Exact solutions in supergravity

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1 July 2005

Lecture 1: Introduction; Supergravities in ten and eleven dimensions
Lecture 2: Supersymmetric vacua, holonomy and Killing spinors
Lecture 3: BPS branes and backgrounds with fluxes

Lecture 3 Outline

- $D = 11$ supergravity with $F_{(4)} \neq 0$

  The $AdS_4 \times S^7$ (Freund-Rubin) background

  The supergravity M2-brane

- Generalized holonomy when $F_{(4)} \neq 0$

  A look at the M2-brane

- The LLM bubbling AdS construction

  G-structure analysis

  Reduction of IIB on $S^3 \times S^3$

  Obtaining the solution

Turning on fluxes

- We now consider the general case with charges or fluxes turned on

  For example, we can consider $D = 11$ supergravity

  $$\hat{\nabla}_M \epsilon = 0 \qquad \hat{\nabla}_M = \nabla_M + \tfrac{1}{288}({\Gamma_M}^{NPQR} - 8\delta_M^N \Gamma^{PQR})F_{NPQR}$$

- Different approaches to solving this system
  - Ansatz based on symmetry
  - Generalized holonomy
  - G-structure analysis
  - Combination of methods, etc.

Use of symmetry: the Freund-Rubin ansatz

- The presence of $F_{(4)}$ hints at a natural compactification to four dimensions, $\mathcal{M}^{11} = \mathcal{M}^4 \times X^7$

$$d\hat{s}_{11}^2 = ds_4^2 + ds_7^2(X^7) \qquad F_{(4)} = m\epsilon_{(4)}$$

- Solve the equations of motion

$F_{(4)}$ eom: $\quad dF_{(4)} = 0$ and $d * F_{(4)} + \frac{1}{2} F_{(4)} \wedge F_{(4)} = 0$

$$\Rightarrow \qquad m = \text{constant}$$

Einstein: $\quad R_{MN} = \frac{1}{12}(F_{MN}^2 - \frac{1}{12} g_{MN} F^2)$

Decompose $D = 11$ indices:

$$\mu, \nu, \ldots = 0, 1, 2, 3 \qquad m, n, \ldots = 4 \ldots 10$$

Freund-Rubin: the bosonic solution

- Use $F^2_{\mu\nu} = -6m^2 g_{\mu\nu}$ and $F^2 = -24m^2$ to obtain

$$\begin{aligned} R_{\mu\nu} &= -3(m/3)^2 g_{\mu\nu} && \text{AdS}_4 \\ R_{mn} &= 6(m/6)^2 g_{mn} && \text{Einstein (e.g. } S^7\text{)} \end{aligned}$$

- Maximum symmetry for the $4+7$ split is

$$\mathcal{M}^{11} = \text{AdS}_4 \times S^7$$

- There are three other interesting cases
  - A $7+4$ split gives $\text{AdS}_7 \times S^4$
  - IIB theory with self-dual $F_{(5)}$ gives $\text{AdS}_5 \times S^5$
  - The D1-D5 system gives $\text{AdS}_3 \times S^3 \times T^4$

Freund-Rubin: Killing spinors

- To explore supersymmetry, we reduce

$$\hat{\nabla}_M = \nabla_M + \tfrac{1}{288}(\Gamma_M{}^{NPQR} - 8\delta_M^N \Gamma^{PQR})F_{NPQR}$$

along the μ and m directions to obtain

$$\begin{aligned} \hat{\nabla}_\mu &= \nabla_\mu - \tfrac{1}{36} m \epsilon_{\mu\nu\rho\sigma}\Gamma^{\nu\rho\sigma} \\ \hat{\nabla}_m &= \nabla_m + \tfrac{1}{288} m \epsilon_{\mu\nu\rho\sigma}\Gamma_m \Gamma^{\mu\nu\rho\sigma} \end{aligned}$$

Introduce $\Gamma^5 = i\Gamma^{0123}$

$$\hat{\nabla}_\mu = \nabla_\mu + \tfrac{i}{2}(m/3)\Gamma_\mu \Gamma^5 \qquad \hat{\nabla}_m = \nabla_m - \tfrac{i}{2}(m/6)\Gamma_m \Gamma^5$$

Killing spinor in AdS — Killing spinor on spheres

Freund-Rubin: Killing spinors

- The basic solution, AdS$_4 \times S^7$, has maximal supersymmetry
- Of course, we could replace S^7 by a manifold with reduced supersymmetry

  7-dimensional Einstein manifold with 'weak' G_2 holonomy yields $\mathcal{N} = 1$ in $D = 4$

- Conversely, we can ask how many backgrounds there are preserving maximal supersymmetry

  Answered by J. Figueroa-O'Farrill and G. Papadopoulos, JHEP 0303, 048 (2003)

