From Green Function to Quantum Field[⋆]

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Abstract

A pedagogical introduction to the theory of a gaussian scalar field which shows firstly, how the whole theory is encapsulated in the Wightman function $W(x,y) = \langle\phi(x)\phi(y)\rangle$ regarded abstractly as a two-index tensor on the vector space of (spacetime) field configurations, and secondly how one can arrive at $W(x,y)$ starting from nothing but the retarded Green function $G(x,y)$. Conceiving the theory in this manner seems well suited to curved spacetimes and to causal sets. It makes it possible to provide a general spacetime region with a distinguished "vacuum" or "ground state", and to recognize some interesting formal relationships, including a general condition on $W(x,y)$ expressing zero-entropy or "purity".

Keywords and phrases: gaussian field, S-J vacuum, Peierls bracket, ground-state condition, purity criterion, entropy of Wightman function

1. Introduction and setup

Since these lectures are intended as a pedagogical introduction to a certain type of quantum field theory, I should begin by explaining in what sense we will be understanding that concept. The specific theory in question will be that of a gaussian (real) scalar field, by which I mean what is often referred to as a free scalar field in a gaussian (or "quasi-free") state. Thus I am including a "choice of vacuum" in the definition of the theory. Although doing this is not unheard of (consider the Wightman axioms, for example), it could seem

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unfamiliar from the "algebraic" perspective according to which the essence of a quantum theory is simply a $\star$-algebra of "observables", with no distinguished "state" being specified.

I have nevertheless chosen to proceed this way for two reasons. First of all, if one takes a quantum dynamics to be given by a quantum measure [1] or decoherence functional [2] (as opposed to a propagator, for example), then one erases any principled distinction between "initial state" and "equation of motion".

Secondly, and more importantly, we will be proceeding in a somewhat unorthodox fashion. Rather than start with the field equations, we will base our construction solely on the retarded Green function and the volume-element $\sqrt{-g}\, d^4x$ of the spacetime in which the field will live, following an approach which arose in the course of defining a quantum field theory on a causal set. [3] [4] The field-operators will then be derived from the two-point correlation function, which in turn will be derived from the Green function. Such an approach builds in "state-information" from the very beginning, in the form of the two-point correlation function or "Wightman function", $W(x, x')$.

In this connection it seems fitting to call to mind some other well known situations in which knowledge of a "vacuum", or of a set of favoured "vacua", plays an important role. Such are Hawking radiation, the Unruh effect, and inflationary scenarios (cf. their attendant "trans-Planckian" worries). In the same vein, one can try to identify a distinguished state or vacuum which is "natural to" the early universe, supposing the underlying metric to be known. If this were done, and if the resulting vacuum turned out to be, for example, thermal with approximately scale-free fluctuations, one would have taken a step toward explaining (with no appeal to inflation) the initial conditions that gave birth to our part of spacetime.

The kinematic framework for these lectures

We begin with a spacetime manifold M, either compact with boundary ∂M, or non-compact without boundary, and we write $\overset{\circ}{M} = M \backslash \partial M$. To signify that $x \in M$ lies in $J^-(y)$, the causal past of $y \in M$, we will write "$x \prec y$", and we will define for $a, b \in M$, $J(a, b) = \{x \in M | a \prec x \prec b\}$. We will assume that M is globally hyperbolic in a sense generalizing that employed in [5]. Namely, we assume that M is causally locally convex, that $J(a, b)$ is compact $\forall a, b \in M$, and in the case with boundary that $J(a, b)$ is disjoint from ∂M when $a, b \in \overset{\circ}{M}$. In addition, one might want to impose some form of completeness on M, for example that every geodesic that doesn't end on ∂M extends to infinite affine parameter.

Our dynamical variable will be a real scalar field $\phi = \phi(x)$ of mass $m \geq 0$, massive or massless but in any case free. (For brevity, I will sometimes write intermediate formulas only for the massless case.)

