FIRST STEPS WITH CAUSAL SETS *

Rafael D. Sorkin
Department of Physics **
Syracuse University
Syracuse, NY 13244-1130

and

Enrico Fermi Institute
University of Chicago
5640 S. Ellis Av.
Chicago, IL 60637-1433

If you are asked to speak in honor of someone's birthday, you may have the problem that nothing you are working on at the moment is related to what the honoree him/herself has done. In the present case my problem is the opposite, in part because I happen to be working on too many things just now, but mainly because Peter has displayed such a breadth of interest in the fundamental questions of physics throughout his life.

In starting to prepare this lecture I had thought of speaking about an interesting logarithmic diffusion law that replaces the familiar $\Delta x \propto \sqrt{\Delta t}$ behavior at very low temperatures, due to quantum effects; and this would have been entirely appropriate in view of Peter's studies in the foundations of statistical mechanics. In fact one of the best discussions of the classical foundations of thermodynamics is still to be found in volume 2 of Peter's book, *Basic Theories of Physics*.

I had also thought of speaking about a new proof of the General Relativistic positive energy theorem based on the focusing properties of gravity, and again this

* Appeared in R. Cianci, R. de Ritis, M. Francaviglia, G. Marmo, C. Rubano, P. Scudellaro (eds.), *General Relativity and Gravitational Physics,* (Proceedings of the Ninth Italian Conference of the same name, held Capri, Italy, September, 1990), 68-90, (World Scientific, Singapore), (1991)

** Permanent address

would have been appropriate because Peter's pioneering work—along with Dirac—on the canonical formulation of gravity provided the setting in which the conserved quantities of General Relativity found their first satisfactory derivation.

Finally I had thought of presenting an analysis of the EPRB experiment from the sum-over-histories standpoint, or perhaps a sum-over-histories approach to decaying systems; and again either one of those topics would have been closely related to Peter's work on the interpretation of Quantum Mechanics. In that work (which also involved another of today's celebrants, Joel Lebowitz) there appeared the famous trace formula for probabilities of strings of observations which figures prominently in current discussions within the Relativity community concerning the interpretation of quantum mechanics.

However, I think that Peter's first love in physics has been the problem of quantum gravity, and it is certainly through his emphasis of its importance that he has had the strongest direct influence on me. It therefore seems most appropriate to devote this lecture to that problem. The view I will adopt in doing so is not one that Peter promoted directly (as far as I know), but I present it in the spirit of trying unorthodox solutions to fundamental difficulties, as Peter himself exemplified when he proposed to fix the gauge difficulties of quantum gravity by using a coordinate-system made from scalar invariants of the metric.

Since the structure I am going to propose as the basis of spacetime is discrete, let me take a moment to recall briefly why many people find an underlying discreteness more natural than the persistence of a continuum down to arbitrarily small sizes and short times. I think the main reasons can be summarized by referring to three self-contradictions or "infinities" which haunt today's physics (ghosts of Zeno perhaps?). The first infinity is that (or really those) of quantum field theory, which one can symbolize by the equation $Z = \infty$, and which the technique of renormalization tries to take care of. The second infinity, symbolized by $R_{abcd} = \infty$, is that of the divergent spacetime curvatures which develop at the singularities of classical General Relativity.

The third infinity belongs to quantum gravity proper, and is perhaps less generally appreciated than the first two, although to me personally it is equally compelling. I mean the infinite black hole entropy which results when one tries to actually count the degrees of freedom of the horizon [1], and which can be expressed by the equation $S_{BH} = \infty$. In addition to contributing this new infinity, quantum gravity makes some of the old ones worse, of course, since it ruins the perturbative renormalizability of the so-called standard model.

It thus seems that the application of quantum ideas to gravity is spawning new contradictions, rather than smoothing out the rough edges of the old ones as had been hoped. A final example of this is the apparent impossibility of measuring the metric on sub-Planckian scales, without the apparatus collapsing into a black hole.

