Exact solutions in supergravity

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Lecture 1: Introduction; Supergravities in ten and eleven dimensions
Lecture 2: Supersymmetric vacua, holonomy and Killing spinors
Lecture 3: BPS branes and backgrounds with fluxes

Lecture 2 Outline

- Conditions for unbroken supersymmetry

  Killing spinors

  Integrability and holonomy

- The Euclidean Taub-NUT solution

- Sasaki-Einstein manifolds

  Five-dimensional examples: S^5, $T^{1,1}$, $Y^{p,q}$ and $L^{p,q,r}$

Supersymmetric configurations

- We are interested in classical solutions of supergravity preserving some or all of the supersymmetries
  - Traditional heterotic string compactified on Calabi-Yau
  - Explore dualities and hidden symmetries of M-theory
  - Classify all supersymmetric vacua;
    progress towards solving the vacuum degeneracy problem
  - Black hole entropy and microstates;
    starting point for exploring near-extremal black holes
- Sometimes we may wish to make a distinction between
  - Supersymmetric vacua, which are generally regular and do not have horizons
  - BPS objects with horizons such as supergravity p-branes and black holes

Conditions for unbroken supersymmetry

- In field theory, for the vacuum to be invariant under a (super)symmetry generated by Q

$$Q|\text{vac}\rangle = 0 \quad \longrightarrow \quad \langle\text{vac}|[Q, \Phi]|\text{vac}\rangle = 0$$

- For a background configuration, this implies that we are seeking a solution to the classical equations of motion such that $\delta\Phi = 0$ for all fields Φ

- Supergravity involves two types of variations

$$\delta(\text{Fermion}) = \partial(\text{Boson})\epsilon, \qquad \delta(\text{Boson}) = \overline{\epsilon}(\text{Fermion})$$

  For a purely bosonic background, the condition $\delta(\text{Boson}) = 0$ is automatic

- Goal: <u>Bosonic field configuration with $\delta(\text{Fermion}) = 0$</u>

Killing spinors

- In a supergravity theory, with graviton and gravitino

$$\begin{aligned} \delta e_\mu^\alpha &= \tfrac{1}{4} i \bar{\epsilon} \Gamma^\alpha \psi_\mu \\ \delta \psi_\mu &= \hat{\nabla}_\mu \epsilon \qquad\qquad \hat{\nabla}_\mu = \nabla_\mu + \tfrac{1}{288} (\Gamma_\mu{}^{\nu\rho\sigma\tau} - 8 \delta_\mu^\nu \Gamma^{\rho\sigma\tau}) F_{\nu\rho\sigma\tau} \\ \delta \lambda &= [\ \cdots\ ] \epsilon \end{aligned}$$

we must solve for $\delta \psi_\mu = 0$ or $\boxed{\hat{\nabla}_\mu \epsilon = 0}$

This is known as the Killing spinor equation

With other fermions, we also solve $\delta \lambda = 0$, etc

- Isometries of the background spacetime $\leftrightarrow$ Killing vector

  Unbroken supersymmetry $\leftrightarrow$ Killing spinor

Pure geometry solutions

- The simplest case to consider is for all fields vanishing except for the metric — pure geometry

  In this case we simply demand $\nabla_\mu \epsilon = 0$ on a d-dimensional Riemannian (or Lorentzian signature) manifold

  Existence of a covariantly constant spinor

- Relation to supersymmetric string compactifications

  If $\mathcal{M}^{10} = \mathbb{M}^{1,3} \times X^6$ where X^6 is a compact manifold admitting a covariantly constant spinor, $\nabla_m \epsilon(y) = 0$, then the resulting system is supersymmetric in four dimensions

  The number of $D = 4$ supersymmetries is related to the number of independent Killing spinors on X^6

Integrability of the Killing spinor equation

- The equation $\nabla_\mu \epsilon = 0$ is first order, but involves derivatives on ϵ

  We may remove derivatives on ϵ by studying the integrability condition

$$\nabla_\mu \epsilon = 0 \quad \longrightarrow \quad [\nabla_\mu, \nabla_\nu]\epsilon = \tfrac{1}{4} R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\epsilon = 0$$

  This expresses conditions on the curvature to ensure the existence of a Killing spinor

