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Virtual Quantum Subsystems, Holonomies & Topological QIP

P.Z., Phys. Rev. Lett. **87**, 077901 (2001)
P. Z., S. Lloyd, Phys. Rev. Lett. **90**, 067902 (2003)
P. Z, D. Lidar, S. Lloyd, Phys. Rev. Lett. **92**, 060402 (2004)

Quantum Tensor Product Structures

H =quantum state space, d=dim(H)

$\phi : H \rightarrow \otimes_i C^{n_j}$ *Unitary mapping* $\boxed{H \cong_\phi \otimes_i C^{n_j}}$

$L(H) \cong_\phi \otimes_i M(n_j)$ *"local observable algebras"*

$d =$ *prime number* ➡ No ***TPS***

$d = \Pi_i p_i^{n_i}$ ➡ Very many possible ***TPSs***

Question : *How a particular TPS is singled out?*

Answer : *it's all about operational resources!*

The Bell basis Example

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), |\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

Let's write them as $|\chi^{\lambda}\rangle = |\chi\rangle \otimes |\lambda\rangle, (\chi = \Phi, \Psi; \lambda = \pm)$

New ***TPS*** on $C^2 \otimes C^2$ *with local subalgebras*

$$A_{\chi} = \{1, \sigma^x \otimes 1, \sigma^y \otimes \sigma^z, \sigma^z \otimes \sigma^z\} \cong u(2)_{\chi}$$

$$A_{\lambda} = \{1, 1 \otimes \sigma^z, \sigma^x \otimes \sigma^y, \sigma^x \otimes \sigma^x\} \cong u(2)_{\lambda}$$

The "swap" operator $S|\alpha\rangle \otimes |\beta\rangle = |\beta\rangle \otimes |\alpha\rangle, (\alpha, \beta = 0,1)$

Un-entangling in the old TPS is now maximally entangling

I.e., $S|\chi^{\lambda}\rangle = (-1)^{\chi\lambda}|\chi^{\lambda}\rangle$ *(C-phase shift)*

Virtual **Subsystems & Sub-Algebras**

$\{A_i\}_{i=1}^{n}$ *Collection of *-subalgebras of* $L(H)$

= algebra of finear operators over the quantum state-space H

(i) *Each* A_i *is independently implementable*

(ii) $\boxed{i \neq j \Rightarrow [A_i, A_j] = 0}$ *Dynamical independence*

(iii) $\boxed{\vee_i A_i \cong \otimes_i A_i \cong L(H)}$ *Completeness*

(i) (ii) (iii) $\boxed{\Rightarrow H \cong \otimes_i H_i}$ $\boxed{A_i \cong 1 \otimes \Lambda \otimes L(H_i) \otimes \Lambda \otimes 1}$

P. Zanardi, D. Lidar and S. Lloyd, Phys. Rev. Lett. (2004)

Virtual multipartiteness: Sub-algebras Chain

$$L(H) \cong B_0 \supset B_1 \Lambda \supset B_n$$

Nested Subalgebras: Iteration of the irrep decomposition

$$H \cong \bigoplus_{J_1,\Lambda,J_n} N_{J_1} \otimes N_{J_2} \otimes \Lambda \otimes N_{J_n} \otimes D_{J_n} \cong \bigoplus_{J_1,\Lambda,J_n} H_{J_1,\Lambda,J_n}$$

$$A_i \cong B_{i-1} \cap B_i' \quad (i = 1,\Lambda,n) \quad \Rightarrow \quad \{A_i |_{H_{J_1,\Lambda,J_n}}\}_{i=1}^{n}$$

Satisfy (ii), (iii)