Global hyperbolicity implies that the retarded Green function

$$G(x, y) \equiv G^{ret}(x, y)$$

exists and is uniquely defined. For definiteness, we will take it to be a biscalar (i.e. a density of weight 0 in both its arguments). We have then

$$(\Box - m^2) G(x, y) = \widehat{\delta}(x, y) \equiv \frac{\delta^{(4)}(x - y)}{\sqrt{-g(y)}} \tag{1}$$

We will often employ matrix or tensor-index notation for G and other two-point functions, writing for example "G^{xy}" instead of "$G(x, y)$". Then $\widetilde{G}$ will denote the transpose of G and $\overline{G}$ its (elementwise) complex conjugate:

$$\widetilde{G}^{xy} = G^{yx} \ , \quad \overline{G}^{xy} = (G^{xy})^*$$

We will denote hermitian conjugate (adjoint) with a star, and write $A \natural B$ to signify that $AB = BA$. Our Lorentzian-metric signature will be that of $(- + + +)$.

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2. Green functions and the commutator function

Recall that G, written without any qualifying superscript, represents the *retarded* Green function G^{ret}. Thus $G(x,y) = 0$ unless $x \succ y$. In what follows it will important to know that the transpose function $\widetilde{G}$ is *also* a Green function, which by definition is equivalent to it being the advanced Green function:

$$\widetilde{G^{ret}} = G^{adv} \tag{2}$$

In flat spacetime this equality holds trivially since $G(x,y) = G(x-y)$ by translation-symmetry, but we do not want to limit ourselves to that case. In the general case, (2) can be proven as follows (see figure 1).

Proof First notice that, just as global hyperbolicity implies the existence and uniqueness of the retarded Green function, it does the same for the advanced Green function. Let $H = G^{adv}$ be this function and consider the integral

$$\begin{aligned} &\int G(x,a)\ (\overleftrightarrow{\Box(x)} - m^2)\ H(x,b)\ dV(x) \\ = &\int G(x,a)\ \overleftrightarrow{\Box}(x)\ H(x,b)\ dV(x) \end{aligned}$$

where the double arrow indicates antisymmetrization as in the equality, $A \overleftrightarrow{\nabla} B = A(\nabla B) - (\nabla A)B$. On one hand (cf. (1)) our integral equals

$$\begin{aligned} &\int G(x,a)\ \widehat{\delta}(x,b)\ dV(x) - \int \widehat{\delta}(x,a) H(x,b) dV \\ = &\ G(b,a) - H(a,b) \end{aligned}$$

On the other hand it equals, by Stokes theorem,

$$\oint_{\partial M} G(x,a)\ \overleftrightarrow{\nabla}^{\mu}(x)\ H(x,b)\ dS_\mu(x)$$

Now for $a, b \in \overset{\circ}{M}$, the last integral necessarily vanishes because its integrand is zero unless $a \prec x \prec b$, whereas the global hyperbolicity of M implies that $J(a,b)$ cannot touch ∂M (or reach infinity).

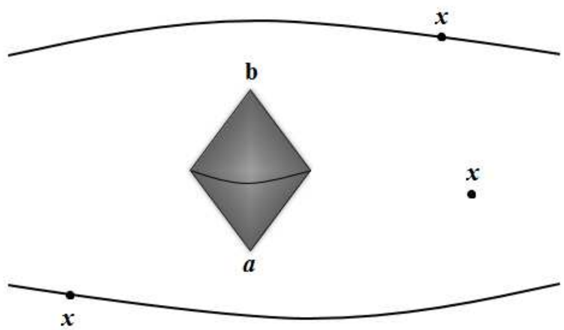

Figure 1. Setup for the proof of (2). The shaded region is $J(a,b)$.

Then by continuity,

$$H(a,b) = G(b,a) = \widetilde{G}(a,b)$$

holds everywhere, as was to be proven. Now let us define

$$\Delta = G - \widetilde{G} \ , \tag{3}$$

noting then that $(\Box - m^2)\Delta = 0$, and that Δ is real and skew: $\widetilde{\Delta} = -\Delta$.

The following lemma is generally useful, and it will help to explain why our construction of the field theory "works", however we won't need it as such below. For this reason I will state it only in a formal sense and will provide only a rough proof. *

Lemma $\;\mathrm{image}\,\Delta = \mathrm{kernel}(\Box - m^2)$

Before trying to prove this, let's state another lemma whose content is essentially the same, but whose demonstration is slightly more direct (see figures 2 and 3).