  1. $\mathbb{M}^{1,10}$ Flat Minkowski space
  2. AdS$_4 \times S^7$ Freund-Rubin
  3. AdS$_7 \times S^4$
  4. Hpp pp-wave geometry $\quad F_{(4)} = \mu\, dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3$

  $$ds^2_{11} = 2dx^+dx^- - \mu^2(\vec{x}^{\,2} + \tfrac{1}{4}\vec{y}^{\,2})(dx^-)^2 + d\vec{x}^{\,2}_3 + d\vec{y}^{\,2}_6$$

Maximal supersymmetry in $D = 11$

- Classification performed through integrability

  Define $M_{MN} \equiv [\hat{D}_M, \hat{D}_N] \equiv \frac{1}{4}\mathcal{R}$

  For maximal supersymmetry, $\mathcal{R}\epsilon$ vanishes for all ϵ

  This implies the matrix $\mathcal{R} = c_M \Gamma^M + c_{MN}\Gamma^{MN} + \cdots + c_{MNPQR}\Gamma^{MNPQR}$ must vanish

  $$\longrightarrow \qquad c_M, c_{MN}, \ldots, c_{MNPQR} \quad \text{all vanish}$$

- Working out the consequences of these conditions is sufficient to prove the result

Use of symmetry: the M2-brane

- The 3-form potential $A_{(3)}$ of $D = 11$ supergravity couples naturally to the worldsheet of the eleven-dimensional supermembrane (M2-brane)
- The M2-brane carries mass (G_{MN}) and charge (electric $F_{(4)}$)

$$\text{BPS} \quad \longrightarrow \quad \text{mass} = \text{charge}$$

- We obtain the M2 geometry by considering a longitudinal + transverse split, $\mathcal{M}^{11} = \mathcal{M}^3 \times \mathcal{M}^8$

Take the ansatz: $ds_{11}^2 = e^{2A(y)} dx_\mu^2 + e^{2B(y)} dy_i^2$

$$F_{(4)} = dx^0 \wedge dx^1 \wedge dx^2 \wedge de^{2C(y)}$$

Goal: obtain a supersymmetric solution for $A(y)$, $B(y)$ and $C(y)$

The M2-brane: supersymmetry

- We may work out the Killing spinor equations

  In the longitudinal direction

$$\begin{aligned}\hat{\nabla}_\mu &= \partial_\mu + \tfrac{1}{2}\omega_{\mu\nu i}\Gamma^{\nu i} - \tfrac{1}{12}F_{\mu\nu\rho i}\Gamma^{\nu\rho i}\\ &= \partial_\mu + \tfrac{1}{6}\Gamma_\mu\Gamma^i e^{-3A}\partial_i(e^{3A} + e^{2C}\Gamma^{012})\end{aligned}$$

  In the transverse direction

$$\begin{aligned}\hat{\nabla}_i &= \partial_i + \tfrac{1}{4}\omega_{ijk}\Gamma^{jk} + \tfrac{1}{72}F_{\mu\nu\rho j}\Gamma_i{}^{\mu\nu\rho j} - \tfrac{1}{36}F_{i\mu\nu\rho}\Gamma^{\mu\nu\rho}\\ &= \partial_i + \tfrac{1}{6}e^{-3A}\partial_i e^{2C}\Gamma^{012} + \tfrac{1}{2}\Gamma_i{}^j(\partial_j B - \tfrac{1}{6}e^{-3A}\partial_j e^{2C}\Gamma^{012})\end{aligned}$$

- We may build a projection by taking $e^{3A} = e^{2C}$ and $B = -\frac{1}{3}C$

$$\Rightarrow \qquad e^{3A} = e^{-6B} = e^{2C}$$

The M2-brane: supersymmetry

- The resulting expressions for the supercovariant derivative are

$$\begin{aligned} \hat{\nabla}_\mu &= \partial_\mu + \tfrac{2}{3}\Gamma_\mu \Gamma^i \partial_i C \, P_+ \\ \hat{\nabla}_i &= \partial_i - \tfrac{1}{3}\partial_i C - \tfrac{1}{3}(\Gamma_i{}^j - 2\delta_i^j)\partial_j C \, P_+ \end{aligned}$$

where $P_\pm = \frac{1}{2}(1 \pm \Gamma^{012})$

- Killing spinors are easily obtained

$$\epsilon = e^{\frac{1}{3}C} P_- \epsilon_0$$

Note that partial supersymmetry does not imply the equations of motion . . . only that the metric and form-field are given in terms of a single function $C(y)$

The M2-brane: supergravity solution

- To obtain a solution, we impose (one) equation of motion

$$d * F_{(4)} + \tfrac{1}{2} F_{(4)} \wedge F_{(4)} = 0 \quad \rightarrow \quad d * F_{(4)} = 0 \quad \rightarrow \quad d *_8 de^{-2C} = 0$$