Example N=3 Qubits

Su(2)-triplet

$$B_1 = CS_6, B_2 = C(S_3 \times S_3)$$

$$C^{64} \cong C^3 \otimes C^9 \oplus \Lambda$$

6-Perm irrep

$$C^9 \cong C^2 \otimes C^2 \oplus \Lambda$$

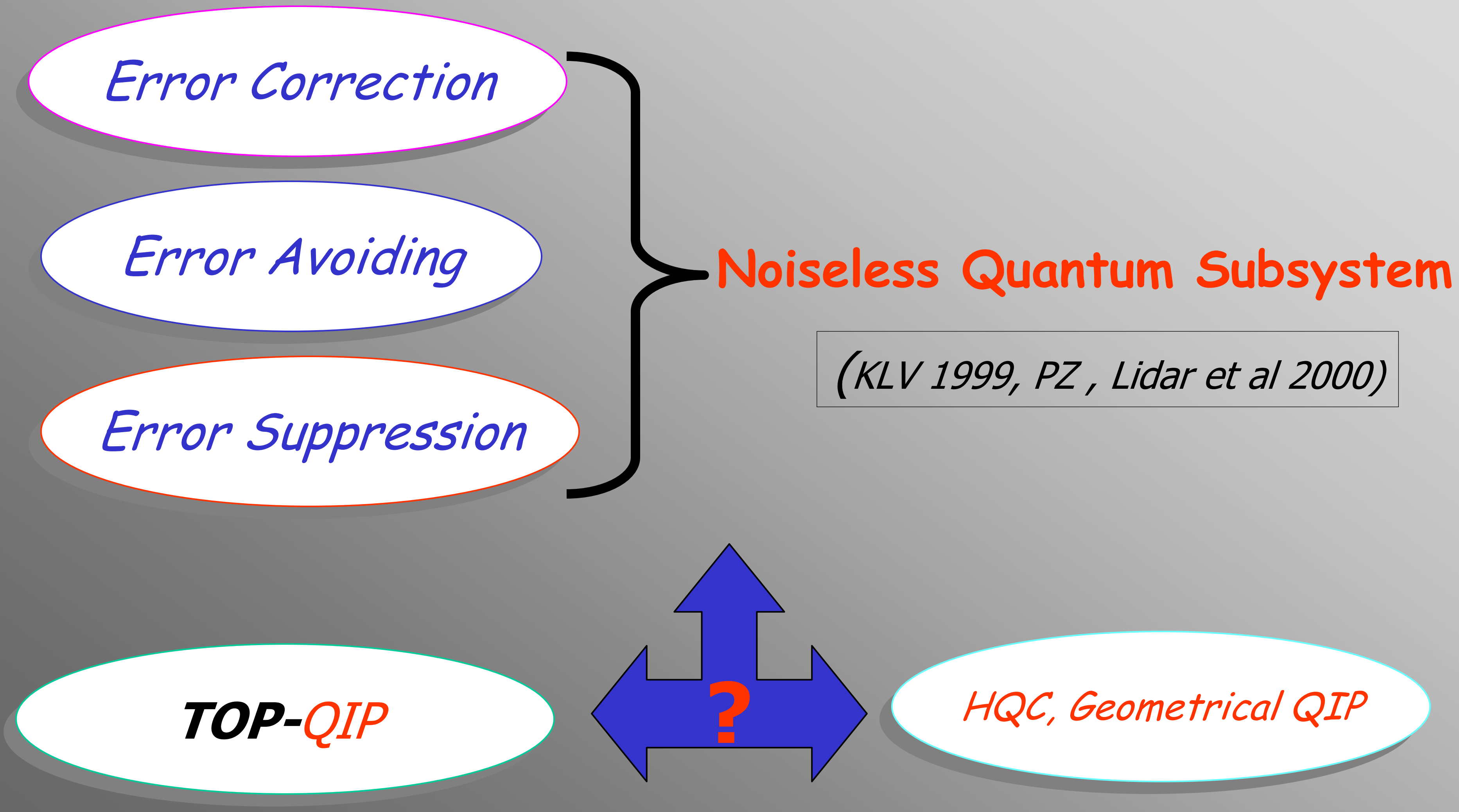
We got the tripartite term

$$C^3 \otimes C^2 \otimes C^2$$

3x3-Perm irrep

Noiseless Quantum Subsystems and TOP-QIP

Fighting Decoherence & control Errors



Error Correction

Error Avoiding

Error Suppression

Noiseless Quantum Subsystem

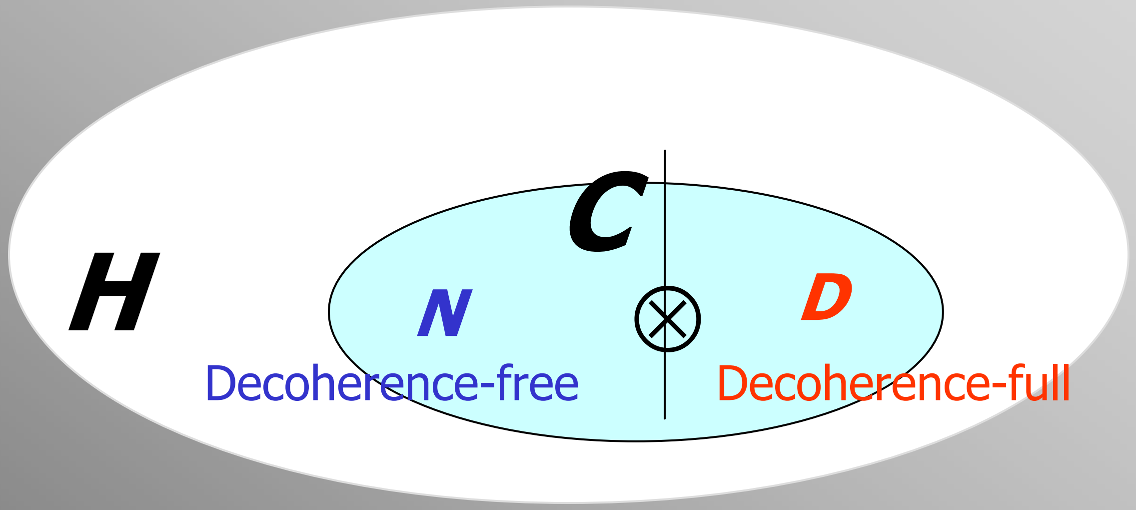
(KLV 1999, PZ , Lidar et al 2000)

TOP-QIP

?

HQC, Geometrical QIP

Noiseless Quantum Subsystem = factor ***N*** of a subspace ***C*** of the state-space ***H*** unaffected by unwanted interactions I.e., errors (KLV 1999)



Errors restricted to ***C*** ***= trivial TENSOR non-trivial***

Symmetry Duality!

Control Operations = ***non-trivial TENSOR trivial***

For ***D =1dim*** → ***N*** = Decoherence-Free Subspace

e.g., Global *SU(2)-singlets for Collective decoherence PZ & Rasetti 1997)*

Canonical Algebra Pairs: Noise & Control

$$\boxed{A \cong \oplus_J 1_{n_J} \otimes M(d_J)}$$

The Noise

Symmetry Duality

$$\boxed{A^{'} \cong \oplus_J M(n_J) \otimes 1_{d_J}}$$

The control

= Noise commutant

State-space splits according irreps of the *Error Algebra A*