Lemma $\;\ker\Delta \perp \ker(\Box - m^2)$

In verifying this lemma, let us take $m = 0$ for convenience, and further let $f \in \ker\Delta$ and $\varphi \in \ker\Box$. We are then required to show that $\int f\varphi dV = 0$. To that end, notice first that $f \in \ker\Delta \iff \Delta f = 0 \iff Gf = \widetilde{G}f \equiv h$. I claim that it follows from

* The lemmas are only formal because we ignore niceties like taking the closure of $\mathrm{im}\,\Delta$ or characterizing the domain of Δ when M is non-compact.

this that both h and its gradient vanish on ∂M. Accepting this for now, notice also that $\Box\, h = \Box\, Gf = f$, since $\Box\, G = 1$ by definition of a Green function. Therefore,

$$\begin{aligned}\int \varphi f dV &= \int \varphi \,\Box\, h dV \\ &= \int \varphi \overleftrightarrow{\Box} h dV + \int (\Box\, \varphi) h dV \\ &= \oint_{\partial M} \varphi \overleftrightarrow{\nabla} h \cdot dS \\ &= 0\end{aligned}$$

where we used $\Box\, \varphi = 0$ in the second line, and then in the fourth line the fact that both h and ∇h vanish on ∂M.

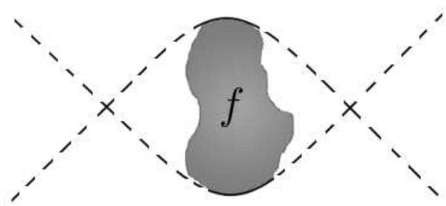

Figure 2. Illustrating the support of f, its past and future

To complete the demonstration, we need to prove this last fact. To that end, observe first that our definition of global hyperbolicity precludes that any finite portion of ∂M is timelike. For simplicity, let's also ignore any null portions as well, and consider, say, $x \in \partial^- M$, the past boundary of M. It's then almost obvious (see figure 3)

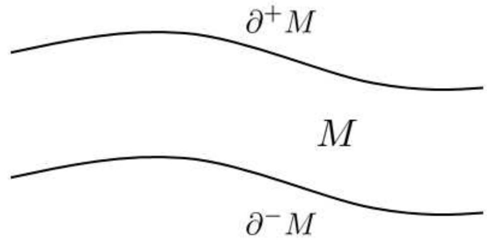

Figure 3. The spacetime M with its past and future boundaries

that $h(x) = \int G(x,y)f(y) = 0$ since only y to the past of x could have contributed. That $\nabla h(x)$ also vanishes is less obvious, but it follows from the observation that, on dimensional grounds the integral giving $h(x)$ must vanish like τ^2 as $x \to \partial^- M$, provided that f is bounded.

We can corroborate this dependence explicitly in both two and four dimensions. In the former case (and for $m^2 = 0$), $G = -1/2$ within the past lightcone (and of course $G = 0$ elsewhere), whence $\int Gf$ will scale like the area of $J^-(x)$, which in turn scales like τ^2 as $\tau \to 0$. In the latter case the corresponding 4D integral looks like $\int\limits_0^\tau dt \delta(r-t)/r\, r^2 dr \sim \int\limits_0^\tau t\, dt \sim \tau^2$, once again.

Problem Clean up this proof.

With the aid of the lemma just proven, we can easily complete the proof of the lemma that precedes it. What we've shown so far is $\ker \Box \subseteq \perp \ker \Delta = \operatorname{im} \widetilde{\Delta} = \operatorname{im} \Delta$. On the other hand, $\operatorname{im} \Delta \subseteq \ker \Box$ trivially since $\Box\, \Delta = 0$. Taken together these say that $\operatorname{im} \Delta = \ker \Box$, as asserted.

Problem Can we relate $\ker \Delta$ to $\operatorname{im} \Box$?

(The lemma relates it to $\operatorname{im} \widetilde{\Box}$, but $\widetilde{\Box}$ is not obviously the same as $\Box$.)