Since all of these contradictions occur around or below the Planck length, they would automatically disappear if at around 10^{-32}cm the continuous manifold of General Relativity gave way to what Riemann called a 'discrete manifold', with only a finite number of degrees of freedom in any finite volume. And in addition there is the possibility (mooted already by Riemann) that the discrete structure might carry its metric relations within itself, which a continuous manifold like $\mathbf{R^4}$ can never do. At the very least, we could expect an explanation of why a fundamental length appears in physics, just as the atomic radius defines a fundamental scale of distance for ordinary matter.

Now the particular discrete manifold I am proposing is known as a *causal set* [2-4]. The way in which its intrinsic order-structure is supposed to relate to (or better give rise to) the metrical continuum of General Relativity can be indicated by the two correspondences,

$$N \longleftrightarrow V$$

$$x \prec y \longleftrightarrow x \in \bar{J}^{-}(y).$$

The first correspondence means that any region of what we ordinarily think of as continuous spacetime is made up of only a finite number, N, of elements of the underlying causal set, and that number is equal to the volume V of the region, when volume is measured in fundamental units. Or to say this the other way around, what we are actually doing when we measure spacetime volume is indirectly just to count the number of causal set elements comprising the region, just as weighing a bar of copper is an indirect way to count the copper atoms comprising the bar.

The second correspondence is equally straightforward, and just states that the macroscopic light-cone structure of spacetime directly reflects the order relation of the underlying causal set. An analogy here might be the way the fracture-planes of a crystal reflect the geometrical structure of its underlying atomic lattice.

Before going any further, I should say more precisely what mathematical structure I am going to be dealing with. Formally, a causal set is a relation satisfying axioms of transitivity, acyclicity, and local finiteness. In symbols these axioms say, respectively,

$$x \prec y \prec z \Longrightarrow x \prec z$$

$$x \prec y \text{ and } y \prec x \Longrightarrow x = y$$

$$\text{cardinality}([x, y]) < \infty.$$

Here $\prec$ is the order relation defining the causal set, and $[x, y]$ is the so-called Alexandrov set of elements lying between x and y with respect to the order, i.e. the set of all elements z such that $x \prec z \prec y$. Taken together, these axioms just say that a causal set is what mathematicians would call a locally finite, partially ordered set. (The local finiteness is expressed by the third axiom, which may be a little stronger than necessary. In its present form it resembles the macroscopic condition that a Lorentzian manifold be globally hyperbolic.)

One obvious motivation for this particular choice of discrete structure is that it relates very directly to macroscopic causality, which, as many workers have sensed, may have a more fundamental character than other macroscopically apparent relations like length. But what is especially appealing about causal sets is that their discreteness is essential to their ability to reproduce macroscopic geometry. If an infinite number of elements were present locally then the correspondence $V = N$ would lose its meaning, and without it we could at best hope to recover the conformal metric, but not the volume element needed to get from the latter to the full metric, g_{ab}.

In my mind, this interdependence between order and discreteness is important evidence that we are on the right track in choosing our fundamental discrete structure to be an order. Another strong encouragement is the prospect of the unification that would accompany a successful theory based on causal sets. In such a theory, the single relationship $\prec$ would unite within itself the topology, the differential structure, the metrical geometry, and of course the causal structure of spacetime. In particular it would explain the Lorentzian signature of the metric (this being the only signature for which distinct past and future directions can be defined), whereas from most other points of view the presence of a minus sign in the signature appears to be mathematically unnatural.

But how should one go about actually constructing a theory (a quantum theory presumably!) of causal sets? In the development of most physical theories one can distinguish two stages, corresponding roughly to what in mechanics are called kinematics and dynamics.

In the present case, the former stage would be concerned first of all with the fact that the main macroscopic properties we would like to make contact with are *emergent* in the sense that they become meaningful only when the causal set is organized in an appropriate way. Concepts such as length, topology and dimension, make little sense for a generic causal set; so it is necessary to understand in what circumstances they do emerge. We would like to be able to recognize these circumstances, and equally importantly, to learn how then we can read information about dimension, etcetera, directly out of the order information itself. To appreciate the difficulty of this task, it is enough to glance at figure 1, which shows three fairly simple causal sets whose dimension is two in a sense we will make more precise below.