Hence e^{-2C} is harmonic in transverse space

$$e^{-2C} = \mathcal{H} = 1 + \sum_i \frac{q_i}{|\vec{y} - \vec{y}_i|^6}$$

- The complete solution

$$ds_{11}^2 = \mathcal{H}^{-2/3} dx_\mu^2 + \mathcal{H}^{1/3} dy_i^2$$

$$F_{(4)} = dx^0 \wedge dx^1 \wedge dx^2 \wedge d\left(\frac{1}{\mathcal{H}}\right) \qquad\qquad \epsilon = \mathcal{H}^{-1/6} P_- \epsilon_0$$

Harmonic superposition for BPS objects

The M2-brane: BPS and time independence

- We may construct a vector from spinor bilinears

$$V^\mu = \bar{\epsilon}\Gamma^\mu\epsilon = \mathcal{H}^{-1/3}\bar{\epsilon}_0\Gamma^\mu\epsilon_0 = \bar{\epsilon}_0\Gamma^{\bar{\mu}}\epsilon_0$$

A constant vector pointing in the time direction, $V = \dfrac{\partial}{\partial t}$

$$\Gamma_M = \text{real} \qquad \Gamma^T_\mu = -C^{-1}\Gamma_\mu C \qquad C = \Gamma^0 \qquad \overline{\psi} = \psi^T C^{-1}$$

- We can prove that V^μ is a Killing vector

$$\begin{aligned} \nabla_M\epsilon &= -\tfrac{1}{288}(\Gamma_M{}^{NPQR} - 8\delta^N_M\Gamma^{PQR})\epsilon\, F_{NPQR} \\ \overline{\nabla_M\epsilon} &= \tfrac{1}{288}\bar{\epsilon}(\Gamma_M{}^{NPQR} + 8\delta^N_M\Gamma^{PQR})F_{NPQR} \end{aligned}$$

$$\rightarrow \qquad \nabla_M V_N = \tfrac{1}{6}F_{MNPQ}\bar{\epsilon}\Gamma^{PQ}\epsilon + \tfrac{1}{6!} * F_{MNPQRST}\bar{\epsilon}\Gamma^{PQRST}\epsilon$$

Generalized holonomy classification

- The presence of $F_{(4)}$ modifies the covariant derivative

$$\hat{\nabla}_M = \nabla_M + \tfrac{1}{288}({\Gamma_M}^{NPQR} - 8\delta^N_M \Gamma^{PQR})F_{NPQR} = \partial_M + \tfrac{1}{4}\Omega_M$$

where

$$\Omega_M = {\omega_M}^{AB}\Gamma_{AB} + \tfrac{1}{72}({\Gamma_M}^{NPQR} - 8\delta^N_M \Gamma^{PQR})F_{NPQR}$$

is a generalized connection taking values in $SL(32;\mathbb{R})$

- Introduce Generalized holonomy $H \subseteq SL(32;\mathbb{R})$ for the generalized connection Ω_M

$$[\mathcal{D}_M(\Omega), \mathcal{D}_N(\Omega)] = \tfrac{1}{4}\mathcal{R}_{MN}(\Omega)$$

The number of preserved supersymmetries = the number of singlets in the decomposition of the $\mathbf{32}$ of $SL(32;\mathbb{R})$ under H

The number of singlets

- For ϵ a 32-component spinor, we may choose a basis of Killing spinors

$$\epsilon = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad \text{etc.}$$

To preserve n supersymmetries, the generalized holonomy must act as

$$\mathcal{R} \in \begin{pmatrix} & & \\ & SL(32;\mathbb{R}) & \\ & & \end{pmatrix} \quad \rightarrow \quad \left(\begin{array}{c|c} \mathbf{1} & \mathbb{R}_{n,32-n} \\ \hline 0 & SL(32-n;\mathbb{R}) \end{array}\right)$$

Hence $H \subseteq SL(32-n;\mathbb{R}) \ltimes \oplus_n \mathbb{R}^{32-n}$

However H does not have to be so large, so long as the singlet counting is correct

Generalized holonomy for the M2-brane

- Given $\hat{\nabla}_M = \partial_M + \frac{1}{4}\Omega_M$, the M2 background yields the generalized connection

$$\Omega_\mu = \tfrac{8}{3}\partial_i C(\Gamma_\mu\Gamma^i P_+) \qquad \Omega_i = \tfrac{4}{3}\partial_i C\Gamma^{012} - \tfrac{4}{3}\partial_j C(\Gamma_i{}^j P_+)$$

- We may view these as Lie algebra generators

$$T_{ij} = \Gamma_{\overline{ij}}P_+ \qquad K_{\mu\, i} = \Gamma_{\overline{\mu}\overline{i}}P_+ \qquad K_{\mu\, ijk} = \Gamma_{\overline{\mu}\overline{ijk}}P_+$$

where $K_{\mu\, ijk}$ arises from second order integrability

- The result is

$$\mathsf{Hol}_{\mathrm{M2}} = SO(8)_+ \ltimes 12\mathbb{R}^{2(\mathbf{8}_s)} \subset SL(16;\mathbb{R}) \ltimes 16\mathbb{R}^{16}$$

G-structure (Killing tensor) analysis

- A powerful means of classifying and constructing new supersymmetric backgrounds was pioneered by Gauntlett, Gutowski, Martelli, Pakis, Sparks, Tod, Waldram. . .

