-1)^F|B\rangle = |B\rangle$
$(-1)^F|F\rangle = -|F\rangle$ } all states are either B or F.
"Z_2 grading"

* Since $(-1)^F$ commutes w/ H, all eigenstates are either B or F.

** Since $(-1)^F$ anticommutes w/ Q_i (Q or Q^+), & the Q_i's commute w/ H, the Q_i's map B-eigenstates to F-eigenstates (& v.v.) of same energy.

So, let $|E\rangle$: $H|E\rangle = |E\rangle E$, $E>0$
$\langle E|E\rangle = 1$ be a (say) B eigenstate

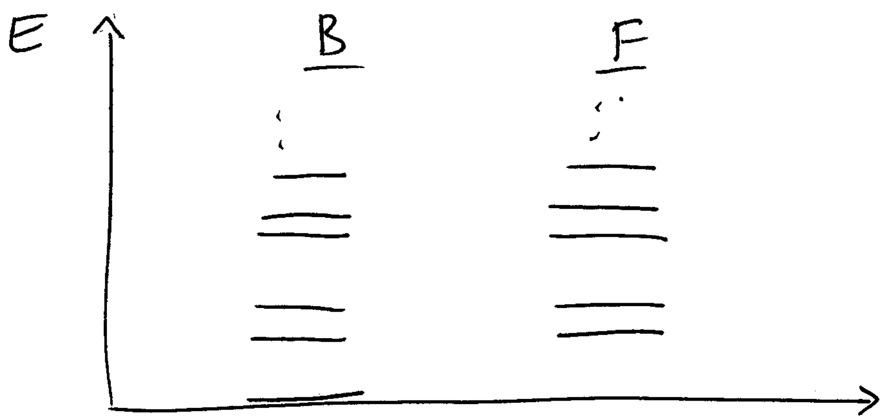
then $|E,1\rangle = Q_1|E\rangle/\sqrt{E}$: $H|E,1\rangle = |E,1\rangle E$; $\langle E,1|E,1\rangle = 1$

so $|E,1\rangle$ is a F eigenstate w/ same energy.

$\Rightarrow$ <u>all $E \neq 0$ eigenstates are paired (B-F)</u> ∎

Note: Acting w/ Q_2 on $|E\rangle$ instead does not produce a new (F-) degenerate state. $\boxed{\text{Show this!}}$ (if you wish, may use the SO(2) symmetry generated by $U_\alpha = e^{i\alpha F}$).

So, the general story, namely that spectrum looks like this:



holds (as for oscillator), but spectrum is not equidistant in general.

Also: 1) B–F degeneracy does not have to hold for $E=0$ state (argument fails, since $Q_1|E\rangle$ not normalizable since $Q_1^2 = 0$ if $E=0$,

(we saw this for harmonic oscillator, where $E=0$ was only non-doubled state).

(2) double degeneracy argument holds for all $E \neq 0$ states, no matter whether ground state is supersymmetric or not (i.e. whether it has $E = 0$ or not).

Which leads us to a discussion of whether we can tell when E_0 is $= 0$ or is > 0.

lowest energy state

unbroken SUSY $E_0 = 0$

broken SUSY $E_0 > 0$

$\boxed{\text{order parameter for } \cancel{\text{SUSY}} \iff E_0}$

For this system, really easy:

$$H = Q_1^2 = Q_2^2$$

$\boxed{\text{SHOW!}}$

$$H|\psi\rangle = 0 \iff Q_i|\psi\rangle = 0 \quad , i = 1, 2$$

whether H has a $E_0 = 0$ state, one can tell by asking whether a normalizable sol'n of $Q_i|\psi\rangle = 0$ exists. $(i = 1, 2)$

<u>Note:</u> $H|\Psi\rangle = 0 \Rightarrow$ 2nd order diff. equ.

$Q_i|\Psi\rangle = 0 \Rightarrow$ 1st order !

$E_0 = 0$ states are studied much more easily!

((more generally, $Q|\Psi\rangle = 0$ states are called "BPS" states $\Leftrightarrow$ easy to study, e.g. in SUGRA, 1st order equs!))

Take, eg. $i = 1$, $Q_1|\Psi\rangle = 0$

$$\Downarrow$$

$$\left(-i\sigma_1 \frac{d}{dx} + \sigma_2 W'(x)\right)\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = 0$$

$\boxed{\text{SHOW!}}$

$$\Psi_0(x) \equiv \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = e^{\sigma_3 W(x)}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}e^{W(x)}c_1\\ e^{-W(x)}c_2\end{pmatrix}$$

<u>$\Psi_0(x)$ normalizable iff</u> :