- In addition, $R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\epsilon = 0$ implies

$$0 = R_{\mu\nu\rho\sigma}\Gamma^{\nu}\Gamma^{\rho\sigma}\epsilon = R_{\mu\nu\rho\sigma}(\Gamma^{\nu\rho\sigma} + 2g^{\nu\rho}\Gamma^{\sigma})\epsilon = -2R_{\mu\sigma}\Gamma^{\sigma}\epsilon$$

  or $R_{\mu\nu}\Gamma^{\nu}\epsilon = 0$

  This is almost the vacuum Einstein equation $R_{\mu\nu} = 0$

Supersymmetry and the equations of motion

- Completely unbroken supersymmetry ($R_{\mu\nu}\Gamma^\nu\epsilon = 0$ for all ϵ) implies $R_{\mu\nu}\Gamma^\nu = 0$

  Using $\mathrm{Tr}(\Gamma^\mu\Gamma^\nu) = 2g^{\mu\nu}\mathrm{Tr}\,1$, we find $R_{\mu\nu} = 0$

  Thus a background with maximal supersymmetry automatically satisfies the equations of motion

- For partially broken supersymmetry, we need a more refined argument

  We would like to remove $\Gamma^\nu\epsilon$ from the condition $R_{\mu\nu}\Gamma^\nu\epsilon = 0$

  But we may only have a single ϵ to work with

Partially broken supersymmetry and the eom

- If ϵ is a Killing spinor, then $R_{\mu_0\nu}\Gamma^\nu\epsilon = 0$ for any fixed μ_0

  Without summing over μ_0, note that $(R_{\mu_0\lambda}\Gamma^\lambda)(R_{\mu_0\nu}\Gamma^\nu)\epsilon = 0$

  $$\rightarrow \quad R_{\mu_0\lambda}g^{\lambda\nu}R_{\mu_0\nu}\epsilon = 0 \quad \rightarrow \quad R_{\mu_0\lambda}g^{\lambda\nu}R_{\mu_0\nu} = 0 \quad \rightarrow \quad ||R_{\mu_0}|| = 0 \; \forall \, \mu_0$$

- For Euclidean signature (positive definite metric) $||R_{\mu_0}|| = 0$ implies all components must vanish

  Since this is true for all μ_0, we are guaranteed that $R_{\mu\nu} = 0$

- But for Lorentzian signature the best we can do is conclude the spacetime is Ricci-null

  A background with partial supersymmetry satisfies some, but not necessarily all, of the equations of motion

Holonomy and supersymmetry

- Another way to understand unbroken supersymmetry

  $[\nabla_\mu, \nabla_\nu]\epsilon$ represents the effect of parallel transport of a spinor around an infinitesimal loop

  $[\nabla_\mu, \nabla_\nu]\epsilon = \frac{1}{2} R_{\mu\nu}{}^{ab} \Sigma_{ab} \epsilon$ where Σ_{ab} is a $SO(n)$ rotation generator

- The group of rotations acting on ϵ is known as the holonomy group

$$H \subseteq SO(n) \qquad \epsilon = \text{spinor of } SO(n)$$

- The number of preserved supersymmetries is then the number of singlets in the decomposition of ϵ under H

Holonomy classification

- M. Berger in Bull. Soc. Math. France **83**, 225 (1955) classified the possible holonomy groups in Euclidean signature

  For an irreducible non-symmetric Riemannian metric, the possible holonomy groups are

dim	H	Ricci flat?	
n	$SO(n)$		generic metric
$2n$	$U(n)$		complex Kähler
$2n$	$SU(n)$	yes	Calabi-Yau
$4n$	$Sp(2n)$	yes	hyperkähler
$4n$	$Sp(2n) \times Sp(2)$		quaternionic Kähler
7	G_2	yes	exceptional holonomy
8	$Spin(7)$	yes	exceptional holonomy

- In Lorentzian signature, the classification is based on either a timelike or a null Killing vector $\xi^\mu = \overline{\epsilon}\Gamma^\mu\epsilon$ [see e.g. R.L. Bryant, math.DG/0004073]

Some examples: flat space

- Perhaps the most trivial example of a supersymmetric background is simply flat space itself

  Maximally symmetric and maximally supersymmetric

$$\nabla_\mu \epsilon = 0 \quad \longrightarrow \quad \partial_\mu \epsilon = 0 \quad \longrightarrow \quad \epsilon(x) = \epsilon_0$$