  Given a Killing spinor ϵ, construct all possible tensors $T_{(n)} = \overline{\epsilon}\Gamma_{(n)}\epsilon$

  Killing spinor $\longrightarrow$ background isometries $\longrightarrow$ specialized coordinates

- Outline of the procedure

  0) Choose initial isometries of the background geometry

  1) Construct all possible spinor bilinears

  2) Derive the algebraic identities (Fierz relations) between bilinears

  3) Obtain the differential identities and identify additional symmetries

  4) Specialize the choice of coordinates and solve the appropriate equations

G-structures

- The resulting supersymmetric backgrounds may be classified according to G-structures

  a principle sub-bundle of the frame bundle with fiber in $GL(n,\mathbb{R})$

  $$\begin{array}{ccccccc} GL(6,\mathbb{R}) & \rightarrow & O(6) & \rightarrow & SO(6) & \rightarrow & SU(3) \\ & & \text{metric} & & \text{orientable} & & \text{Calabi-Yau } (J_{(2)},\ \Omega_{(3)}) \end{array}$$

  The failure of $\hat{\nabla}$ to be compatible with the G-structure is measured by the intrinsic torsion (in this case $dJ_{(2)}$ and $d\Omega_{(3)}$)

  See e.g. Gauntlett et al., hep-th/0411194

- We will not focus on the classification, but will instead illustrate the construction with an example

  Bubbling $\mathrm{AdS}_5 \times S^5$ geometry [Lin, Lunin and Maldacena, hep-th/0409174] (LLM)

Motivating the LLM construction

- By focusing on the near-horizon geometry of a stack of N D3-branes

  Strings on AdS$_5 \times S^5$ $\longleftrightarrow$ $\mathcal{N}=4$ super-Yang Mills

- Both sides have the identical isometry group

$$SU(2,2|4) \supset SO(2,4) \times SO(6)$$

- The AdS/CFT conjecture then relates
  - States in $\mathcal{N}=4$ super-Yang Mills
  - String configurations/giant gravitons in AdS$_5 \times S^5$
  - Exact supergravity backgrounds

- Investigate the 1/2 BPS sector of the theory

  . . . given by operators, states or configurations with $\Delta = J_1$

1/2 BPS states in $\mathcal{N} = 4$ SYM

- Consider the super-Yang-Mills fields A_μ, $4\chi^r$, $6\phi^i$

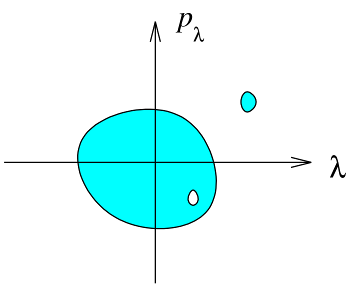
  1/2 BPS chiral primaries are built from operators with conformal dimensional equal to R-charge, $\Delta = J_1$

$$X = \phi^1 + i\phi^2, \qquad Y = \phi^3 + i\phi^4, \qquad Z = \phi^5 + i\phi^6$$

$$\mathrm{Tr}(X^J), \mathrm{Tr}(X^n)\mathrm{Tr}(X^{J-n}), \text{etc.}$$

- Reduce the system to matrix quantum mechanics $X(x^\mu) \to X(t)$

Free fermion phase space



1/2 BPS states in supergravity

- A $\Delta = J_1$ state still preserves $SO(4) \times SO(4)$ symmetry

$$SO(2,4) \times SO(6) \supset SO(2)_\Delta \times SO(4) \times SO(2)_{J_1} \times SO(4)$$

Consider writing AdS$_5 \times S^5$ as

$$\begin{aligned} ds_{10}^2 &= [-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2] + [\cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta\, d\widetilde{\Omega}_3^2] \\ &= [-\cosh^2\rho\, dt^2 + \cos^2\theta\, d\phi^2 + d\rho^2 + d\theta^2] + [\sinh^2\rho\, d\Omega_3^2 + \sin^2\theta\, d\widetilde{\Omega}_3^2] \end{aligned}$$

Giant gravitons rotate on S^5 (t,ϕ) but may expand either in AdS$_5$ (Ω_3) or S^5 ($\widetilde{\Omega}_3$)

- We thus seek a supergravity solution preserving $SO(4) \times SO(4)$ isometry

IIB sugra in $D = 10$ $\longrightarrow$ Effective $D = 4$ model

(breathing mode reduction on $S^3 \times S^3$)