(1) $c_1 = 0$ $W(x) \to +\infty$ as $|x| \to \infty$

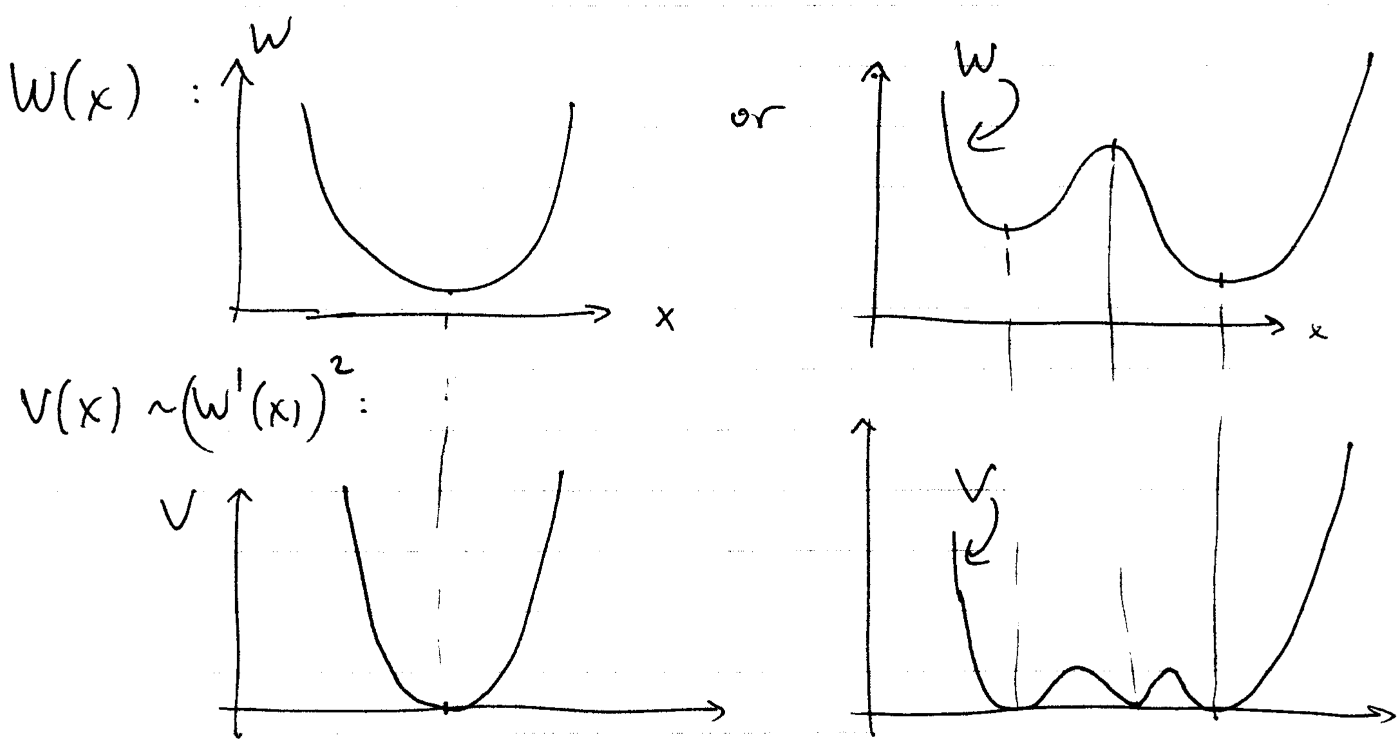
(2) $c_2 = 0$ $W(x) \to -\infty$ as $|x| \to \infty$

<u>i.e. $W(x)$ is "even at infinity"</u>

otherwise, if $W(x)$ is "odd @ ∞"

no normalizable $E_0 = 0$ state exists &

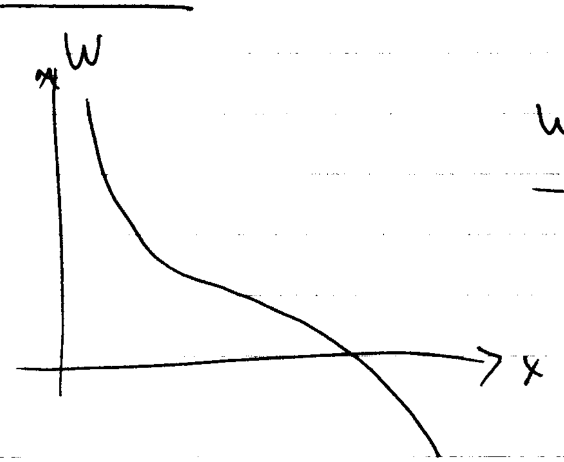
SUSY breaks.



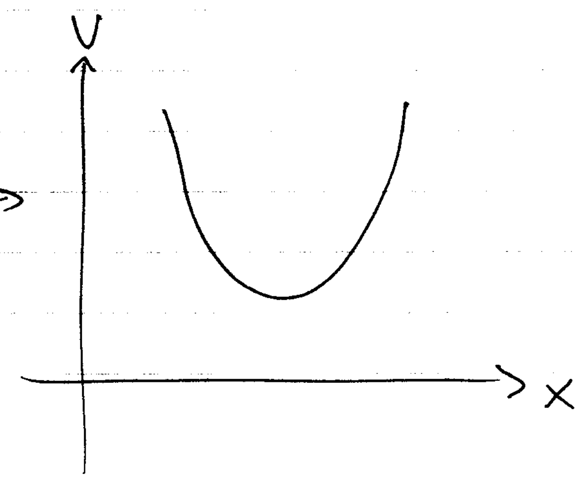
∴ if $V(x)$ has an odd # of $E=0$ classical minima, SUSY is unbroken, as normalizable $\psi_0 = 0$ exists (both perturbatively & exactly)