$$\boxed{H = \oplus_J N_J \otimes D_J}$$

$d_J = \dim D_J$

$n_J = \dim N_J$

J= *irrep label*

$A^{'} \vee A \cong A^{'} \otimes A$ *in each* $N_j \otimes D_j$

Noise/Control algebras define a bunch of **QVS**s

Noiseless Quantum Subsystems III

Collective Decoherence (N qubit systems)

Error Algebra = *Totally Symmetric Operators*
(permutation Symmetry) I.e., algebra generated by Collective *SU(2)*

Control Algebra = *linear combinations of permutations*
I.e., algebra generated by the permutation group S_N

J = total angular momentum

$$d_J = 2J + 1$$

$$n_J = \frac{(2J+1)N!}{(N/2+J+1)!(N/2-J)!}$$

N=4 one Noiseless qubit sub-space

PZ & Rasetti PRL ***79****, 3306 (1997)*

N=3 one Noiseless qubit sub-system

KLV PRL **84, 2525** (1999)

Experimental verification: Ion Traps, Q-Optics, NMR,…

Holonomic Quantum Computation

PZ, M Rasetti, Phys. Lett. **264** (1999) 94

$$\{H(\lambda)\}_{\lambda \in M}$$

Iso-degenerate Hamiltonian Family

Example: Isospectral family i.e.,

$$H(\lambda) \equiv U_\lambda H_0 U_\lambda{}^+$$

$$H(\lambda) = \sum_{r=1}^{R} \varepsilon_r \Pi_r(\lambda), \quad \Pi(\lambda) \equiv \sum_{\alpha=1}^{n_r} | \psi_\alpha(\lambda) \rangle \langle \psi_\alpha(\lambda) |$$

$$\{ | \psi_\alpha(\lambda) \rangle \}_{\alpha=1}^{n_r}$$

Non-Trivial λ -dependence!