A second task of this kinematical stage of investigation would be just to gain familiarity with the new mathematics needed to describe causal sets, much of which belongs to the branch of combinatorial theory devoted to partial orders. Unfortunately this branch of mathematics is unfamiliar to most physicists, and it is also true that the questions mathematicians have concentrated on are not always those which are likely to be the most relevant physically. So a significant period of kinematical development is likely to be needed before we possess the concepts which will help us survey the different possibilities for constructing a quantum dynamics for causal sets.

The dynamical stage of development of causal set theory would, of course, be the one in which we would understand their "laws of motion". But because time itself is discrete in this case, we could not hope to write down some Hamiltonian generator of time-evolution for the causal set. Indeed it is not even clear what 'configuration space' could mean for causal sets, and therefore unclear what Hilbert space such a Hamiltonian could act on. The only available framework to work in thus appears to be that of the sum-over-histories. In this framework our job is to assign an amplitude to each causal set, and then figure out how to sum these amplitudes in order to obtain physically meaningful probabilities.

In looking for the correct amplitude function, I think we have a couple of requirements to guide us. Most obviously there is the requirement that the dynamics should possess a classical limit in which the causal set would look like a smooth manifold. Since classical limits arise from constructive interference of many histories, this presumably means that Lorentzian manifolds should be in some sense, "stationary phase" points of the causal set amplitude. Conversely the causal sets which are far from looking like manifolds (and they are the vast majority) should be points at which the amplitude is in some sense "extremely rapidly varying". (Notice, however, that the amplitude-function can have no actual derivative because a causal set cannot be varied continuously. This means that a classical dynamics for individual causal sets will probably not be definable at all, seemingly a good sign.)

The other requirement, related in more than one way to the first, is what I would call "locality". By this I mean the requirement that the effective Action contributed by the family of all causal sets corresponding to a given Lorentzian manifold (M, g), should look like the integral of a locally defined quantity in M. For reasons related to their inherent Lorentz invariance, this is much more difficult to achieve with causal sets than with ordinary sorts of lattices. On the other hand, if it *is* achieved then one can argue that the needed *form* of the resulting integrand (namely, $-\Lambda + \frac{1}{2\kappa}R$) follows almost of its own accord. Hence the thing to focus on in searching for the right amplitudes seems to be not the specific form of the Einstein equation, but just its local character in spacetime.

In the remainder of this lecture I would like to survey some of the progress that has been made in constructing a theory of causal sets along the lines I have just indicated. To be consistent with the historical scheme I advocated earlier, I ought to spend all of my time discussing kinematics; in fact I can not resist talking about dynamics as well. But perhaps that is more than excusable in the present case, because the lack of clearly relevant experimental or observational evidence to guide us, means that we really will have little indication whether we have been traveling in the right direction until we reach the territory of dynamics proper.

The first task of causal set kinematics, should really be to verify that it is possible, even in principle, for a structure as elementary as a causal set to give rise to a Lorentzian manifold in some suitable approximation. Because the whole theory would be stillborn were it not true, we call this belief the "Hauptvermutung", after a conjecture that at one time was thought to be central to the theory of manifolds. (I do hope, however, that this new Hauptvermutung has a better fate than its namesake, which got whittled down and whittled down, until finally it turned out to be completely false!)

In order to see in what way we could formulate a conjecture of this type, let us return to the basic correspondences involving volume and causal relationship from which our theory sets off. In effect these correspondences are telling us what it means for a manifold M with Lorentzian metric g to approximate a given causal set C, and the criterion they are expressing is this:

> The manifold (M, g) "emerges from" the causal set C iff C "could have come from sprinkling points into M at unit density."

Here unit density means with respect to the fundamental unit of volume (which is unknown as yet, but expected to be around the Planckian value of $10^{-139}\mathrm{cm}^3\mathrm{sec}$), and of course the sprinkled points inherit their causal ordering from the light-cones of g. Actually, I probably should have added the qualifier 'modulo coarse-graining' to the words 'could have come', but let me ignore that complication here.

Now if we interpret sprinkling to mean random generation of points according to a Poisson process in M then we can actually prove a limiting version of the Hauptvermutung (work with David Meyer). Namely let C_1 be the result of one such process and let C_2, C_3, etc. be the results of sprinkling in additional points at ever higher densities (i.e. we are imagining the fundamental length to go to zero).