<u>Otherwise</u>:

a

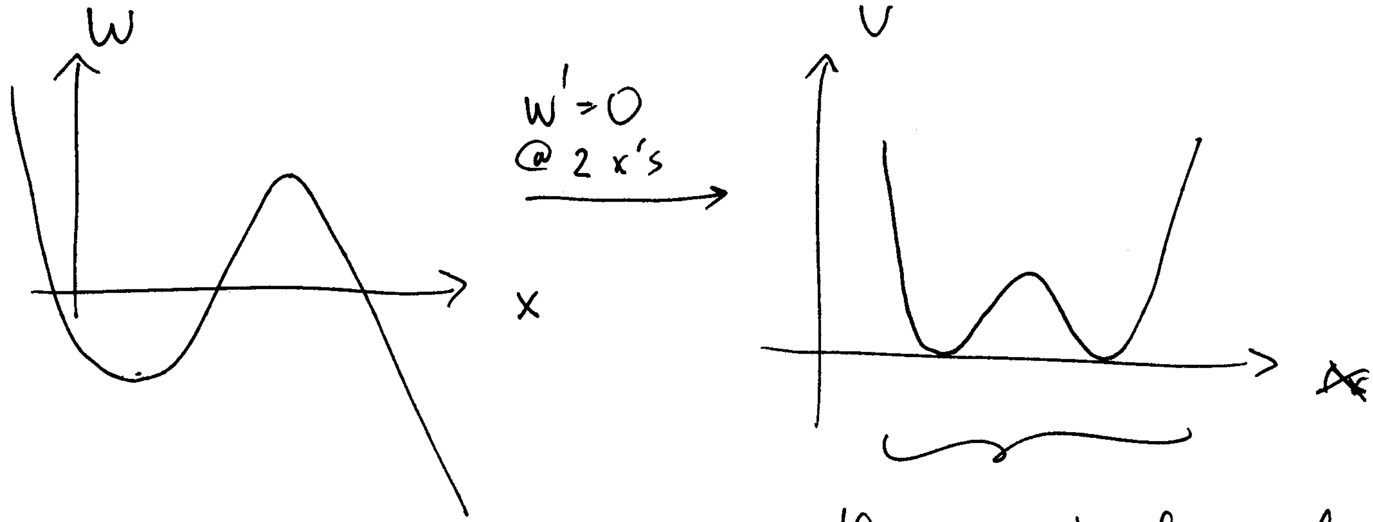


W' never $= 0$



: $\cancel{\text{SUSY}}$ classically & QM-cally

(b.)



there exist classical
$E_0 = 0$ states
(or perturbatively)

→ but no $E_0 = 0$ normalizable state exists

⇒ $\cancel{\text{SUSY}}$ is nonperturbative (tunneling – instantons!)

* very interesting for QFT & applic'ns.

* reason: instantons generate small E_0 ....
($\propto E_0 \ll \omega$, where ω is classical frequency of motion in wells).

Moral: * SUSY can break nonperturbatively

* behavior of superpotential @ $|x| \to \infty$ (a topological property) can tell us whether this is at all possible (whether a normalizable $E=0$ state exists).

* a lot of this persists in SUSY QFT

* so, introduce some more "technology"...

Witten index:

$$\begin{cases} \mathrm{tr}(-1)^F \equiv \sum_E n_B(E) - n_F(E) \quad \left(\equiv \sum_E \langle E|(-1)^F|E\rangle \right) \\ \quad = n_B(E=0) - n_F(E=0), \text{ since all } E>0 \text{ states are B-F degenerate.} \end{cases}$$

- if $\mathrm{tr}(-)^F \neq 0$, then SUSY is <u>unbroken</u>
- if $\mathrm{tr}(-1)^F = 0$, SUSY may or may not be broken, since it may be that $n_B(0) = n_F(0) \neq 0$, or $n_B(0) = n_F(0) = 0$.

* Utility is, then, in the ability to argue that if $\mathrm{tr}(-)^F \neq 0$, SUSY is unbroken for that theory.

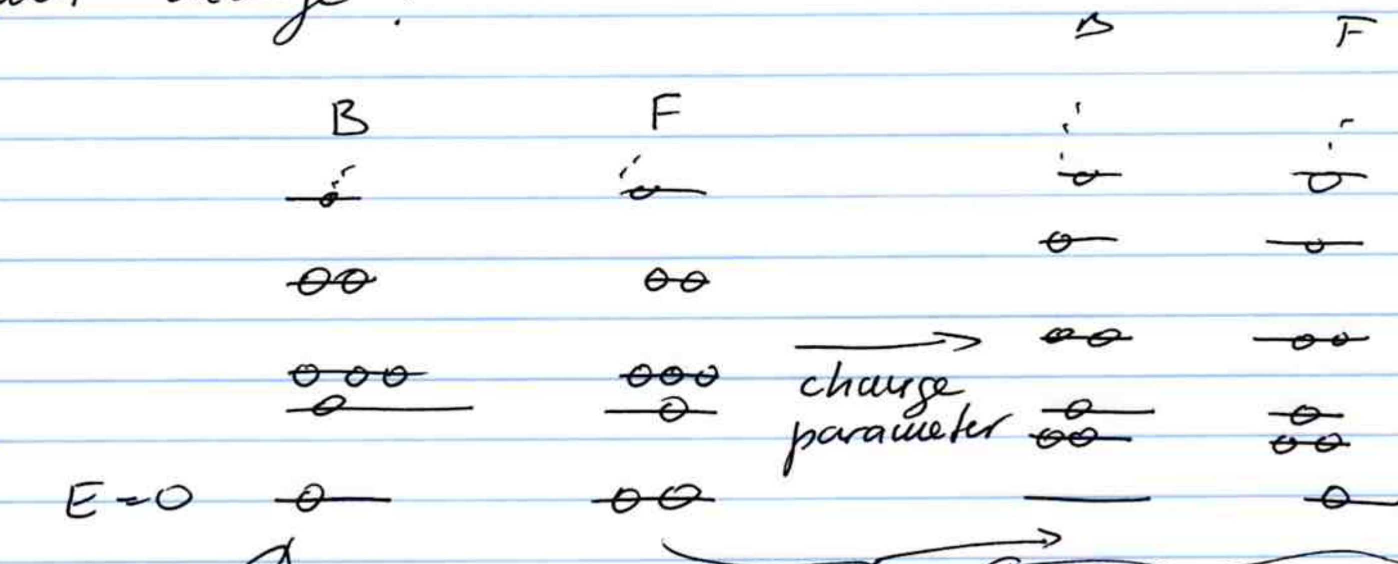
* Even more so, $\mathrm{tr}(-)^F$ is "easy" to calculate in many cases because it is a topological quantity, or more precisely an "index"; and is invariant under continuous deformations of the theory.