IIB supergravity on $S^3 \times S^3$

- Start with IIB theory in ten dimensions

$$(g_{\mu\nu}, B_{(2)}, \phi, \quad C_{(0)}, C_{(2)}, C_{(4)})$$

- Focus on D3-branes dissolving into fluxes

$$\begin{aligned} ds_{10}^2 &= g_{\mu\nu} dx^\mu dx^\nu + e^H (e^G d\Omega_3^2 + e^{-G} d\widetilde{\Omega}_3^2) \\ F_{(5)} &= F_{(2)} \wedge d\Omega_3 - *_4 e^{-3G} F_{(2)} \wedge d\widetilde{\Omega}_3 \end{aligned}$$

- The reduction on $S^3 \times S^3$ yields an effective $D = 4$ Lagrangian

$$e^{-1}\mathcal{L}_4 = e^{3H}[R + \tfrac{15}{2}\partial H^2 - \tfrac{3}{2}\partial G^2 - \tfrac{1}{4} e^{-3(H+G)} F_{\mu\nu}^2 + 12 e^{-H} \cosh G]$$

- Use of symmetry reduces the system to a simpler one

$$g_{\mu\nu}, \quad H, \quad G, \quad F_{(2)} \qquad \text{in} \qquad D = 4$$

Effective $D = 4$ supersymmetry

- We reduce the IIB gravitino variation (only non-trivial variation)

$$\delta\psi_M = [\nabla_M + \tfrac{1}{16\cdot 5!} i F_{NPQRS}\Gamma^{NPQRS}\Gamma_M]\varepsilon$$

With the reduction ansatz, this turns into

$$\begin{aligned}
\delta\psi_\mu &= [\nabla_\mu - \tfrac{1}{16} e^{-\frac{3}{2}(H+G)} F_{\nu\lambda}\Gamma^{\nu\lambda}\Gamma^{(3)}\Gamma_\mu]\varepsilon \\
\delta\psi_a &= [\hat\nabla_a + \tfrac{1}{4}\Gamma_a\Gamma^\mu\partial_\mu(H+G) - \tfrac{1}{16} e^{-\frac{3}{2}(H+G)} F_{\mu\nu}\Gamma^{\mu\nu}\Gamma^{(3)}\Gamma_a]\varepsilon \\
\delta\psi_{\tilde a} &= [\hat\nabla_{\tilde a} + \tfrac{1}{4}\Gamma_{\tilde a}\Gamma^\mu\partial_\mu(H-G) - \tfrac{1}{16} e^{-\frac{3}{2}(H+G)} F_{\mu\nu}\Gamma^{\mu\nu}\Gamma^{(3)}\Gamma_{\tilde a}]\varepsilon
\end{aligned}$$

where $\Gamma^{(3)} = -i\Gamma^{456}$ lives on the first S^3 and we have used the fact that IIB spinors have definite Γ^{11} chirality

Effective $D = 4$ supersymmetry

- There is one final step in obtaining effective four-dimensional Killing spinor equations

  Choose a Dirac matrix decomposition

$$
\begin{aligned}
\Gamma_\mu &= \gamma_\mu \times 1 \times 1 \times \sigma_1 \\
\Gamma_a &= 1 \times \sigma_a \times 1 \times \sigma_2 \\
\Gamma_{\tilde{a}} &= \gamma_5 \times 1 \times \sigma_{\tilde{a}} \times \sigma_1
\end{aligned}
$$

  along with

$$
\varepsilon = \epsilon \times \chi \times \widetilde{\chi} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

- We also demand that η and $\widetilde{\eta}$ are Killing spinors on $S^3 \times S^3$

$$
[\hat{\nabla}_a + \tfrac{1}{2} i \eta \hat{\sigma}_a]\chi = 0 \qquad [\hat{\nabla}_{\tilde{a}} + \tfrac{1}{2} i \widetilde{\eta} \hat{\sigma}_{\tilde{a}}]\widetilde{\eta} = 0 \qquad (\eta, \widetilde{\eta} = \pm 1)
$$

The Killing spinor equations

- The result may be written as

$$\begin{aligned}
\delta\psi_\mu &= [\nabla_\mu + \tfrac{1}{16} i e^{-\frac{3}{2}(H+G)} F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu]\epsilon \\
\delta\chi_H &= [\gamma^\mu\partial_\mu H + e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} - i\widetilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\epsilon \\
\delta\chi_G &= [\gamma^\mu\partial_\mu G - \tfrac{1}{4} i e^{-\frac{3}{2}(H+G)} F_{\mu\nu}\gamma^{\mu\nu} + e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} + i\widetilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\epsilon
\end{aligned}$$

  This looks vaguely like $D = 4$ gauged supergravity coupled to matter

  (but this is not a consistent truncation)