- The fraction of preserved supersymmetries is 1

  We can choose a basis

$$\epsilon^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \epsilon^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad \epsilon^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \quad \text{etc.}$$

  The details depend on the dimension and type of spinor

Some examples: $SU(2)$ holonomy

- In four Euclidean dimensions, the tangent space group is

$$SO(4) = SU(2)_+ \times SU(2)_-$$

- This allows us to decompose

$$R_{\mu\nu}{}^{ab}\Gamma_{ab}\epsilon = R_{\mu\nu}{}^{ab}(\tfrac{1}{2}(1+\Gamma^5)\Gamma_{ab} + \tfrac{1}{2}(1-\Gamma^5)\Gamma_{ab})\epsilon$$

Note the projections $P_\pm = \frac{1}{2}(1 \pm \Gamma^5)$

Also, since $\Gamma^5\Gamma_{ab} = -\epsilon_{abcd}\Gamma^{cd}$, we have

$$R_{\mu\nu}{}^{ab}\Gamma_{ab}\epsilon = (R^{+}_{\mu\nu}{}^{ab}\Gamma_{ab}P_- + R^{-}_{\mu\nu}{}^{ab}\Gamma_{ab}P_+)\epsilon$$

where

$$R^{\pm}_{\mu\nu}{}^{ab} = \tfrac{1}{2}(\delta^c_a\delta^d_b \pm \tfrac{1}{2}\epsilon_{ab}{}^{cd})R_{\mu\nu cd}$$

Some examples: $SU(2)$ holonomy

- We may solve for a Killing spinor by picking a self-dual connection, i.e. by setting $R^{-}_{\mu\nu}{}^{ab} = 0$

  The resulting condition is $R^{+}_{\mu\nu}{}^{ab}\Gamma_{ab}P_{-}\epsilon = 0$ where $\Gamma_{ab}P_{-}$ clearly generates the $SU(2)_{-}$ subgroup of $SO(4)$

- Holonomy: The spinor of $SO(4)$ is $\mathbf{4} = (\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2})$ which decomposes as $\mathbf{4} \to \mathbf{2} + \mathbf{1} + \mathbf{1}$ under $SO(4) \supset SU(2)_{-}$ (two singlets)

- Killing spinor: An arbitrary spinor ϵ decomposes into positive and negative chirality spinors $\epsilon_{\pm} = P_{\pm}\epsilon$
  The ϵ_{+} spinors are Killing spinors since they are annihilated by $R^{+}_{\mu\nu}{}^{ab}\Gamma_{ab}P_{-}$ (the ϵ_{-} spinors are fermion zero modes)

- In either case, we conclude that this is a $\frac{1}{2}$-BPS background

Some examples: $SU(2)$ holonomy

- A compact manifold of $SU(2)$ holonomy is known as a K3 surface
  See e.g. P.S. Aspinwall, *K3 Surfaces and String Duality*, hep-th/9611137

- Non-compact examples include

Euclidean Taub-NUT asymptotically locally flat ($\mathbb{R}^3 \times S^1$)

$$ds_4^2 = H[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] + H^{-1}(dy - q\cos\theta d\phi)^2$$

$$H = 1 + q/r$$

Eguchi-Hanson asymptotically locally Euclidean ($\mathbb{R}^4/Z_2$)

$$ds_4^2 = h^{-1}dr^2 + \tfrac{1}{4}r^2[d\theta^2 + \sin^2\theta\, d\phi^2 + h(d\psi + \cos\theta d\phi)^2]$$

$$h = 1 - a^4/r^4$$

Some examples: $SU(3)$ holonomy

- In six Euclidean dimensions, the tangent space group is

$$SO(6) = SU(4)$$

- Consider the $SU(3)$ subgroup

$$SO(6) \supset U(3) \supset SU(3)$$

The complex spinor decomposes as

$$\mathbf{4} \longrightarrow \mathbf{3}_{-1} + \mathbf{1}_3 \longrightarrow \mathbf{3} + \mathbf{1}$$

so this is a $\frac{1}{4}$-BPS background: Calabi-Yau (complex) 3-fold

Note the presence of a covariantly constant $(3,0)$-form $\Omega_{\mu\nu\rho} = \overline{\epsilon}\Gamma_{\mu\nu\rho}\epsilon$