Then $C_1 \subseteq C_2 \subseteq C_3 \ldots$, there is a well defined sense in which the limit can be taken, and we have with probability one that

$$(M, g) = \lim_{j \to \infty} C_j.$$

This theorem is encouraging, but really we would like to dispense with the limiting process, since in reality the fundamental volume is small but not zero. Luca Bombelli has made some progress in this direction, but difficulties remain, even in knowing how precisely to formulate the desired theorem. For now, let me just indicate the kind of result we would hope for. Let C be a causal set and let $f : C \to M$ and $f' : C \to M'$ be faithful embeddings (i.e. embeddings which are sprinkling-like according to some suitable criterion) of C into (M, g) and (M', g'). Then there exists an approximate isometry $h : M \to M'$ such that $f' = h \circ f$.

Assuming—as seems very likely—that causal sets do possess a structure rich enough to give us back a macroscopically smooth Lorentzian geometry, our real task would be to figure out how in practice we can extract geometrical information from an order relation. But before we can speak of a geometry we must have a manifold, and the most basic aspect of a manifold's topology is its dimension. So is there a good way to recognize the effective continuum dimension of a causal set (or more precisely of a causal set that is sufficiently "manifold like" for the notion of its dimension to be meaningful)? In fact *two* workable methods seem to exist, and I would like to speak briefly about each of them.

The first method is perhaps more traditional in character, and relies on the existence of a family of relatively small causal sets which can serve to characterize the dimension of any Minkowski spacetime via embeddings. Since the sets in question are small, it would be inappropriate to worry about uniformity of embedding, so we will require of an embedding only that it induce the correct order relations on the embedded points.

To get an idea of what such causal sets might be like, take a look at the prototypes for dimensions 1 through 3 shown in figure 2. It is obvious that set (a) for $d = 1$ will embed in $\mathbf{M}^1$ but not in $\mathbf{M}^0$ (where $\mathbf{M}^d$ is Minkowski space of d-dimensions); and it is equally obvious that set (b) for $d = 2$ will go into $\mathbf{M}^2$ but not into $\mathbf{M}^1$. Similarly set (c) for $d = 3$ will pretty clearly embed in $\mathbf{M}^3$ (think of the diagram as a perspective drawing), and almost as clearly not embed in $\mathbf{M}^2$ (as can be rigorously proven with a little trial and error).

Now it turns out [5-7] that there exist, for each spacetime-dimensionality d, analogous causal sets which will embed in $\mathbf{M}^d$ but not in $\mathbf{M}^{d-1}$. A systematic construction of one such family has been given in [7]. For dimension d the set in question comprises the 2^d possible subsets S of the d-element set $\{1, 2, 3, \ldots d\}$, ordered according to the rule,

$$S_1 \prec S_2 \Longleftrightarrow S_1 \subseteq S_2 \ \ \textbf{and} \ \ \mathrm{card}(S_1) = 1 \text{ or } \mathrm{card}(S_2) = d - 1.$$

(There is no time here to indicate the proof that this family does indeed behave as claimed, but let me just mention one incidental surprise that came out of this study, namely the conjecture that the so-called binomial poset on 6 elements, B_6 cannot be embedded in a Minkowski spacetime of *any* dimension. As a set B_6 is defined just as specified above for $d = 6$, but in the definition of its order we drop the restriction on the cardinalities of S_1 and S_2.)

Now the notion of dimension that results from considering embeddability in Minkowski space may be called "flat conformal dimension", because it would not care if we replaced the flat metric on $\mathbf{M}^d$ by any other metric in the same conformal equivalence class. It does not directly apply to the kind of very large causal set we are actually interested in, because we certainly do not want to limit ourselves to the case where the continuum approximation to the causal set in question is free of curvature. On the other hand it is true by definition that any continuum manifold is flat on small scales, so it would be natural to try to extract the desired dimensional information by looking at suitable small subsets of our causal set. If among them we found ones of maximum "flat conformal dimension" d, in the above sense, then we could conclude that the effective continuum dimension of our causal set was d.