↳ what we mean by that?

- change of couplings ("deformation"); since inv., can calculate @ weak coupling and use to argue behavior @ strong coupling, for one... or at finite volume & deduce $V \to \infty$...

the reason for deformation inv. is that, since we already argued, all states of $E > 0$ are partnered (each B has an F)

- So as one varies parameters, $E>0$ energies can change, as well as the number of those states (but always paired!)
- but $E=0$ states can become $E>0$ only in pairs, so $n_B(0) - n_F(0)$ can not change!

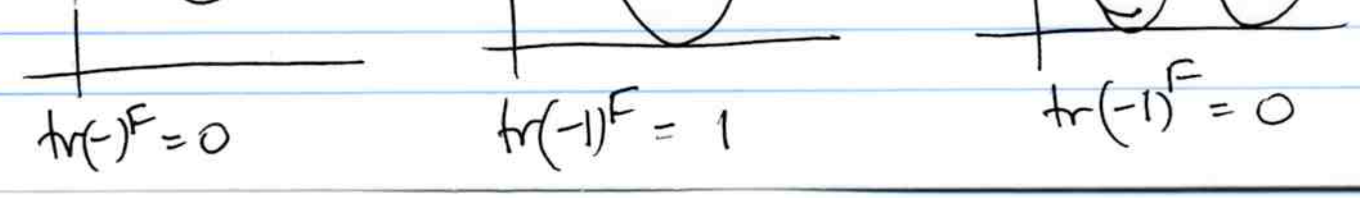


(—o— denotes # of states of that E)

two $E=0$ states became $E>0$ but in a B–F pair!

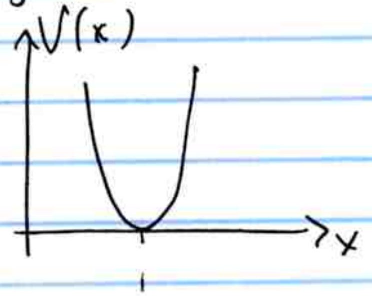
QED

Ex:

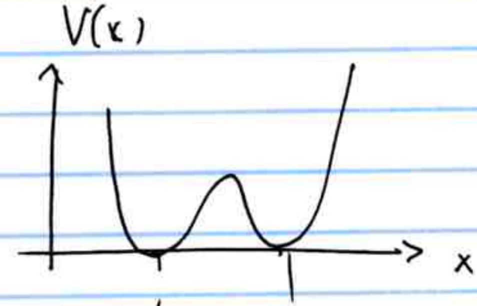


$\mathrm{tr}(-)^F = 0$ $\qquad$ $\mathrm{tr}(-1)^F = 1$ $\qquad$ $\mathrm{tr}(-1)^F = 0$

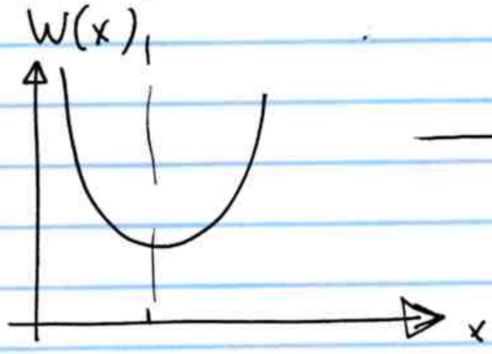
changing



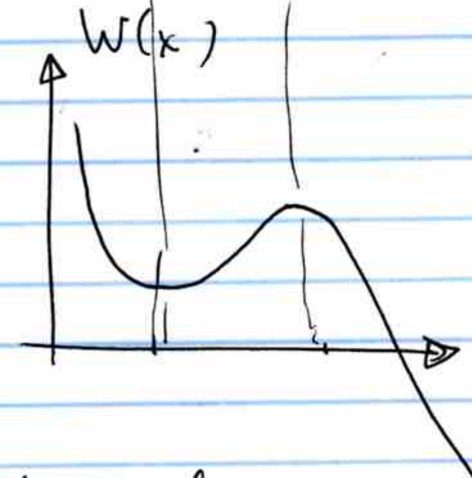
to



means changing



→



which changes the asymptotics of $W(x)$ @ $|x| \to \infty$ and leads to the appearance of new vacua "from infinity".