Remark The lemma implies that the image of Δ consists entirely of solutions to the wave equation, $\Box\, \phi = 0$. In this sense the kernel ("nullspace") of Δ, being orthogonal to its image, has to be enormous, because the solutions ϕ are "very few" in relation to the set of all functions ϕ on M. In other words, $\operatorname{im}\Delta$ is "very small", while $\ker\Delta$ is "very big". In the case of a causal set, this relationship would be meaningful without the quotation marks, because the dimensions of both $\operatorname{im}\Delta$ and $\ker\Delta$ would be finite, but it usually turns out to hold only in an approximate manner: Δ has very many tiny eigenvalues, but only a handful of *exact* 0-modes! One might say in such cases that the equation of motion for ϕ holds only approximately in the causal set. It turns out that this complicates the definition of entanglement entropy in a way that is discussed further in [6].

The commutation relations in covariant form (Peierls bracket)

With a free field, the commutator of any two field-operators $\widehat{\phi}(x)$ is a c-number, and we have the luxury of being able to express it in a manifestly covariant form known as the Peierls bracket, namely as $[\widehat{\phi},\widehat{\phi}] = i\Delta$, or more explicitly:

$$[\widehat{\phi}(x),\widehat{\phi}(y)] = i\Delta(x,y) \tag{4}$$

In order to relate this form to the more traditional equal-time commutation relations, let us first examine a simple example in $\mathbb{M}^2$ with Cartesian coordinates $t = x^0$ and $x = x^1$. If, starting from the covariant form $[\widehat{\phi}(x),\widehat{\phi}(y)] = i\Delta(x-y)$ with $y = 0$, we take the derivative $\partial/\partial t$ and then set $t = 0$, we obtain

$$[\dot{\widehat{\phi}}(t,x),\widehat{\phi}(0,0)] = i\dot{\Delta}(t,x)|_{t=0} \ ,$$

which we want to equal $-\delta(x)$. Equivalently, we want

$$\frac{\partial\Delta}{\partial t}|_{t=0} = -\delta(x) \ .$$

But we know (defining $u = t-x$, $v = t+x$) that, as illustrated in figure 4,

$$G(u,v) = -\frac{1}{2}\theta(u)\theta(v)$$

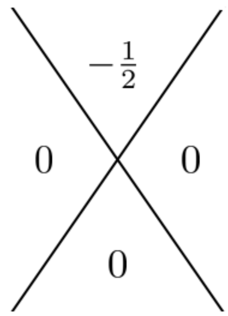

Figure 4. The massless Green-function $G(x)$ for $\mathbb{M}^2$

whence $-2\Delta = \theta(u)\theta(v) - \theta(-u)\theta(-v)$. Noticing also that $\dot{u} = \dot{v} = 1$ and that at $t = 0$, $\delta(u) = \delta(v) = \delta(x)$, we easily compute

$$\begin{aligned} -2\frac{\partial \Delta}{\partial t} &= \delta(u)\theta(v) + \theta(u)\delta(v) + \delta(u)\theta(-v) + \theta(-u)\delta(v) \\ &= \delta(u)[\theta(v) + \theta(-v)] + (u \leftrightarrow v) \\ &= \delta(u) + \delta(v) \\ &= 2\delta(x) \quad [t = 0] \\ \frac{\partial \Delta}{\partial t} &= -\delta(x) \end{aligned}$$

as required.

The general case is best handled in a coordinate-free fashion with the aid of the Klein-Gordon inner product, defined for solutions f and g of the wave equation by †

$$f \bullet g = \int_\Sigma f \overleftrightarrow{\nabla}^\mu g \, dS_\mu$$

where Σ, which can be any Cauchy surface, is externally oriented toward the future (i.e. the sign of dS_μ is chosen to agree with that of $\partial_\mu t$, where t is chosen to vanish on Σ and be positive in the future of Σ, see figure 5).

† In the lectures, I denoted this inner product by "KG" with a circle around it, but I don't know how to make such symbol in plain Tex.

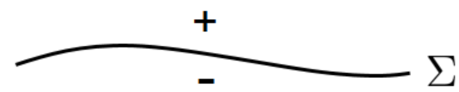

Figure 5. Orientation of the surface-element dS_μ for the Klein-Gordon-inner product

Lemma $(\Box - m^2) f = 0 \quad \Rightarrow \quad \Delta \bullet f = f$

Proof (see figure 6) Written out in full, the equation we are trying to prove says, at $x \in M$,

$$\int\limits_{y \in \Sigma} \Delta(x,y) \overleftrightarrow{\nabla}^{\mu}(y)\, f(y)\, dS_\mu(y) = f(x)$$

