# QFT in curved spacetime HW # 2
# To do after (or during) the lectures

1) Show that for the Harmonic oscillator (i.e., a 0+1 scalar field) there is a unique choice of symplectic (i.e., "block diagonal") basis on the space of solutions such that the associated vacuum state is an eigenstate of the Hamiltonian. Show that this basis leads to the usual creation and annihilation operators of the Harmonic oscillator.

2) Find a two particle state in the usual Fock space of a free scalar quantum field on Minkowski space such that the normal-ordered stress-energy tensor of a free quantum scalar field is negative at *some* spacetime events.

3) Consider the expression

$$|0\rangle_{Mink} = \exp\left(\frac{1}{2}\sum_{modes} e^{-\pi\omega/\kappa} a_R^\dagger a_L^\dagger)\right) |0\rangle_{Rindler} \tag{1}$$

for the Minkowski vacuum state in terms of the Rindler vacuum state. Here the notation implies that one considers a basis for the space of solutions consisting of pairs of modes, with one mode ($R$) supported in the right Rindler wedge and with the associated mode ($L$) supported in the left Rindler wedge. Both modes are eignenmodes of the Rindler Hamiltonian and so have a definite frequency $\omega$. The operators $a_R^\dagger, a_L^\dagger$ are the associated *creation* operators, while the corresponding annihilation operators $a_R, a_L$ annihilate the Rindler vacuum: $a_R|0\rangle_{Rindler} = 0$, $a_L|0\rangle_{Rindler} = 0$.

*The problem:* Trace the density matrix $\rho_{Mink} := |0\rangle_{Mink\,Mink}\langle 0|$ over the states supported in the left Rindler wedge to obtain an effective density matrix for the right Rindler wedge. Show that the result has the thermal form

$$\rho_{right} = Tr_{left}\rho_{Mink} = Ne^{-\beta H_{Rindler}}, \tag{2}$$

where $\beta = \kappa/2\pi$ is the inverse Unruh temperature and $N$ is an appropriate normalization factor.

4) *The spectrum of inflationary perturbations.* The metric

$$ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2) \tag{3}$$

plays a special role in modern cosmology. Here, $H > 0$ is a constant known as the Hubble constant. This metric describes a so-called "inflating" spacetime. You can see immediately that in some sense the space part gets bigger at an exponential rate. This is just de Sitter space in so-called flat coordinates. We are interesting in a (free) scalar field $\phi$ on the spacetime (3).

a) Calculate the scalar Laplacian on this spacetime. Show that $H > 0$ acts like a friction term, while $H < 0$ is "anti-friction."

b) Now take $H > 0$. Suppose that our scalar field satisfies the wave equation $\nabla^2\phi = 0$. Consider a plane wave in space: $\phi = f_{\vec{k}}(t)e^{ik\cdot x} = f_{\vec{k}}(t)e^{i(k_x x+k_y y+k_z z)}$. With enough work, one can solve for $f_{\vec{k}}(t)$ exactly. However, let's try to understand the solution by WKB techniques: Write $f_{\vec{k}}(t) = \exp(\alpha_{\vec{k}}(t))$ and work out the equation of motion for $\alpha_{\vec{k}}(t)$. Now, assume that the second derivative term $\alpha''$ can be neglected, and solve the remaining equation for $\alpha'$. You can now express $\alpha$ as a fairly simple integral. Show that $f_{\vec{k}}(t)$ is oscillatory only when the "proper wavelength" is sufficiently small. How small must it be?

   *Note:* Cosmologists describe this phenomenon using the phrase that:

   > "In an inflating spacetime, modes larger than the *horizon size* freeze out."

   The "horizon size" is $H^{-1}$, which you may recall is the size of the de Sitter horizon.

c) Quantize above scalar field and construct the (appropriately normalized!) creation and annihilation operators for the modes above. Define the vacuum state associated with these modes, and use your WKB approximation above to calculate (for each wave vector $\vec{k}$) the expectation value

$$\langle\phi(\vec{k},t)\phi(-\vec{k},t)\rangle$$

   at a late time $t$, long after the $\vec{k}, -\vec{k}$ modes have "frozen out." Here the notation $\phi(\vec{k},t)$ means that we have Fourier transformed the field operator $\phi(\vec{x},t)$ in space but not in time.

   Note that the above two-point function is proportional to $k^{-3}$ and is thus scale-invariant. This is the famous scale-invariant spectrum of inflationary density perturbations.

*Hint:* If you need help, consult Ted Jacobson's lecture notes where this calculation is sketched qualitatively in. Ted also gives references where more details can be found.

*Note:* One might wonder whether the vacuum state associated with the above modes is obviously the physically relevant initial state for a quantum field on an inflating background. One can find long discussions of this in the literature, and the question has been a topic of continuing research. However, the short version is that we do not understand in detail what determines the state, but that there is a sort of attractor mechanism such that a wide assortment of quantum states tend in the asymptotic future toward the same result obtained above.

d) In general, we could consider a case where $H$ is a function of time. In particular, we might have $H = \textit{constant} \neq 0$ until some $t = t_1$, but then $H$ drops suddenly to zero at $t = t_1$. Consider any mode that "freezes out" at some $t = t_0 \ll t_1$. When $H$ becomes sufficiently small, that mode will start to oscillate again. Use your observations above to argue that *all* such modes begin *in phase*. Essentially this phenomenon is responsible for the famous "acoustic peaks" in the Cosmic Microwave Background radiation (see, e.g., http://map.gsfc.nasa.gov/m_mm/sg_parameters.html). Hint: Just give a rough argument. Don't do any new calculations. Of course, if you *did* extend the calculations of (4d) in detail to this case (with a realistic model of $H(t)$), it would lead to something like the acoustic peaks shown on the website above.

**Final note:** Remember that more advanced material on QFT in curved spacetime can be found in the suggested references and in the works that they cite.