In QFT $\mathrm{tr}(-)^F$ can be calculated in some cases; in SYM w/ vectorlike matter ONLY it has been calculated (finite small Volume, weak-coupling calc'n) and it is not $= 0 \to$

→ so vectorlike matter ONLY SYM does not break SUSY.

→ most SUSY theories are chiral (exception ∃...)

Change gears $\Longrightarrow$ want to study QFT ---
$\Longrightarrow$ to make QM look like $d=1$ QFT, let's work out Lagrangian ---

$$H = \frac{1}{2} p^2 + \frac{1}{2}\left(W'(x)\right)^2 + \frac{1}{2} W''(x)\, \sigma_3$$

now, note:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\bar{\Psi}} \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\Psi} - \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\Psi} \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\bar{\Psi}}$$

$$\Psi \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \; ; \quad \Psi^2 = 0$$

$$\bar{\Psi} = (\Psi)^{+} \Longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; ; \quad \bar{\Psi}^2 = 0$$

$$\{\Psi, \bar{\Psi}\} \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\{\Psi, \bar{\Psi}\} = 1$$

So, $$H = \frac{1}{2} p^2 + \frac{1}{2}\left(W'(x)\right)^2 + \frac{1}{2} W''(x) \underbrace{\left(\bar{\Psi}\Psi - \Psi\bar{\Psi}\right)}$$

this looks like a Dirac hamiltonian; mass term only since in $d=1$ no $\vec{\nabla}$ available!

$\{\Psi, \bar{\Psi}\} = 1$ means that $i\bar{\Psi} = \Pi_\Psi$ so fermion term is $\frac{1}{2} W''(\bar{\Psi}\Psi - \Psi\bar{\Psi}) =$

$$= -\frac{i}{2} W''(x)\left(\Pi_\Psi \Psi - \Psi \Pi_\Psi\right)$$

Using $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$,

find that $\dot{\psi} = \frac{\partial H}{\partial \Pi_\psi} = -i\, W''(x)\, \psi$

& that

$$\mathcal{L} = \Pi_\psi \dot{\psi} - H = (\text{keep } \psi, \bar{\psi}\text{- part ONLY}) =$$

$$= i\, \bar{\psi}\dot{\psi} - \frac{1}{2} W''(x)(\bar{\psi}\psi - \psi\bar{\psi}), \quad \text{so } (+\textstyle\int \text{parts})$$

$$\mathcal{L}_F = \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \frac{1}{2} W''(x)(\bar{\psi}\psi - \psi\bar{\psi}), \text{ so}$$

$$\boxed{\mathcal{L}_{\substack{SUSY\\QM}} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(W'(x))^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \frac{1}{2} W''(x)(\bar{\psi}\psi - \psi\bar{\psi})}$$

→ here, of course $\bar{\psi}\psi - \psi\bar{\psi} = 2\bar{\psi}\psi$, this is a "classical" $\mathcal{L}$ of "1d SUSY QFT" that includes classical grassman field $\psi(t)$, $\bar{\psi}(t)$, $\{\psi, \bar{\psi}\} = 0$, $\psi(t)^2 = \bar{\psi}(t)^2 = 0$.

[note also, $(\psi\chi)^+ = \chi^+\psi^+$ $\forall\, \chi, \psi$ - grassman
$\frac{\partial}{\partial\psi}(\psi\chi) = \chi$, $\frac{\partial}{\partial\chi}(\psi\chi) = -\psi$; $\psi\chi = -\chi\psi$]

CHECK:

① $\mathcal{L}$ is inv't under

$$\delta x = \epsilon \bar{\psi} - \bar{\epsilon} \psi$$

$$\delta \psi = \epsilon \left(i \dot{x} + h'(x) \right)$$

$$\delta \bar{\psi} = \bar{\epsilon} \left(-i \dot{x} + h'(x) \right), \qquad \bar{\epsilon} = (\epsilon)^{\dagger}.$$

(or, more precisely $\delta L = \frac{\partial}{\partial t}(--)$ so S is inv't.)

② $$\left[\delta_1, \delta_2 \right](\dots) = 2i \left(\epsilon_1 \bar{\epsilon}_2 - \epsilon_2 \bar{\epsilon}_1 \right) \frac{\partial}{\partial t}(\dots)$$

Technology to find this $\mathcal{L}$, (& other,) (& generalize it!)