Supersymmetry of Taub-NUT

- The $D6$-brane of IIA theory lifts to pure geometry in $D = 11$

  $U(1)$ bundled over $\mathbb{R}^3$

- In general, this is a multi-centered Taub-NUT configuration

$$ds_4^2 = H(\vec{x})d\vec{x}^2 + H^{-1}(\vec{x})(dy + A_{(1)})^2$$

  where $F_{(2)} = dA_{(1)}$ and $F_{(2)} = *_3 dH$ (which requires $d *_3 dH = 0$)

- To examine the supersymmetry, we may examine the Killing spinor equation $\nabla_\mu \epsilon = 0$ where

$$\nabla_\mu = \partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}$$

Computing the spin connection

- Introduce a vielbein basis

$$g_{\mu\nu} = e_\mu{}^a \delta_{ab} e_\nu{}^b \qquad \text{or} \qquad ds^2 = e^a \otimes e^a$$

- The spin connection may be computed by demanding the torsion free condition $de + \omega \wedge e = 0$

The Cartan structure equation $\quad de^a = \frac{1}{2} c_{bc}{}^a e^b \wedge e^c$

In terms of the anholonomy coefficients $\quad \omega_{abc} = \frac{1}{2}(c_{abc} - c_{bca} + c_{cab})$

- Note that there is a shortcut for diagonal metrics

If $e_\mu{}^a = f_\mu \delta_\mu^a$ so that $g_{\mu\nu} = f_\mu^2 \delta_{\mu\nu}$ then

$$\omega_{\mu\nu\lambda} = g_{\mu[\nu,\lambda]} \equiv \frac{1}{2}(\partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda})$$

The spin connection for Taub-NUT

- Given

$$ds_4^2 = H(\vec{x})d\vec{x}^2 + H^{-1}(\vec{x})(dy + A_{(1)})^2$$

we choose a natural vielbein basis

$$e^a = H^{1/2}dx^a \qquad e^4 = H^{-1/2}(dy + A_{(1)})$$

- Then we obtain

$$\begin{aligned}
\omega_{abc} &= H^{-3/2}\delta_{a[b}\partial_{c]}H \\
\omega_{ab4} &= \tfrac{1}{2}H^{-3/2}F_{ab} \\
\omega_{4ab} &= -\tfrac{1}{2}H^{-3/2}F_{ab} \\
\omega_{4a4} &= \tfrac{1}{2}H^{-3/2}\partial_a H
\end{aligned}$$

This allows us to write down the covariant derivative

The Killing spinor projection

- For the above metric, the Killing spinor equation becomes

$$
\begin{aligned}
0 &= \nabla_m \epsilon = [\partial_m + X_{\overline{m}} + A_m X_4]\epsilon \\
0 &= \nabla_y \epsilon = [\partial_y + X_4]\epsilon
\end{aligned}
$$

where

$$X_a = \tfrac{1}{4} H^{-1}(\Gamma_a{}^b \partial_b H + F_{ab}\Gamma^{b4}) \qquad X_4 = \tfrac{1}{4} H^{-2}(\Gamma^{a4}\partial_a H - \tfrac{1}{2} F_{ab}\Gamma^{ab})$$

With some manipulation, we obtain

$$X_a = \tfrac{1}{4} H^{-1}\Gamma^{b4}(F_{ab} + \epsilon_{ab}{}^c \partial_c H \Gamma^5) \qquad X_4 = \tfrac{1}{4} H^{-2}\Gamma^{a4}(\partial_a H + \tfrac{1}{2}\epsilon_a{}^{bc} F_{bc}\Gamma^5)$$

where $\Gamma^5 = \Gamma^{1234}$

The Killing spinor projection

- These matrices admit zero eigenvalues when $F_{(2)} = *_3 dH$

$$\begin{aligned} \nabla_m \epsilon &= [\partial_m + \tfrac{1}{2} H^{-2} (H \epsilon_{\overline{m}a}{}^b + A_m \delta_a^b) \partial_b H \Gamma^{a4} P_+] \epsilon \\ \nabla_y \epsilon &= [\partial_y + \tfrac{1}{2} H^{-2} \partial_a H \Gamma^{a4} P_+] \epsilon \end{aligned}$$