- Our goal is to solve these Killing spinor equations in order to obtain all 1/2 BPS solutions

Supersymmetry analysis

- We now apply the G-structure (Killing tensor) analysis
  Gauntlett et al. CMP 247, 421 (2004); CQG 20, 4587 (2003);
  CQG 20, 5049 (2003); CQG 21, 4335 (2004)

- Start with the spinor bilinears

  For ϵ a four-dimensional Dirac spinor, we define the real quantities

$$f_1 = \bar{\epsilon}\gamma_5\epsilon \qquad f_2 = i\bar{\epsilon}\epsilon \qquad K^\mu = \bar{\epsilon}\gamma^\mu\epsilon \qquad L^\mu = \bar{\epsilon}\gamma^\mu\gamma_5\epsilon \qquad Y^{\mu\nu} = i\bar{\epsilon}\gamma^{\mu\nu}\gamma_5\epsilon$$

  In addition, we may also consider expressions involving $(\epsilon^c \cdots \epsilon)$ where $\epsilon^c = \epsilon^T C$

- The algebraic identities give relations between these tensors

$$L^2 = -K^2 = f_1^2 + f_2^2 \qquad \text{etc.}$$

  $\Rightarrow \quad K^\mu$ is timelike and L^μ is spacelike

The differential identities

- Start with the gravitino variation

$$\begin{aligned}\nabla_\mu \epsilon &= -\tfrac{1}{16} i e^{-\frac{3}{2}(H+G)} F_{\nu\lambda} \gamma^{\nu\lambda} \gamma_\mu \epsilon \\ \overline{\nabla_\mu \epsilon} &= \tfrac{1}{16} i e^{-\frac{3}{2}(H+G)} F_{\nu\lambda} \bar{\epsilon} \gamma_\mu \gamma^{\nu\lambda}\end{aligned}$$

- For $K_\nu = \bar{\epsilon}\gamma_\nu \epsilon$ we find

$$\nabla_\mu K_\nu = \tfrac{1}{4} e^{-\frac{3}{2}(H+G)} (f_2 F_{\mu\nu} - f_1 * F_{\mu\nu}) \qquad \rightarrow \qquad \nabla_{(\mu} K_{\nu)} = 0$$

so K^μ is a (timelike) Killing vector (take $K = \partial/\partial t$)

- For $L_\nu = \bar{\epsilon}\gamma_\nu \gamma_5 \epsilon$ we find

$$\nabla_\mu L_\nu = \tfrac{1}{4} e^{-\frac{3}{2}(H+G)} (\tfrac{1}{2} g_{\mu\nu} F_{\lambda\rho} Y^{\lambda\rho} - 2F_{(\mu}{}^{\lambda} Y_{\nu)\lambda}) \qquad \rightarrow \qquad dL_{(1)} = 0$$

so $L_{(1)}$ is a (spacelike) closed 1-form (take $L_{(1)} = dy$)

Pinning down f_1 and f_2

- The gravitino variation also yields

$$\partial_\mu f_1 = \tfrac{1}{4} e^{-\frac{3}{2}(H+G)} * F_{\mu\nu} K^\nu \qquad \partial_\mu f_2 = -\tfrac{1}{4} e^{-\frac{3}{2}(H+G)} F_{\mu\nu} K^\nu$$

magnetic — electric

- This may be combined with additional 'differential' identities obtained from $\delta\chi_H$ and $\delta\chi_G$

$$\begin{aligned} f_1 \partial_\mu (H - G) &= \tfrac{1}{2} e^{-\frac{3}{2}(H+G)} * F_{\mu\nu} K^\nu \\ f_2 \partial_\mu (H + G) &= -\tfrac{1}{2} e^{-\frac{3}{2}(H+G)} F_{\mu\nu} K^\nu \end{aligned}$$

$$\rightarrow \qquad d[e^{-\frac{1}{2}(H-G)} f_1] = 0 \qquad d[e^{-\frac{1}{2}(H+G)} f_2] = 0$$

$$\rightarrow \qquad f_1 = b e^{\frac{1}{2}(H-G)} \qquad f_2 = a e^{\frac{1}{2}(H+G)}$$

where a and b are constants

Specializing the metric

- We consider one more constraint from the $\delta\chi_H$ equation

$$\bar{\epsilon}: \qquad \gamma^\mu \partial_\mu H \epsilon = -e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} - i\widetilde{\eta}\gamma_5 e^{\frac{1}{2}G})\epsilon$$

$$\rightarrow \qquad K^\mu \partial_\mu H = i\left(\eta e^{-\frac{1}{2}(H+G)} f_2 + \widetilde{\eta} e^{-\frac{1}{2}(H-G)} f_1\right)$$

$$\rightarrow \qquad K^\mu \partial_\mu H = 0 \qquad \text{and} \qquad a\eta + b\widetilde{\eta} = 0$$