$\Rightarrow$ superfields

SUSY algebra $$\{Q, \bar{Q}\} = 2H$$ $$Q^2 = \bar{Q}^2 = 0$$ (in $Q, \bar{Q}$, not $Q_{1,2}$ basis; $2H$ instead of H

recall $H \leftrightarrow -i \frac{\partial}{\partial t}$: generator of time translations

$$\left.\begin{aligned} \{Q, \bar{Q}\} &= -2i \frac{\partial}{\partial t} \\ Q^2 = \bar{Q}^2 &= 0 \end{aligned}\right\} \sim \text{"square root" of } \frac{\partial}{\partial t}$$

idea of <u>superspace</u> : just like $H = -i\frac{\partial}{\partial t}$, translation in space, add new "coordinates" to space, which are translated by $Q, \bar{Q}$; these have to be fermionic, as many as there are supercharges:

$(t, \theta, \bar{\theta})$ – coordinates of a point in superspace

$$\theta^2 = \bar{\theta}^2 = 0 \,, \{\theta, \bar{\theta}\} = 0$$

$$\bar{\theta} = (\theta)^+$$

realize the algebra $\{Q, \bar{Q}\} = 2H$, $Q^2 = \bar{Q}^2 = 0$ i.t.o. differential operators acting on functions of $(t, \theta, \bar{\theta})$:

$$H = -i\frac{\partial}{\partial t}$$

CHECK!

$$Q = \frac{\partial}{\partial \theta} + i\bar{\theta}\frac{\partial}{\partial t} \quad , \quad Q^2 = 0$$

$$\bar{Q} = -\frac{\partial}{\partial \bar{\theta}} - i\theta\frac{\partial}{\partial t} \quad , \quad \bar{Q} = (Q)^+; \ \bar{Q}^2 =$$

these obey $\{Q, \bar{Q}\} = -2i\frac{\partial}{\partial t}$:

$$Q\bar{Q} = \left(\frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}\right)\left(-\frac{\partial}{\partial\bar{\theta}} - i\,\theta\frac{\partial}{\partial t}\right) =$$

$$+ \quad = -\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}} - i\frac{\partial}{\partial t} + i\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial t} - i\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial t} + \bar{\theta}\theta\frac{\partial^2}{\partial t^2}$$

$$\bar{Q}Q = -\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta} - i\frac{\partial}{\partial t} + i\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial t} - i\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial t} + \theta\bar{\theta}\frac{\partial^2}{\partial t^2}$$

$$\{Q\,\bar{Q}\} = -2i\frac{\partial}{\partial t}$$

---

there op's act on fns of $(t, \theta, \bar{\theta})$: what's that??

def'd by Taylor exp. in $\theta, \bar{\theta}$:

$$f(t,\theta,\bar{\theta}) = \underbrace{f_0(t) + \theta f_1(t) + \bar{\theta} f_2(t) + \theta\bar{\theta} f_3(t)}$$

With a θ & a $\bar{\theta}$ only, the most one can have is <u>4 fields</u> (f-ns of t) collected into one <u>superfield</u> $f(t,\theta,\bar{\theta})$

there can be restrictions on the type of $f(t,\theta,\bar{\theta})$ — whether $\mathbb{R}$ or $\mathbb{C}$ etc, <u>B or F</u>, we'll take <u>real superfields</u>, <u>bosonic</u>

---

real bosonic SF $\underset{\sim}{\phi(t,\theta,\bar\theta)}$:

$$\phi(t,\theta,\bar\theta) = x(t) + \theta\bar\Psi(t) - \bar\theta\Psi(t) + \theta\bar\theta f(t)$$

$\boxed{\text{SHOW}}$: ϕ is real!

($x, \psi, \bar\psi$ appear in our $\mathcal{L}$; f doesn't ... we'll see..

SUSY transforms on ϕ:

generated by $\delta = \epsilon Q - \bar\epsilon\bar Q \equiv \delta_\epsilon + \delta_{\bar\epsilon}$

where $\bar\epsilon = (\epsilon)^+$ are Grassman SUSY transform. parameters & $Q \equiv \frac{\partial}{\partial\theta} + i\bar\theta\frac{\partial}{\partial t}$, $\bar Q = (Q)^+$.

are the SUSY generators.

The transforms on the component fields are easy to find:

$$\delta\phi(t,\theta,\bar\theta) = (\epsilon Q - \bar\epsilon\bar Q)\phi(t,\theta,\bar\theta) =$$

$$= \left(\begin{smallmatrix}\text{since result}\\ \text{is a SF, still}\end{smallmatrix}\right) = \delta x(t) + \theta\,\delta\bar\Psi(t) - \bar\theta\,\delta\Psi(t) + \theta\bar\theta\,\delta f(t)$$

by comparing l.h.s. w/ r.h.s.!

USEFUL EXCERSIZE: show that $\delta_\epsilon\phi$ yields:

$$\left| \begin{aligned} \delta_\epsilon x &= \epsilon \bar{\Psi} \\ \delta_\epsilon \bar{\Psi} &= 0 \\ \delta_\epsilon \Psi &= -i\epsilon \dot{x} - \epsilon f \\ \delta_\epsilon f &= -i\epsilon \bar{\Psi} \end{aligned} \right|$$

& find $\delta_{\bar{\epsilon}}(\cdots)$ by c.c.!

Just like for ordinary fields, a products of SFs is a SF, a f-n of SF – a SF, etc...