The second method of recognizing dimensionality that I referred to has a statistical character; and, since it yields dimension values which need not be integral, I will call it 'fractal'. Unlike the method just discussed, this one uses volume information as well as conformal information, and seems to be somewhat more efficient for that reason. Actually there can be several distinct variants of this method, but the basic idea always involves counting suitable substructures of a given Alexandrov subset of our causal set.

Given such a subset $A = [x, y]$, the simplest thing to count is of course the number N of elements of A, and the next simplest is probably the number R of *relations* it contains, i.e. the number of pairs of elements $u, v \in A$ such that $u \prec v$. If the full causal set can be faithfully embedded in $\mathbf{M}^d$ then the typical values of N and R will be expectation values with respect to a Poisson process taking place within an Alexandrov neighborhood of $\mathbf{M}^d$ itself. Now, by Lorentz invariance, these expectation values can depend only on the volume of this Alexandrov neighborhood (or "double light cone") and, of course, on the dimension d itself. Forming a ratio to make the volume drop out, and performing a short calculation, we find

$$\frac{< R >}{< N >^2} = \frac{3}{4} \frac{(3d/2)!}{d!(d/2)!}$$

Because the right-hand-side is a monotonically decreasing function of d, the equation can be inverted to give a unique "fractal dimension" for each value of the ratio R/N^2. For suitable small Alexandrov subsets of a "manifold-like" causal set, the fractal dimension should average out to the true continuum dimension. For larger ones it would be necessary to take the curvature into account (c.f. [8]).

Computer tests of this method of recognizing dimension [8] show that it works reasonably well in low dimensions ($d = 2, 3, 4$), with the computed dimension converging rapidly to the true one as the number of sprinkled points becomes larger than about 300-400. In high dimensions more points would be needed of course, but not as many as you might think. In fact $N \sim (27/16)^d$ will suffice, as follows from computing the variance of the number of relations, R. By comparison the special causal sets discussed earlier require 2^d points to characterize dimension d.

Beyond recognizing dimensionality, there is the problem of extracting information about global connectivity, and ultimately about the metric itself. In particular one may ask how to estimate the geodesic distance between two causal set elements (assuming as always that the causal set resembles a continuum geometry (M, g).) Thus let x and y be elements of the causal set C, and let $x \prec y$. The idea which suggests itself most readily would be just to define a "distance" from x to y by counting the number of elements in the longest chain joining these two elements, where a chain is by definition a succession of elements, $x \prec z_1 \prec z_2 \prec z_3 \ldots \prec y$. Clearly a maximal chain in this sense is analogous to a timelike geodesic, which also maximizes the distance between its endpoints.

Now, how does length defined in this way compare to geodesic length? Once again, a precise answer is known only for a causal set sprinkled into Minkowski space, but that answer [9] is very encouraging (as also were the results of some computer experiments done by David Meyer involving chains of up to about 20 links). In the asymptotic limit of large distances there is a strict proportionality $L \sim cT$, where T is the Minkowski distance between two sprinkled points x, y, and L is the number of links in the longest chain joining these points. The constant c depends on the spacetime dimension and is not known exactly, although fairly tight bounds on its value are available. Assuming the proportionality holds up when curvature is present, we will have a way to extract timelike distances directly from the order relation $\prec$. No equally direct method appears to exist for spacelike distances, but there would seem to be a good prospect of getting them from the ensemble of timelike ones.

There are also other ways of deducing geometrical information from order information. For example, the type of counting technique we used to define fractal dimension will in certain situations yield the value of the scalar curvature, and in other situations the radius of the "internal circle" of a Kaluza-Klein vacuum-metric [5,8]. However all these methods have been developed only on the assumption that the causal set is faithfully embeddable in *some* continuum spacetime. What we still lack is a way to judge directly whether or not this is the case; i.e. whether the causal set will give rise to any Lorentzian manifold at all.

Before leaving kinematics and turning briefly to the question of dynamics, I would like to describe some progress that has been made on what one can call "counting problems". We have already seen how geometrical insight can be gleaned just by counting suitable substructures, but I think such enumeration is likely to be important for another reason. I think it gives us a way to associate to a causal set, numbers in terms of which the amplitude entering into the sum-over-histories could be defined. In any case let me mention some counting problems whose solution is known, as well as some which are clearly important, but which have not been solved so far.

