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An energy bound deduced from the vanishing of the radiation reaction force during uniform acceleration *

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Abstract

We deduce from energy conservation a lower bound on the mass of any system capable of imparting a constant acceleration to a charged body. We also point out a connection between this bound and the so called dominant energy condition of general relativity.

A well known peculiarity of the radiation reaction force on a charged particle is that (in flat spacetime) it vanishes when the particle accelerates uniformly. But this raises a paradox. An accelerating charge radiates, and the longer the acceleration continues, the greater the total energy radiated. If one asks where this energy comes from in the case of uniform acceleration, the usual answer is that it is "borrowed" from the near field of the particle and then "paid back" when the acceleration finally ceases. But this "debt" can be arbitrarily great if the acceleration remains uniform for a long enough time. What, then, if the agent causing the acceleration decides not to repay the borrowed energy? What if, in fact, it does not even possess enough energy to pay its immense debt at that time? If we believe in conservation of energy the respective answers must be that the accelerating agent must not be at liberty to avoid transferring the required energy and that it must always possess the necessary amount to cover its accumulated debt.

I claim that this last requirement implies a lower bound on the mass-energy of the device that impels the accelerating charge forward. The form of this inequality might be guessed on dimensional grounds (given that it involves only the mass, the charge, and the

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magnitude of the acceleration), but the situation is simple enough that one can analyze it more exactly.

To make the story more vivid, let us conceive of the accelerating particle as a billiard ball carrying a fixed surface charge e of static electricity and of the agency causing the acceleration as a long rubber band wrapped around the billiard ball (and being tugged on by a rocket, say). The acceleration can be very slight, so that our everyday intuition will be a good guide to what happens. Suppose that at $t = 0$ in some inertial frame, the ball is at rest and we begin to pull the rubber band with a constant force (in the rest frame of the rubber band), so that uniform acceleration ensues. After some years of proper time, we slip the rubber band off the billiard ball. It is obvious that nothing much will happen, and the ball will continue on its way with a constant velocity equal to the velocity it had at the point of release. Yet in this process of release, an immense energy must have been transferred.

Of course the billiard ball itself has acquired an immense kinetic energy because it is now moving near the speed of light, as is the rubber band. However the work done before the release — given that the force of radiation reaction has been zero all along — is by definition no more than would have been needed to accelerate an uncharged body of equal mass. Nothing extra has been provided for the energy radiated. Therefore, there is indeed an immense debt.

At least the debt appears immense when energy is reckoned with respect to the original reference frame, in which the system began at rest. When referred to the comoving frame at the moment of release, the debt might be of a very different magnitude, and in fact one can guess that it is rather small in that frame. Nevertheless, it is not zero, and this means that the rubber band (in which we henceforth include the rest of the accelerating agency, i.e. the rocket or whatever) must possess enough energy to transfer the required minimum, willy nilly, to the system consisting of the billiard ball and its self field.

What is the magnitude of this minimum energy? For simplicity let the motion follow the curve $x^2 - t^2 = 1/a^2$ in some system of Galilean coordinates (so a is the magnitude of the proper acceleration). The energy-momentum radiated per unit proper time is known to be

$$\frac{dP^a}{d\tau} = \frac{e^2 a^2}{6\pi} \frac{dx^a}{d\tau} \tag{1}$$

(in units such that $c = \epsilon_0 = 1$). Hence the total 4-momentum radiated from beginning to end is

$$P^a_{radiated} = \int \frac{dP^a}{d\tau} d\tau = \frac{e^2 a^2}{6\pi} \int \frac{dx^a}{d\tau} d\tau = \frac{e^2 a^2}{6\pi} \Delta x^a \tag{2}$$

where $\Delta x^a = (x_{final} - x_{initial})^a$. On the other hand, if the final energy transferred is Δm in the comoving frame (and if, for example, no momentum is transferred), then the spacetime vector of transferred four-momentum is

$$P^a_{transferred} = \Delta m \frac{dx^a}{d\tau}|_{final} \tag{3}$$

Now, in our coordinate system, the components of the final 4-velocity $u^a = dx^a/d\tau$ are $(u^0, u^1) = a\,(x, t)$, as is easy to derive from the facts that u^a is normalized and orthogonal to the radius vector; and the components of Δx^a are $(\Delta x^0, \Delta x^1) = (\Delta t, \Delta x) = (t, x - 1/a)$. Furthermore, after the particle has been moving at the speed of light for a while, we have $t \approx x \gg 1/a$. Hence

$$\frac{dx^a}{d\tau}|_{final} \approx a\,\Delta x^a \tag{4}$$

Comparing equations (2) and (3) in view of (4), we see that

$$\Delta m\, a \approx \frac{e^2 a^2}{6\pi} \quad \Rightarrow \quad \Delta m \approx \frac{e^2 a}{6\pi}$$

In order for this to be possible, the rubber band must have at least the mass Δm, so we arrive at the promised lower bound,

$$m \gtrsim \frac{e^2 a}{6\pi} \tag{5}$$

governing the mass-energy of a rubber band impressing an acceleration of magnitude a on a charge of magnitude e.

In the form given above, the derivation of (5) does little to suggest an underlying reason for this bound. However, one can see that there exists, at least heuristically, a connection between our result and the energy conditions utilized in general relativity. We explore this connection further in the Appendix.

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Appendix

Let F be the tension in the rubber band or equivalently the force it exerts on the billiard ball, and let μ be its mass per unit length. The "dominant energy condition", requires that $F \leq \mu$. Furthermore, if r is the radius of the billiard ball, then in order to wrap around the ball, the rubber band's length should at least be in the vicinity of $2r$, whence its energy m will be bounded below by $2r\mu$. Assuming that the energy condition is satisfied, this implies $m \gtrsim 2rF$.[†]

Now let M be the mass of the billiard ball itself. From the force equation $F = Ma$ we can conclude $M = F/a \lesssim m/2ra$. But since the electric field surrounding the ball contributes an energy of $e^2/8\pi r$ to M, we also have $M \gtrsim e^2/8\pi r$. Concatenating these inequalities yields $m \gtrsim e^2 a/4\pi$, which is substantially the same as (5).

[†] One might think to shorten the rubber band by gluing it to the billiard ball instead of wrapping it, but in this case the stresses set up in the ball — which we have been neglecting — might induce a similar mass increment. We have been neglecting any such effect by assuming tacitly that the inertial mass of the billiard ball is unaffected by acceleration. If one wished to relax this assumption, then perhaps the quantity m in (5) should be re-interpreted as the net mass of rubber band plus mass excess induced in the ball.