Consider the following <u>functional of $\phi(t,\theta,\bar{\theta})$</u>:

$$S = \int dt\, d\theta\, d\bar{\theta}\, \mathcal{F}(\phi_i(t,\theta,\bar{\theta}))$$

(many fields!)

arbitrary f-n

How does S transform under $\phi_i \to \phi_i + \delta\phi_i$?

$$\delta S = \int dt\, d\theta\, d\bar{\theta} \left(\mathcal{F}(\phi(t,\theta,\bar{\theta}) + \delta\phi) - \mathcal{F}(\phi) \right) =$$

$$= \int dt\, d\theta\, d\bar{\theta}\, \delta\phi_i \frac{\partial \mathcal{F}}{\partial \phi_i}$$

consider δ_ϵ (add $\delta_{\bar{\epsilon}}$ yourself!) $\longrightarrow$

$$\delta_\epsilon S = \int dt\, d\theta\, d\bar\theta \; \delta_\epsilon \phi_i \frac{\partial \mathcal{F}}{\partial \phi_i} =$$

$$= \int dt\, d\theta\, d\bar\theta \left(\epsilon \frac{\partial}{\partial\theta} \phi_i + i \epsilon \bar\theta \frac{\partial}{\partial t} \phi_i \right) \frac{\partial \mathcal{F}}{\partial \phi_i}$$

$$= \int dt\, d\theta\, d\bar\theta \left(\epsilon \frac{\partial}{\partial \theta} + i \epsilon \bar\theta \frac{\partial}{\partial t} \right) \mathcal{F}(\phi(t,\theta,\bar\theta))$$

Now, recall that $\int d\theta = 0$

$\int d\theta\, \theta = 1$

So, 1 $\sim \frac{\partial}{\partial\theta}$ -term vanishes, since $\frac{\partial}{\partial\theta}$ gets rid of θ & nothing's there to 'soak' up $\int d\theta$.

2 $\sim \frac{\partial}{\partial t}$ - term is a total (space-)time derivative — drop

[periodic b.c.; vanishing fields @ $|x_\mu| \to \infty$; boundary term $\to$!]

[<u>Var'n of a SUSY action always a total derivative !</u>]

Same for $\delta_{\bar\epsilon}$! $\Longleftrightarrow$ $\int dt\, d\theta\, d\bar\theta\, \mathcal{F}(\phi_i(t,\theta,\bar\theta))$ is inv't under $\phi_i \to \phi_i + \delta\phi_i$

hence, integrals over the full superspace $(t, \theta, \bar\theta)$
(or, in $d>1$, $(x^\mu, \theta, \dots, \bar\theta, \dots)$)
of f-us of superfields (local) are
invariant under SUSY transforms of
the fields. this is the main principle
we write SUSY actions ((these days, in many
cases...))

For our single superfield case consider the
following* superspace action:

$$S_{(W)} = \int dt \, d\theta \, d\bar\theta \; W(\phi(t, \theta, \bar\theta))$$

to see the physical (component) action, need to
evaluate the $\theta, \bar\theta$ integrals; another $\boxed{\text{USEFUL}}$
$\boxed{\text{EXCERSIZE}}$!!

$$W(\Phi) = W(x) + (\theta\bar\Psi - \bar\theta\Psi) W'(x) + \theta\bar\theta f W'(x)$$
$$+ W''(x) \bar\Psi \Psi \theta\bar\theta$$

$\int d\theta \, d\bar\theta \; \theta\bar\theta = -1$, otherwise 0, so only $\theta\bar\theta$ parts matter

& we find

$$S_{(W)} = \int dt\, d\theta\, d\bar\theta\, W(\phi) = \int dt \left(-f W'(x) - W''(x) \bar\Psi \Psi \right)$$

$$\int dt\, \mathcal{L}(\phi)$$

$$\mathcal{L}(\phi) = \int d\theta\, d\bar\theta\, W(\phi)$$

$$= \int \text{"} d^2\theta \text{"}\, W(\phi)$$

this is the component action corresponding to $\mathcal{L}_{(W)} \sim \int d^2\theta\, W(\phi)$

So, $W'' \bar\Psi \Psi$ part looks right but:

(a) we've got $f W'$ as well (?)

(b) where are the kinetic terms ??

we want to be able to reproduce at least $\mathcal{L}_{\substack{SUSY \\ QM}}$ of p. 19 !!

Some dim analysis always helps!

$$\boxed{\begin{aligned} [t] &= -1 \\ [\theta] &= -1/2 \\ [d\theta] &= +\tfrac{1}{2} \\ \left(\int d\theta\, \theta = 1 \right) & \\ \left[\tfrac{\partial}{\partial\theta}\right] &= +\tfrac{1}{2} \end{aligned}}$$

$[x(t)] = -\frac{1}{2}$ so that $\left(\left[\int dt\, \dot{x}^2 \right] = 0 \right)$

$[\Psi(t)] = 0$ so $\longrightarrow \left(\left[\int dt\, \bar\Psi \dot\Psi \right] = 0 \right)$

hence:

$$\left[\int dt\, d\theta\, d\bar\theta \right] = 0.$$

$$[\phi(t, \theta, \bar\theta)] = [x] = -\frac{1}{2}$$

→ so, to get kinetic terms, we need smthing bilinear in ϕ of dim 0:

* $\underset{\substack{\downarrow \\ \text{mass parameter of dim 1}}}{m}\,\phi\,\phi$ is already in $W(\phi)$

* $\left(\frac{\partial}{\partial\theta}\phi\right)\left(\frac{\partial}{\partial\theta}\phi\right)^{+}$ has right dim'n.