Without loss of generality we can assume that $\Sigma \prec x \prec \Sigma'$, where Σ' is a second Cauchy surface introduced to the future of x, as shown. Then for $y \in \Sigma$, $\Delta(x,y) = G(x,y) - G(y,x) = G(x,y)$, whence the integral we are evaluating becomes

$$\begin{aligned}
&\int\limits_{y \in \Sigma} G(x,y) \overleftrightarrow{\nabla}^{\mu}(y)\, f(y)\, dS_\mu(y) \\
&= -\Big(\int\limits_{y \in \Sigma'} - \int\limits_{y \in \Sigma} \Big)\; G(x,y) \overleftrightarrow{\nabla}^{\mu}(y)\, f(y)\, dS_\mu(y) \qquad \text{(since } \Sigma' \succ x) \\
&= - \int_{\Sigma}^{\Sigma'} G(x,y) \overleftrightarrow{\Box}(y) f(y)\, dV(y) \qquad \text{(Stokes theorem)} \\
&= \int_{\Sigma}^{\Sigma'} \widehat{\delta}(x,y)\, f(y)\, dV(y) \\
&= f(x)
\end{aligned}$$

(The minus sign in the third line arises because the Stokes theorem requires that dS_μ be oriented *outward*, which in the case of Σ means pastward, given that Σ constitutes the past boundary of the region between Σ and Σ'. [7] Notice in this connection that the "boundary at spatial infinity", if any, does not contribute, because $G(x,y)$ vanishes outside some finite radius.) This completes the proof of the lemma.

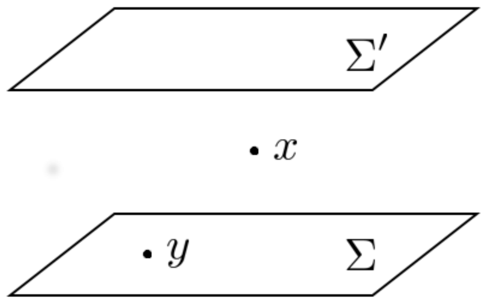

Figure 6. Setup for proof of the lemma

Now let f and g again be solutions and consider the expression

$$f \bullet \Delta \bullet g \tag{5}$$

Evaluating it first directly and second with the aid of the lemma, we can demonstrate that (4) is equivalent to the equal-time commutation relations that one commonly sees. In preparation, recall first that the surface-element $dS_\mu(x)$ equals $n_\mu(x)d^3V(x)$, where n_μ is the unit normal to Σ at x and $d^3V(x)$ is the volume-element of Σ. And let a dot denote normal time-derivative, so that given our sign conventions, $\dot{F} = -n^\mu \nabla_\mu F$ because n^μ points pastward, when n_μ has the same sign as $\partial_\mu t$. Let us also make the ansatz that on Σ, $\dot{f} = 0$ and $g = 0$. (This ansatz is always possible, since any choice of f and $\dot{f}$ yields valid Cauchy data for the wave equation.) For (5) we find then

$$\int\int_\Sigma f(x) \overleftrightarrow{\nabla}^\mu(x)\ \Delta(x,y) \overleftrightarrow{\nabla}^\nu(y)\ g(y)\ n_\mu(x)d^3V(x)\ n_\nu(y)d^3V(y)$$
$$= \int\int f(x)\dot{\Delta}(x,y)\dot{g}(y)d^3V(x)d^3V(y)$$

On the other hand, the lemma teaches us that (5) is equal to $f \bullet g$

$$
\begin{aligned}
&= \int f \overleftrightarrow{\nabla}_\mu \, g \; n^\mu d^3V \\
&= \int (-f\dot{g} + \dot{f}g) d^3V \\
&= \int -f\dot{g} \, d^3V \\
&= -\int f(x) \, \widehat{\delta}^{(3)}(x,y) \, \dot{g}(y) \; d^3V(x) d^3V(y)
\end{aligned}
$$

Comparing the two expressions then reveals that on Σ, $\dot{\Delta}(x,y) = -\widehat{\delta}^{(3)}(x,y)$. (Recall that we defined $\widehat{\delta}(x,y)$ to be a bi-scalar.)