We have already encountered the notion of a chain, in connection with the definition of timelike distance. A natural question is how many distinct chains exist between two specified elements. For a causal set sprinkled into *two-dimensional* Minkowski space it is possible to evaluate in closed form the expected number of chains joining two sprinkled points separated by the time lapse T, and one finds $<\text{chains}> = \mathrm{I}_0(\sqrt{2}T)$, where $\mathrm{I}_0(x) = \mathrm{J}_0(ix)$ is a so-called modified Bessel function of the first kind. (Notice that mathematicians have defined 'chain' to mean *any* totally ordered sequence of elements, even if the sequence "skips over" intermediate elements. Thus a chain need not consist of "links", and, e.g., any subset of a chain is also a chain.)

One can also ask for the number of chains of a *fixed* length occurring as subsets of a given causal set, say one sprinkled into an Alexandrov neighborhood of $\mathbf{M}^d$. In this case the expected number $< C_k >$ of k-element chains is known for all dimensions d. [5,8] Also known is a formula for the expected number of k-element chains in a causal set sprinkled into a Lorentzian manifold of constant curvature.

In this connection, I might mention some mysterious identities that have been found to relate certain expectation values of the sort we have been considering. Thus [5,8] in $\mathbf{M}^d$ we have that $< V >< R >=< C_3 >< N >$, where 'V' stands for the number of three-element subsets such that one element precedes the other two, and 'C_3' stands for the number of 3-element chains. This identity relates the numbers of certain specified sub-causal-sets of a sprinkled causal set, and is rigorously true in any dimension. A similar but empirical identity has turned up in $\mathbf{M}^2$ in computer work by Jorge Pullin and Eric Woolgar, and states that $< L >^2= 2 < W >< N >$, where now L is the number of links (unrefinable two-element chains) and W the number of "V's" each of whose "arms" is a link.

It would be easy to pose many other counting problems of equal potential interest to those just referred to, but I'm sure several will occur to you without my mentioning them explicitly. However I *would* like to mention a slightly different counting problem of obvious importance for understanding the classical limit of causal set theory. That is the problem of determining the total number of distinct causal sets which can result from sprinkling N points into a given Lorentzian manifold (M, g). Clearly this is related to the question of how far a given sprinkling can be perturbed without changing the induced order, but, aside from a poorly justified estimate based on such considerations, this is a counting problem which is still awaiting a solution.

There are some other kinematical issues which I would have liked to discuss had there been time, especially coarse-graining and the related possibilities of scale-dependent dimensionality, emergent matter fields, and the like. Rather than spend my remaining minutes on such topics, however, I would like to report on some intriguing results and preliminary work concerning the dynamical stage of causal set theory.

As I said earlier, the sum-over-histories framework appears to be the most natural point of departure for anyone trying to develop a plausible "quantum equation of motion" for causal sets. In this framework each causal set C would carry an amplitude $A(C)$; and dynamics would be contained in the amplitude-function $A(\cdot)$, together with the combining rules telling us how to use the amplitudes to construct meaningful probabilities. Without yet possessing these latter rules in their final form, we can still investigate some of the consequences of particular choices of $A(\cdot)$, and thereby try to gain insight into what features an adequate choice would have to have.

Consider, for example an amplitude of the form $A = \exp(i\beta R)$, R being the total number of relations, as before. This looks like a familiar path-integral amplitude with trivial measure, and with R and β playing the roles of "classical Action" and "coupling constant" respectively. To prevent the amplitude-sums from diverging, one can take the total number of causal set elements to be a fixed integer N. Now, if we were to go over to the corresponding "statistical mechanics" problem by continuing β to imaginary values, then we would be dealing with a random causal set with N elements and with probability-weight given by the "Boltzmann factor", $\exp(-\beta R)$. It happens that just this problem has been studied in connection with a certain "lattice-gas" model [10]. This, of course, was one reason I chose it as an example, but another was the suggestiveness of the intimate relation between R and dimensionality that we observed earlier (cf. Myrheim [4]).