<u>Problem</u>: $\phi'(t,\theta,\bar\theta)=\frac{\partial}{\partial\theta}\phi(t,\theta,\bar\theta)$ is <u>NOT a superfield</u>.

e.g. under SUSY $\delta_{\bar\epsilon}\phi = -\bar\epsilon\bar Q\,\phi$, $\forall$ superfield (ϕ, ϕ^2, $\phi_1\phi_2$, ...)

however $\delta_{\bar\epsilon}\phi' = -\frac{\partial}{\partial\theta}(\bar\epsilon\bar Q\,\phi)$

$\neq -\bar\epsilon\bar Q\left(\frac{\partial}{\partial\theta}\phi\right)$ because $\left\{\frac{\partial}{\partial\theta},\bar Q\right\}\neq 0$

in other words $\int d\theta\, d\bar\theta\, \left(\frac{\partial}{\partial\theta}\phi\right)\left(\frac{\partial}{\partial\bar\theta}\phi\right)$ will not be SUSY-c.

<u>Way out</u>: <u>Covariantize $\frac{\partial}{\partial\theta}$!</u>

$\frac{\partial}{\partial\theta}\longrightarrow D$, such that $\{D,\bar Q\}=\{D,Q\}=0$

$\partial/\partial\bar\theta\longrightarrow \bar D$, s.t. $\{\bar D,Q\}=\{\bar D,\bar Q\}=0$.

by demanding $\left\{ \frac{\partial}{\partial \theta} + ?.., \bar{Q} \right\} = 0$, etc..,

$\boxed{\text{SHOW:}}$ that the correct covt. derivatives

are

$$\mathcal{D} = \frac{\partial}{\partial \theta} - i \bar{\theta} \frac{\partial}{\partial t}$$

$$\bar{\mathcal{D}} = -\frac{\partial}{\partial \bar{\theta}} + i \theta \frac{\partial}{\partial t}$$

obeying $\{D, Q\} = \{D, \bar{Q}\} = \{\bar{D}, Q\} = \{\bar{D}, \bar{Q}\} = 0$

& show that $\{D, \bar{D}\} = 2i \frac{\partial}{\partial t}$; also $D^2 = \bar{D}^2 = 0$.

(as opposed to $\{Q \bar{Q}\} = -2i \, \partial/\partial t$)

So, since $\{D, Q\} = \{D, \bar{Q}\} = 0$, we can

argue that $S_{kin.} = \int dt \, d\theta \, d\bar{\theta} \; \mathcal{D}\phi \, \bar{\mathcal{D}} \phi \, (\#)$ ← to be fixed

is
- supersymmetric
- real

(& has right dim'n)

Let's work it out:

$$D\phi = \bar{\psi} + \bar{\theta}(f - i\dot{x}) - i \bar{\theta}\theta \dot{\bar{\psi}}$$

$$\bar{D}\phi = \psi + \theta(f + i\dot{x}) + i \bar{\theta}\theta \dot{\psi}$$

$$\int d\theta \, d\bar{\theta} \, D\phi \, \bar{D}\phi \Big|_{\text{bosonic only}} = \int d\theta \, d\bar{\theta} \, \bar{\theta}\theta \, (f - i\dot{x})(f + i\dot{x}) = f^2 + \dot{x}^2$$

hence $\# = \frac{1}{2}$

So we have

$$S[\phi] = \int dt\, d\theta\, d\bar{\theta} \left(\frac{1}{2} D\phi \bar{D}\phi + W(\phi) \right)$$

$$= \underbrace{\int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2} f^2 - f W'(x) - W''(x)\bar{\psi}\psi + i\bar{\psi}\dot{\psi} \right)}$$

(almost) same as $\mathcal{L}$ of p. 19

$\hookrightarrow$? f

- SUSY measure is (in path $\int$)

$$\mathcal{D}x\, \mathcal{D}f\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}$$

- f appears w/out derivatives $\rightarrow$ $\int$ out

$$f = W' \Rightarrow \frac{1}{2}f^2 - fW' = -\frac{1}{2}\left(W'(x)\right)^2$$

& then we get EXACTLY $\mathcal{L}$ of p. 19

"f" $\Leftrightarrow$ auxiliary field, SUSY algebra closes off-shell w/out use of E.O.M., then.

– Clearly, above action NOT most general one; e.g.

any $f(\alpha\phi)\, D\phi\, \bar{D}\phi$ is allowed

$[\alpha] = +1/2$

$\underbrace{\phantom{f(\alpha\phi)\, D\phi\, \bar{D}\phi}}$
nonlinear "sigma" model (QM on a non-flat manifold) ...

Lecture "2" $N=1$ 4d SUSY

1. algebra
2. chiral and vector superfields
3. chiral SF actions
4. gauge theory actions

Why $N=1$ 4d SUSY?

- lowest (# of SUSY's) algebra in 4d
- only SUSY theories allowing chiral fermions
- dynamically can break supersymmetry & generate small scales — — hierarchy problem
- lots of interesting dynamics — confinement, symmetry breaking, Seiberg dualities ...

1. algebra

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2i\,\sigma^m_{\alpha\dot\alpha} P_m \qquad [P_m, Q] = 0$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \qquad [P_m, \bar{Q}] = 0$$

$\alpha, \dot\alpha$: $SL(2,\mathbb{C})$ indices, $\psi_\alpha \to \psi'_\alpha = M_\alpha{}^\beta \psi_\beta$ under Lorentz $SL(2,\mathbb{C}$

[Wess & Bagger notations throughout.]