(and similarly for $F_{(2)} = - *_3 dH$, but with P_-) where

$$P_\pm = \tfrac{1}{2}(1 \pm \Gamma^5)$$

- This is a $\frac{1}{2}$-BPS projection, with Killing spinors

$$\epsilon = P_- \epsilon_0$$

- Note that we must satisfy the Bianchi identity

$$dF_{(2)} = 0 \quad \rightarrow \quad d *_3 dH = 0 \quad \rightarrow \quad H = 1 + \sum \frac{q_i}{|\vec{x} - \vec{x}_i|}$$

Relation to $SU(2)$ holonomy

- This provides an explicit realization of $SU(2)_+ \times SU(2)_- \longrightarrow SU(2)_+$

$$SU(2)_+ : \quad J_+^{ab} = \tfrac{1}{2} i \Gamma^{ab} P_+ \qquad SU(2)_- : \quad J_-^{ab} = \tfrac{1}{2} i \Gamma^{ab} P_-$$

- In this case, the 4-component spinor ϵ has $SU(2)_+$ eigenvalues

$$\{\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0\} \quad \longrightarrow \quad \mathbf{4} \rightarrow \mathbf{2} + \mathbf{1} + \mathbf{1}$$

In general, each compatible projection reduces the supersymmetry by half

$\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS, $\frac{1}{8}$-BPS, etc.

Taub-NUT as a cone over S^3

- Another look at the $q = 1$ Taub-NUT metric

$$\begin{aligned} ds_4^2 &= H[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] + H^{-1}(d\psi + \cos\theta\, d\phi)^2 \\ &= H dr^2 + H r^2(\sigma_1^2 + \sigma_2^2) + H^{-1}\sigma_3^2 \end{aligned}$$

Where the σ^a are left-invariant 1-forms on S^3 satisfying $d\sigma^a = -\frac{1}{2}\epsilon_{abc}\sigma^b \wedge \sigma^c$

$$\begin{aligned} \sigma^1 &= -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi \\ \sigma^2 &= \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi \\ \sigma^3 &= -d\psi - \cos\theta\, d\phi \qquad\qquad \theta \in [0,\pi],\ \phi \in [0, 2\pi],\ \psi \in [0, 4\pi] \end{aligned}$$

- In general, we may obtain Kähler metrics as cones

$$d\hat{s}_d^2 = dr^2 + r^2 ds_{d-1}^2(X)$$

Sasaki-Einstein manifolds

- For the metric $d\hat{s}_d^2 = dr^2 + r^2 ds_{d-1}^2(X)$

  Compute $\hat{\omega}_m{}^{ab} = \omega_m{}^{ab}$ $\hat{\omega}_m{}^{a\bar{r}} = e_m{}^a$

  This results in $\hat{\nabla}_r = \partial_r$ $\hat{\nabla}_m = \nabla_m + \frac{1}{2}\Gamma_m\Gamma^{\bar{r}}$

  This yields a modified $(d-1)$-dimensional Killing spinor equation

$$\hat{\epsilon} = \epsilon(x^m) \quad \text{with} \quad \mathcal{D}_m\epsilon = 0 \quad \text{where} \quad \mathcal{D}_m = \nabla_m + \tfrac{1}{2}\Gamma_m\Gamma^{\bar{r}}$$

- Study the integrability

$$[\mathcal{D}_m, \mathcal{D}_n] = [\nabla_m, \nabla_n] + \tfrac{1}{4}[\Gamma_m\Gamma^{\bar{r}}, \Gamma_n\Gamma^{\bar{r}}] = \tfrac{1}{4}[R_{mnpq} - (g_{mp}g_{np} - g_{mq}g_{np})]\Gamma^{pq}$$

  Take a Γ-trace to obtain

$$\Gamma^n[\mathcal{D}_m, \mathcal{D}_n] = -\tfrac{1}{2}[R_{mn} - (d-2)g_{mn}]\Gamma^n$$

Sasaki-Einstein manifolds

- As we have seen before, this tensor must vanish (in Euclidean signature) given a Killing spinor
- We conclude
  - The space X^{d-1} is Einstein, $R_{mn} = \Lambda g_{mn}$ with $\Lambda = d - 2$
  - Integrability demands $[\mathcal{D}_m, \mathcal{D}_n]\epsilon = \frac{1}{4} C_{mnpq}\Gamma^{pq}\epsilon = 0$
- Existence of a Killing spinor on X^{d-1} is related to Killing spinors on $\mathcal{M}^d$

  $\mathcal{M}^d$ is Ricci-flat and Kähler (Calabi-Yau) if and only if X^{d-1} is Einstein and preserves a Killing spinor (Sasaki-Einstein)