We have thus established the nontrivial one of the three equal time commutation relations, the counterpart of $[q,p] = i$. The other two may be left as an exercise:

PROBLEM Show that the equal-time commutation relations $[\phi,\phi] = 0$ and $[\dot{\phi},\dot{\phi}] = 0$ follow similarly if one chooses $f = g = 0$, respectively $\dot{f} = \dot{g} = 0$.

As stated earlier, we will build up our scalar field $\widehat{\phi}$ in a manner suggested originally by the needs of causal set theory, where certain features of the (local) continuum theory seem to be available only in approximate form. Essential to our work will be the recognition that one can characterize a gaussian field with no other input than a two-point function $W(x,x')$, and it is to this remarkable characteristic of gaussian theories that we now turn.

3. Gaussian fields

What, at a minimum, does it take to define a field theory? If the field in question is *gaussian*, then it suffices to have the one- and two-point correlation functions,

$$\langle \phi(x) \rangle \qquad \text{and} \qquad W(x,y) = \langle \phi(x)\phi(y) \rangle$$

Or if, as is most often the case, and as we will assume herein,

$$\langle \phi(x) \rangle = 0 \ ,$$

one can make do with $W(x,y)$ alone. Once this "Wightman function" is given, the remaining n-point functions follow via Wick's rule:

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n) \rangle = \sum \langle \phi\phi \rangle \langle \phi\phi \rangle \cdots \langle \phi\phi \rangle \tag{6}$$

where the sum is taken over all pairings of the arguments, $\phi(x_1)\cdots\phi(x_n)$. (In accord with the rule, the n-point functions with n odd all vanish.)

For present purposes we may take (6) as the meaning of gaussianity, inasmuch as in the analogous situation with probability distributions, this "Wickian" pattern of moments is known to characterize the gaussian distributions as a family, and individually to determine any such distribution fully (compare also the Wightman axioms for flat-space quantum field theory). An analogous relationship for quantum field theory may be found in [8], where it is shown how Wick's rule leads to a path-integral (more precisely a *quantum-measure*) with a manifestly gaussian form. Notice here that, as advertised in the introduction, we are regarding the expectation-value functional, $\langle\cdot\rangle$, as part of the definition of our theory, which is why I have been speaking of a "gaussian theory" or "gaussian field", and not for example, a "gaussian state".

Although it is true that W determines the theory, it far from true that it can be specified freely. On the contrary $W(x,y)$ needs to satisfy the condition that, regarded as a matrix — or better as a quadratic form — it is positive-semidefinite: [♭]

$$(\forall f) \qquad \int dV(x)\, f^*(x)\, W(x,y)\, f(y)\, dV(y) \;\geq\; 0 \tag{7}$$

This condition (or rather the attendant condition of "strong positivity of the quantum measure") is perhaps best regarded as an irreducible axiom of quantum theory. However it can also be derived as a theorem, provided that one knows that $\langle\cdot\rangle$ can be expressed as expectation value with respect to a state-vector $|0\rangle$ (or a density-matrix) in some Hilbert space. This follows immediately from the positivity of $||\psi||^2 = \langle\psi|\psi\rangle$ with $\psi = \int dV(x) f(x)\widehat{\phi}(x)|0\rangle$. Among its other consequences, (7) guarantees, via the fluctuation-dissipation theorem, the "passivity" condition that one cannot extract work from a system in thermal equilibrium by purely mechanical means (cf. [9]).

Granted positivity, we can obtain a Fock space and field operators $\widehat{\phi}(x)$ thereon simply by diagonalizing W and introducing the corresponding "annihilation operators" $\widehat{a}$, as we will see in a moment. By doing so, we can verify that nothing is missing from our formulation that one might want to include under the rubric of "field theory".

[♭] If f and g are taken to be scalar densities rather than scalars then the volume-element dV will not appear in this condition.

First however, let me introduce a notation and a viewpoint that will allow us to conduct the passage from W to $\widehat{\phi}$ in a more abstract and general setting. For us so far, a field has been a function $f : M \to \mathbb{R}$, but if we abstract from this particular context, we can view it simply as an element of a real vector-space V, and accordingly view W as a tensor in $V \otimes V \otimes \mathbb{C}$.