Actually the model studied was not exactly that defined by the above "Boltzmann factor", but instead by the corresponding "micro-canonical ensemble", in which R is fixed and every causal set with that R (and with N elements) is weighted equally. The first result of interest is that, in the "thermodynamic limit", $N \to \infty$, at least two, and probably an infinite number, of phase transitions occur as R/N^2 (correspondingly βN^2) is varied. For small values of this parameter the most probable causal sets possess only two "layers" (i.e. no chain has more than three elements), and the phase transitions occur as successively greater numbers of layers begin to contribute. In some some very general sense the causal set is thus becoming more manifold-like with each such transition (but so far there is no real evidence that genuine manifold behavior sets in for any value of the parameter).

One other noteworthy event that accompanies the 2-level to 3-level phase transition is a spontaneous breaking of time-reversal symmetry. In the 2-level phase the most probable configurations look similar to their T-reversals, but in the initial 3-level phase the causal sets of high-probability have very unequal numbers of elements in their top and bottom layers. I obviously would not want to claim that this effect was at the root of the cosmological time-asymmetry, but it does demonstrate the possibility that something of the sort could ultimately emerge from a better understanding of causal set dynamics.

The other work I want to report on here concerns not the causal set's own dynamics, but rather that of a "'scalar field" living in a fixed (or "background") causal set. Such additional degrees of freedom might be necessary to incorporate "matter", but even if they are not, their study can serve to clarify issues like locality which will be crucial to the choice of amplitude for the causal set itself. So let C be a given causal set and let ϕ be a real-valued function on C, or in other words a "real scalar field". One would like to discover an Action $S(\phi)$, defined purely in terms of ϕ and the order $\prec$, which in appropriate situations will reproduce the known behavior of a quantum scalar field in curved spacetime.

In what we have done so far (work with Jorge Pullin and Eric Woolgar) we have limited ourselves to a causal set sprinkled into 2-dimensional Minkowski space, $\mathbf{M}^2$, and searched for an $S(\cdot)$ which would reproduce the Action of a free scalar field. To that end, consider within C any "wedge", by which I mean a triple of elements x_0, x_1, x_2 such that the pairs $x_0 \prec x_1$ and $x_0 \prec x_2$ are both links. (I referred to this structure earlier in connection with counting problems.) Letting ϕ_1 denote $\phi(x_1)$, etc., I can form the following quadratic expression as a sum over all wedges: $S_W = -\sum(\phi_1 - \phi_0)(\phi_2 - \phi_0)$; and in the same way the sum over all links, $S_L = -\sum(\phi_1 - \phi_0)^2$.

We chose the first expression because it resembles the square of the gradient expressed in u-v-coordinates, if you take the two links of the wedge as rightward null and leftward null. The second expression also resembles the continuum Lagrangian to some extent, but clearly can't be correct by itself since it can never be positive. Now one can argue that there should exist *some* linear combination of these expressions that will yield (on average) the correct Action for arbitrary *linear* ϕ. What is surprising, however, is that the simple difference $S = S_W - S_L$ appears to be the combination that works (modulo an overall normalization that depends on N). The tests are still unfinished, but for the three test fields $\phi \equiv t, \phi \equiv x$, and $\phi \equiv t + x$, we find errors in $< S >$ of about 15% for 500 points and 6 runs, and of about 1% for 10,000 points and 10 runs.

For some nonlinear functions including $\phi \equiv x^2$ and $\phi \equiv \sin(t)$ the results are also not too bad, but for more rapidly oscillating ones like $\phi \equiv \sin(5t)$ they are awful (Here the points are sprinkled into an Alexandrov neighborhood whose height in the t-direction is unity.), and they don't get noticeably better as the number of sprinkled points is increased.

In fact the difficulty was to be expected as a manifestation of the problem with locality that I alluded to earlier, and can be seen as a manifestation of a very general conflict among locality, discreteness, and Lorentz invariance. What happens in this case is that the links and wedges, which we would like to be rather "small", can instead be "very long and skinny". For example, a link which in one reference-frame looks to be purely timelike and of length 1, will in a highly-boosted frame appear to be very long and almost null. By including such links (and the analogous wedges) in our expression for $S(\phi)$ we make S depend on very large finite differences instead of only on small ones that could furnish a good approximation to the gradient of ϕ. In fact S_W alone would in some sense already have been a good approximation to S had not this problem been present, and one can probably understand the effect of subtracting S_L as a partial cancellation of the contribution of the "long and skinny wedges". What is needed, then, is a full cancellation of their contribution, leaving behind an expression for the Action that one could expect to be accurate for a much wider range for test functions.