Choose, e.g., $a = b = \eta = -\widetilde{\eta} = 1$ or $a = b = \eta = -\widetilde{\eta} = -1$

This gives a $1/4 + 1/4 = 1/2$ BPS configuration

- This normalization yields

$$L^2 = -K^2 = f_1^2 + f_2^2 = e^H(e^G + e^{-G}) = 2e^H \cosh G$$

Specializing the metric

- We now have enough information to write down the metric in a convenient coordinate system (where $K = \partial/\partial t$ and $L_{(1)} = dy$)

$$ds_4^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(e^{2\gamma}(dx^i)^2 + dy^2)$$

Combining $L_{(1)} = dy$ with $L_{(1)} = de^H$ obtained from $\delta\chi_H$, we obtain

$$e^H = y \qquad h^{-2} = 2y\cosh G$$

- We may also perform a similar analysis on the 1-form $\omega_\mu = \epsilon^c \gamma_\mu \epsilon$ to find $d\omega = 0$ and that it has normalized components along x^1 and x^2

$\longrightarrow$ we are allowed to choose coordinates such that $\gamma = 0$

The form field $F_{(2)}$

- What remains is a determination of $F_{(2)}$ and dV

  Recall the differential identities

$$\partial_\mu f_1 = \frac{1}{4} e^{-\frac{3}{2}(H+G)} * F_{\mu\nu} K^\nu \qquad \partial_\mu f_2 = -\frac{1}{4} e^{-\frac{3}{2}(H+G)} F_{\mu\nu} K^\nu$$

magnetic electric

- With $f_1 = e^{\frac{1}{2}(H-G)}$ and $f_2 = e^{\frac{1}{2}(H+G)}$, we obtain

$$F_{(2)} = -de^{2(H+G)} \wedge (dt + V) - h^2 e^{3G} *_3 de^{2(H-G)}$$

  where we may substitute in $e^H = y$ and $h^{-2} = 2y \cosh G$

The metric vector $V_{(1)}$

- Finally, note that the antisymmetric part of the differential identity $\nabla_\mu K_\nu$ yields

$$dK = \tfrac{1}{2} e^{-\frac{3}{2}(H+G)} (f_2 F_{(2)} - f_1 * F_{(2)})$$

  This may be combined with $K = -h^{-2}(dt + V)$ to give

$$dV = -2h^4 e^H *_3 dG \qquad \text{or} \qquad dV = -\tfrac{1}{2} y^{-1} *_3 d\tanh G$$

- Define $z = \frac{1}{2}\tanh G$ so that $dV = -y^{-1} *_3 dz$
- We now obtain the consistency condition $d^2 V = 0$ or

$$d\left(\frac{1}{y} *_3 dz\right) = 0$$

Interpretation of the solution

- What have we learned?

$$
\begin{aligned}
ds^2 &= -h^{-2}(dt+V)^2 + h^2(dx_1^2 + dx_2^2 + dy^2) + y(e^G d\Omega_3^2 + e^{-G} d\widetilde{\Omega}_3^2) \\
F_{(2)} &= -\left[d(y^2 e^{2G}) \wedge (dt+V) + h^2 e^{3G} *_3 d(y^2 e^{-2G})\right]
\end{aligned}
$$

with

$$
h^{-2} = 2y\cosh G, \qquad z \equiv \tfrac{1}{2}\tanh G, \qquad dV = -y^{-1} *_3 dz
$$

where

$$
\left[\partial_1^2 + \partial_2^2 + y\,\partial_y \frac{1}{y}\partial_y\right] z(x_1, x_2, y) = 0 \qquad \text{harmonic in } \mathbb{H}_3
$$

or

$$
\left[\partial_1^2 + \partial_2^2 + \frac{3}{y}\partial_y + \partial_y^2\right]\left(\frac{z}{y^2}\right) = 0 \qquad \text{harmonic in } \mathbb{R}^2 \times \mathbb{R}^4
$$

Linear superposition and bubbling

- Underlying the 1/2 BPS solution is a linear system

$$\Box_6 \left(\frac{z(x_1, x_2, y)}{y^2} \right) = 0$$

- This admits a Green's function solution

$$z(\vec{x}, y) = \frac{y^2}{\pi} \int \frac{1}{(|\vec{x} - \vec{x}'|^2 + y^2)^2} z(\vec{x}', 0) d^2\vec{x}'$$

where boundary conditions are imposed at $y = 0$

- Because $ds^2 = \cdots + y(e^G d\Omega_3^2 + e^{-G} d\widetilde{\Omega}_3^2)$ boundary conditions must be chosen to ensure a regular solution

Regularity of the geometry at $y = 0$

- When $y \to 0$ the volume of $ds_6^2 = e^G d\Omega_3^2 + e^{-G} d\widetilde{\Omega}_3^2$ goes to zero
- To be smooth, only a single S^3 can collapse