To diagonalize the tensor W means to cast it into the form

$$W = \sum_k w_k \otimes \overline{w}_k \tag{8}$$

Written in an indicial notation, (8) reads $W^{xy} = \sum\limits_k w_k^x \, \overline{w}_k^y$, which of course would be interpreted in our specific context as $W(x,y) = \sum\limits_k w_k(x) \, \overline{w}_k(y)$. (Because W is positive, all of the terms in (8) necessarily take the form $+w \otimes \overline{w}$ and none the form $-w \otimes \overline{w}$.) On the basis of (8) we can introduce in the familiar manner, ladder-operators $\widehat{a}_k$ such that

$$[\, \widehat{a}_k, \widehat{a}_l^{\,*} \,] = \delta_{kl} \qquad \text{(say)} \tag{9}$$

and a "Fock vacuum" $|0\rangle$ such that $\widehat{a}_k|0\rangle = 0$, defining then

$$\widehat{\phi} = \sum_k (w_k \, \widehat{a}_k + \overline{w}_k \, \widehat{a}_k^*) \tag{10}$$

That these definitions correctly reproduce W is not hard to show. Indeed, with $\langle \cdot \rangle$ taken to be $\langle 0| \cdot |0\rangle$, we have

$$\begin{aligned}
\langle \widehat{\phi} \otimes \widehat{\phi} \rangle &= \sum_k \langle (w_k \widehat{a}_k + \overline{w}_k \widehat{a}_k^*) \otimes (w_l \widehat{a}_l + \overline{w}_l \widehat{a}_l^*) \rangle \\
&= \sum_k (w_k \otimes \overline{w}_l \langle \widehat{a}_k \widehat{a}_l^* \rangle + 0 + 0 + 0) \\
&= \sum_k w_k \otimes \overline{w}_k \\
&= W
\end{aligned}$$

It is equally straightforward to demonstrate the "canonical commutation relations" in their abstract form:

$$[\, \widehat{\phi}^x, \ \widehat{\phi}^y \,] = W^{xy} - W^{yx} \tag{11}$$

[The expectation value of (11) follows immediately from the calculation just above, but to derive it as relation among operators requires a separate proof. Notice also that we have not written the right hand side of (11) as $i\Delta^{xy}$. We could have done so, but in the present development, that would count as the definition of Δ, not a relation between W and some independently defined object.]

Problem Equation (11) can also be written as $\widehat{\phi}\wedge\widehat{\phi} = W - \overline{W}$, where $a\wedge b \equiv a\otimes b - b\otimes a$. Prove it in this form, with the aid of (8) and (9).

In (10), we have written a real (selfadjoint) operator $\widehat{\phi}$ in terms of complex operators $\widehat{a}$ and complex vectors w. One can also work exclusively with real operators $\widehat{q}$ and $\widehat{p}$, and real vectors u and v for which

$$W = \sum \frac{1}{2}(u \otimes u + v \otimes v + iu \wedge v) \tag{12}$$

and $[q, p] = i$. The "purely real" form of (10) is then

$$\widehat{\phi} = \sum \, (u\,\widehat{q} + v\,\widehat{p}) \ , \tag{13}$$

while the relation between the two representations is given by $w = (u - iv)/\sqrt{2}$ and $\widehat{a} = (\widehat{q} + i\widehat{p})/\sqrt{2}$.

The construction of vacuum-vector and field-operators from W is now complete, but a few further comments are in order, before returning to the more concrete context where our field ϕ is a function on the manifold M.

A question that might arise in relation to the decomposition (8) is why we didn't bother to normalize the vectors w_k. The answer in this case is that the question presupposes some reference metric with respect to which such a normalization could take place. In this section, we have been aiming at generality, and therefore have taken care to avoid introducing any metric on V. When V consists of functions on M, however, a metric is available in the form of the L^2-inner product, and we will in fact use it in the next section to formulate a certain "ground-state condition" that can serve to determine W uniquely in certain cases. Here, though, we have simply taken W as given, without asking where it might have come from.

A second question concerns the uniqueness of the Fock representation we have just set up. The only input was supposed to be W, but what would have happened had we used a different set of vectors w_k to diagonalize W? To this the answer is that nothing would have happened. Given that any second set of vectors w'_k will be related to the first by a unitary matrix (i.e. $w'_k = \sum U_{kl} w_l$), it is not hard to see that one arrives thereby at a unitarily equivalent Fock representation. That is, our representation, $\widehat{\phi}$-cum-$|0\rangle$, is unique up to unitary equivalence.