In fact, there is at the moment, the prospect of obtaining such a cancellation by using "diamonds" instead of "wedges". But the main point I want to make is that locality in this sense might be the guiding star leading us to an appropriate dynamics for causal sets, and not just for whatever fields may inhabit it. If we can achieve locality for such "matter" fields, then we could acquire a new type of "random lattice" approximation to quantum field theory in curved (or even flat!) spacetimes. But also—and more importantly for our present purposes—we might begin to understand how to build local amplitudes for the causal set itself. Indeed it could even turn out that the resulting ϕ-dynamics would directly yield an effective dynamics for the underlying causal set in the manner of "induced gravity", that is, via "integrating out" the ϕ-fluctuations in the sum-over-histories.

This might seem an adequately optimistic note on which to conclude, but before doing so, I want to throw out a more speculative idea that illustrates how the assumption of a discrete substratum for spacetime can suggest otherwise unexpected routes to the resolution of basic puzzles like the smallness of the observed cosmological constant. Suppose, in fact, that this smallness of Λ were a statistical effect due to the very large number of elements making up the relevant portion of the universe. Then we might expect whatever cancellations were responsible for this smallness to take place not exactly, but just with some statistically imposed accuracy. This suggests the formula

$$\Lambda \sim \frac{1}{\sqrt{N}},$$

N being the number of elements in question. Well, for the observable universe (to date) N is of the order of 10^{240}, and Λ would thus be somewhere around 10^{-120} in natural units. Might it turn out to be more than a coincidence that this is currently the largest value not yet ruled out by astronomical data?

This research was partly supported by NSF grants PHY 9005790 and 8918388

References

[1] Bombelli, L., Koul, R.K., Lee, J. and Sorkin, R.D., "A Quantum Source of Entropy for Black Holes", *Phys. Rev.* **D34**, 373-383 (1986) [2] Bombelli, L., Lee J., Meyer, D. and Sorkin, R.D., "Spacetime as a causal set", *Phys. Rev. Lett.* **59**, 521 (1987).

[3] Bombelli, L., *Space-time as a Causal Set*, Ph.D. thesis, Syracuse University (1987);
Sorkin, R.D., "A Specimen of Theory Construction from Quantum Gravity", in *How Theories are Constructed: The Methodology of Scientific Creativity* (Proceedings of the Thirteenth Annual Symposium in Philosophy, held Greensboro, North Carolina, March, 1989) (to appear).

[4] Finkelstein, D., "The space-time code", *Phys. Rev.* **184**, 1261 (1969);
Myrheim, J., "Statistical geometry," CERN preprint TH-2538 (1978);
't Hooft, G., "Quantum gravity: a fundamental problem and some radical ideas,", in *Recent Developments in Gravitation* (Proceedings of the 1978 Cargese Summer Institute) edited by M. Levy and S. Deser (Plenum, 1979).

[5] Meyer, D.A., *The Dimension of Causal Sets*. Ph.D. thesis, M.I.T. (1988).

[6] Meyer, D.A., "The Dimension of Causal Sets I: Minkowski dimension," Syracuse University preprint (1988);
"Spherical containment and the Minkowski dimension of partial orders," Syracuse University preprint (1989).

[7] Brightwell, G. and Winkler, P., "Sphere Orders", *Order* **6**, 235 (1989).

[8] Meyer, D.A., "The Dimension of Causal Sets II: Hausdorff dimension," Syracuse University preprint (1988).

[9] Brightwell, G. and Gregory, R., "The Structure of Random Discrete Spacetime", preprint (1990).

[10] Dhar, D., "Entropy and phase transitions in partially ordered sets", *J. Math. Phys* **19**:1711-1713 (1978); and "On phase transitions in posets", *Pacific J. Math.* **90**: 299-305 (1980);
Kleitman, D.J. and Rothschild, B.L., "A Phase Transition on Partial Orders", *Physica* **96A**: 254-259 (1979).