Examples: Cones over S^{d-1}

- Perhaps the simplest Einstein space is a sphere $X^{d-1} = S^{d-1}$

  The cone over round S^{d-1} is trivially flat space $ds_d^2 = dr^2 + r^2 d\Omega_{d-1}^2$

- For anything other than a round sphere, the tip of the cone ($r = 0$) is singular

- More generally, Sasaki-Einstein metrics always admit a constant norm Killing vector, which may be taken as $K = c^{-1}\partial/\partial\psi$

  This allows us to write (at least locally)

$$ds_{d-1}^2 = ds_{d-2}^2 + c^2(d\psi + A)^2$$

  where ds_{d-2}^2 is Kähler-Einstein (and dA is related to the Käher form)

Examples: $T^{1,1}$ space

- The spaces $T^{p,q}$ are obtained by fibering $U(1)$ over $S^2 \times S^2$

$$\begin{aligned} ds_5^2 &= \Lambda_1^{-1}(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2) + \Lambda_2^{-1}(d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2) \\ &\quad + c^2(d\psi + p\cos\theta_1\, d\phi_1 + q\cos\theta_2\, d\phi_2)^2 \end{aligned}$$

Note $F = dA = -p d\Omega_2 - q d\tilde{\Omega}_2$

- With tangent indices, we find

$$R_{ab} = \Lambda_1(1 - \tfrac{1}{2}c^2p^2\Lambda_1)\delta_{ab} \qquad R_{\tilde{a}\tilde{b}} = \Lambda_2(1 - \tfrac{1}{2}c^2q^2\Lambda_2)\delta_{\tilde{a}\tilde{b}}$$

$$R_{55} = \tfrac{1}{2}c^2(p^2\Lambda_1^2 + q^2\Lambda_2^2)$$

Λ_1, Λ_2 and c can be chosen to make the space Einstein ($R_{mn} = 4g_{mn}$)

Examples: $T^{1,1}$ space

- Only $T^{1,1}$ is supersymmetric

$$\begin{aligned} ds_5^2 &= \tfrac{1}{6}(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2) + \tfrac{1}{6}(d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2) \\ &\quad + \tfrac{1}{9}(d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2)^2 \end{aligned}$$

homogeneous space $SU(2) \times SU(2)/U(1)$ with $SU(2) \times SU(2) \times U(1)$ isometry

- The cone over $T^{1,1}$ is known as the conifold

Since $T^{1,1}$ is topologically $S^2 \times S^3$, the tip is singular with shrinking 2 and 3 cycles

  – small resolution: blowing up the S^2

  – deformation: blowing up the S^3

Examples: $Y^{p,q}$ space

- Recently constructed by Gauntlett, Martelli, Sparks and Waldram, hep-th/0403002

$$ds_5^2 = \tfrac{1}{6}(1 - cy)(d\theta^2 + \sin^2\theta\, d\phi^2) + (wv)^{-1} dy^2 + \tfrac{1}{36} wv(d\beta - c\cos\theta\, d\phi)^2$$
$$+\tfrac{1}{9}(d\psi + \cos\theta\, d\phi + y(d\beta - c\cos\theta\, d\phi))^2$$

where

$$w = \frac{2(a - y^2)}{1 - cy} \qquad v = \frac{a - 3y^2 + 2cy^3}{a - y^2} \qquad (y_1 < y < y_2)$$

Note that $T^{1,1}$ is obtained by setting $a = 3$, $c = 0$ and taking $y = \cos\xi$

a and c much be chosen to obtain integral periods p and q over $S^2 \times S^2$

Now only $SU(2) \times U(1) \times U(1)$ isometry (cohomogeneity 1)

Examples: $L^{p,q,r}$ space

- Additional non-singular generalization to $L^{p,q,r}$ by Cvetič, Lü, Page and Pope, hep-th/0504225

  generically cohomogeneity 2, with isometry $U(1) \times U(1) \times U(1)$

  Obtained by analytically continuing the Kerr-AdS$_5$ black hole with two independent rotation parameters

- Also many examples of 7-manifolds with G_2 holonomy

Next time

- Backgrounds with non-trivial charges or fluxes

  Freund-Rubin compactification to AdS$\times S$

  p-brane solutions in supergravity

- Methods for obtaining new supersymmetric solutions