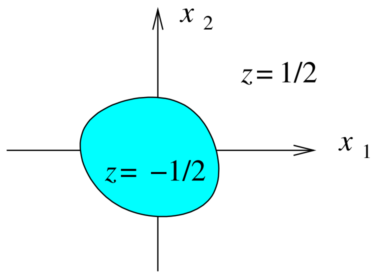
  Either $e^G \to 0$ or $e^{-G} \to 0$

  $\Rightarrow \quad G = \pm\infty$ or $z \equiv \frac{1}{2}\tanh G = \pm\frac{1}{2}$ as $y \to 0$

  Suppose $z \to \frac{1}{2}$ as $y \to 0$

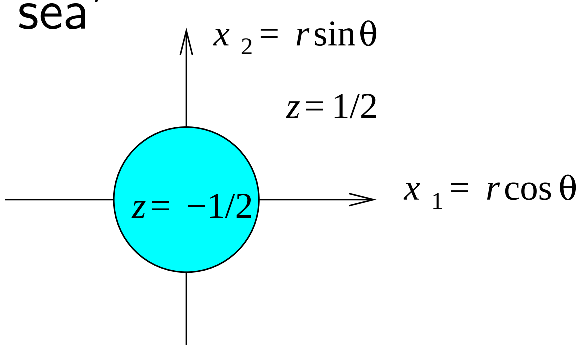
  Solving the harmonic equation gives an expansion $e^G \sim y^{-1}$
  so that $h^2 dy^2 + y(e^G d\Omega_3^2 + e^{-G} d\widetilde{\Omega}_3^2) \sim dy^2 + y^2 d\Omega_3^2 + d\widetilde{\Omega}_3^2$

- Boundary conditions: $z(x_1, x_2, 0) = \pm\frac{1}{2}$



Example: The AdS$_5 \times S^5$ background

- We recover AdS$_5 \times S^5$ by filling the 'Fermi sea'



$$z \equiv \frac{1}{2}\tanh G = \frac{r^2 + y^2 - \ell^2}{2\sqrt{(r^2+y^2-\ell^2)^2 + 4y^2\ell^2}}$$

$$V = -\frac{r^2 + y^2 + \ell^2}{2\sqrt{(r^2+y^2-\ell^2)^2 + 4y^2\ell^2}} d\phi$$

- Make the change of coordinates $y = \ell \sinh\rho \sin\theta \qquad r = \ell\cosh\rho\cos\theta$

$$\rightarrow \qquad z = \frac{1}{2}\frac{\sinh^2\rho - \sin^2\theta}{\sinh^2\rho + \sin^2\theta} \qquad e^G = \frac{\sinh\rho}{\sin\theta}$$

and $\quad h^{-2} = \ell(\cosh^2\rho - \cos^2\theta) \qquad V = -\frac{1}{2}\frac{\cosh^2\rho + \cos^2\theta}{\cosh^2\rho - \cos^2\theta}$

Example: The AdS$_5 \times S^5$ background

- Look at the metric

$$
\begin{aligned}
ds^2 &= -h^{-2}(dt+V)^2 + h^2(dr^2 + r^2 d\phi^2 + dy^2) + y(e^G d\Omega_3^2 + e^{-G} d\widetilde{\Omega}_3^2) \\
&= \ell\Big\{-(\cosh^2\rho - \cos^2\theta)\Big(dt - \frac{1}{2}\frac{\cosh^2\rho + \cos^2\theta}{\cosh^2\rho - \cos^2\theta} d\phi\Big)^2 + d\rho^2 + d\theta^2 \\
&\qquad + \frac{\cosh^2\rho\cos^2\theta}{\cosh^2\rho - \cos^2\theta} d\phi^2 + \sinh^2\rho\, d\Omega_3^2 + \sin^2\theta\, d\widetilde{\Omega}_3^2\Big\} \\
&= \ell\Big\{-\cosh^2\rho(dt - \tfrac{1}{2}d\phi)^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 \qquad \text{AdS}_5 \\
&\qquad + \cos^2\theta(dt + \tfrac{1}{2}d\phi)^2 + d\theta^2 + \sin^2\theta d\widetilde{\Omega}_3^2\Big\} \qquad S^5
\end{aligned}
$$

- Note the mixing between t and ϕ

motion of the giant gravitons along the equator of S^5

Some additional resources

- A rather incomplete list. . .

M. Duff, B. Nilsson and C. Pope, *Kaluza-Klein supergravity*, Phys. Rep. **130**, 1 (1986).

P. Candelas and X. de la Ossa, *Comments on Conifolds*, Nucl. Phys. **B342**, 246 (1990).

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P. Aspinwall, *K3 surfaces and string duality*, hep-th/9611137.

K. Stelle, *BPS branes in supergravity*, hep-th/9803116.

J. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS Backgrounds in String and M-theory*, hep-th/0411194.

- Happy Canada Day!