Orthonormal Basis of the r-th eigenspace

Adiabatic Loops

$$\lambda(0) = \lambda(T), \quad \eta T^{-1} << \min_{r,s} |\varepsilon_r - \varepsilon_s|$$

NO jumps among different eigenspaces

$$|\Psi\rangle_{in} \in \Pi_r C^N \Rightarrow |\Psi\rangle_{out} = \Gamma_A(\gamma)|\Psi\rangle_{in} \in \Pi_r C^N$$

Holonomy operator

$$\boxed{\Gamma_A(\gamma) \equiv P\exp(\oint_\gamma A)}$$

$u(n_r)$ –valued Connection
1- Form over M

$$A = \sum_\mu A_\mu d\lambda_\mu, A_\mu = -A_\mu^+ \in M_{n_r}(C)$$

$$\boxed{(A_\mu)_{\alpha\beta} = \langle\psi_\alpha(\lambda)|\frac{\partial}{\partial\lambda_\mu}|\psi_\beta(\lambda)\rangle}$$

F Wilczek, A Zee PRL 52, 2111 (1984)

Set of realizable transformations
= **Holonomy Group**

$$\boxed{Hol(A) \equiv \{\Gamma_A(\gamma)\} \subset U(n_r)}$$

Curvature 2-Form

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

Hol(A) is generated by F and its covariant derivatives

$$n_r^{\ 2} \geq \dim Hol(A) \geq \dim span\{F_{\mu\nu}\}$$

A irreducible $\forall U \in U(n_r), \varepsilon > 0 \Rightarrow \exists \gamma \ \ \| U - \Gamma_A(\gamma) \| \leq \varepsilon$

Any computation can be approximated by holonomies

Question: How many loops? Answer (existential): a pair!

S Lloyd PRL 75, 346 (1995)

TOPOLOGICAL QIP

(Kitaev 1997, Freedman 2000)

Encoding = degenerate & Gapped Ground-state ***C***

Degeneration Topologically robust against local perturbations
i.e., tunneling amongs GSs exponentially suppressed

Leakage gap-suppressed

Processing *= Creation of excitations, braiding, fusing*
⇒ *non-trivial Op on the GS manifold **C***

C= *representation space for the Braid-group I.e, b=braid*

⇒ $X(b): C \longrightarrow C$ *(modular functor)*

$\{X(b)\}_b$ = *Topologically Robust Operations* **!**

Error Algebra= *local perturbations O* *trivial topological content*

$$\Psi_i, \Psi_j \in C \Rightarrow \langle \Psi_i | O | \Psi_j \rangle = f(O)\delta_{i,j} \Rightarrow A_{loc}C \cong D_{loc} \otimes C$$

NB *Thermodynamical limit, f does not depend on i,j*

Control Algebra = *braiding operations= Holonomies*
over a statistical connection ***Non****-trivial topological content*

NB non-trivial topology of the ambient space e.g., torus (Kitaev 1997)

TOP-QIP is based on *topologically generated* ***NS*** *over which Robust computations are performed by means of holonomies*

P. Zanardi, S. Lloyd, Phys. Rev. Lett. **90**, 067902 (2003)

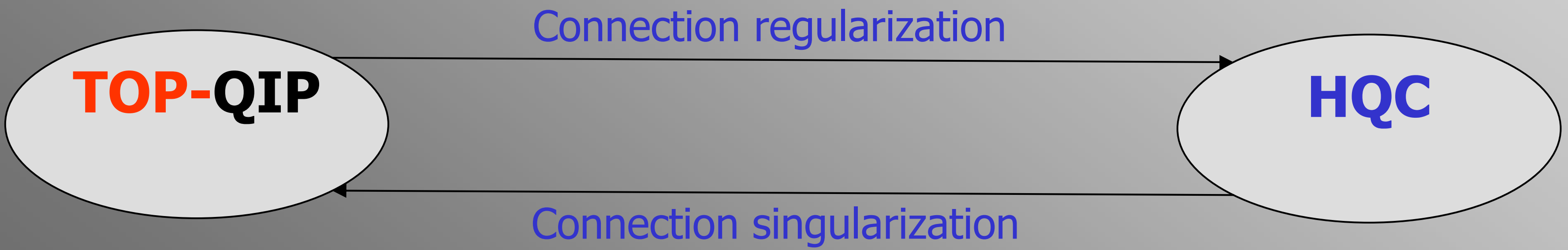
Optimizing Connections: from geometry to Topology

(1) statistical connection → curvature form singular on a line(s)

e.g., AB-effect

(2) geometrical connection → curvature form singular on a point(s)
e.g., Berry phase

Topology of the set of regular point (=ambient space – singular points)
(1)=NON-trivial, (2) trivial e.g., I Homotopy group



The more topological the better !