Problem Show that if one defines the "tensor operator" $\widehat{a}$ by

$$\widehat{a}^{\,x} = \sum_k w^x_k \, \widehat{a}_k$$

then our ansatzes for $\widehat{a}_k$, $\widehat{\phi}$, and $|0\rangle$ result in equations from which any reference to the specific vectors w_k introduced to diagonalize W has dropped out, to wit:

$$\widehat{\phi}^{\,x} = \widehat{a}^{\,x} + (\widehat{a}^{\,x})^*$$
$$[\,\widehat{a}^{\,x},\, (\widehat{a}^{\,y})^*] = W^{xy}$$
$$\widehat{a}^{\,x}|0\rangle = 0$$

A third question that one might ask is not disposed of so simply, because it emphasizes something that we have *not* claimed about our construction. What if the tensor W from which we started were that of an "impure" gaussian theory, such as that of a system in thermal equilibrium? The entropy would then have to be nonzero, whereas our Fock vacuum seems to be that of the "pure state" $|0\rangle$. The resolution to this paradox is that (as with the closely related case of GNS representations of *-algebras) our Fock vacuum can be impure if the operators $\widehat{\phi}$ defined by (10) fail to act irreducibly in the Fock space. Of course the operators $\widehat{a}$ and $\widehat{a}^{\,*}$ do act irreducibly, essentially by definition, but that does not necessarily imply that the $\widehat{\phi}$ do so as well, because one might not be able to recover the former from the latter. This *would* follow if, with respect to some hermitian reference metric, the w_k and the $\overline{w}_k$ were all orthogonal to each other (which actually will be one of our "ground-state conditions" below), but it need not be true always.

That the $\widehat{a}$ might not be recoverable from the $\widehat{\phi}$ is perhaps easiest to understand in finite dimensions, where the index on ϕ^x can be taken to run from 1 to N. The decisive question, then, is how many w_k there are in relation to the ϕ^x. Let k run from 1 to M, where M is the rank of W as a matrix (the dimension of its image). From N components $\widehat{\phi}^x$, we need to recover $2M$ operators, $\widehat{a}_k$ and $\widehat{a}^*_k$, which is conceivable only when $2M \le N$. Thus, purity (zero entropy) requires that rank(W) be sufficiently small. A simple example is $N = 2$, with $\phi^1 = q$ and $\phi^2 = p$, where purity corresponds to the rank of W being 1 rather than 2.

A final comment is that if W happens to satisfy (exactly) some "equation(s) of motion" (which means abstractly that $v_x W^{xy} = 0$ for some covector or set of covectors v_k), then plainly our field operator $\widehat{\phi}$ will satisfy the same equation(s). But since we never needed any such equation of motion, our construction will go through equally well in the causal set case where such equations might hold only approximately. This of course was one motivation for proceeding in the way we have.

In the present section, we have raised our whole structure on the foundation of a real vector space V and a Wightman tensor $W \in V \otimes V \otimes \mathbb{C}$. But where does W come from in practice? One answer to this question is provided by the so-called SJ-ansatz, and it is to that which we turn next.

4. The "S-J vacuum"

Returning to the setting of scalar functions on a globally hyperbolic spacetime, let us assume that we are given a function $G = G^{ret}$ to play the role of the retarded Green function. We will derive a distinguished Wightman function W from G by way of Δ. In contrast, recall the usual story as symbolized by the progression,

$$\Box - m^2 \;\to\; G \;\to\; \Delta \;\to\; [\widehat{\phi}, \widehat{\phi}] \;\xrightarrow{+\text{ freq}}\; \widehat{a} \;\to\; |0\rangle \;\to\; W$$

Here we will tread the shorter path,

$$G \to \Delta \to W \ ,$$

in whose following out the Klein-Gordon field equation will play no role, except implicitly through its connection with the Green function G from which Δ is formed.

To obtain Δ from G, we simply recall the Peierls prescription,

$$\Delta = G - \widetilde{G} \ ,$$

which then furnishes information on W itself via the relation,

$$W - \overline{W} = i\Delta$$

Note here that since $\widetilde{\Delta} = -\Delta$ the tensor $i\Delta$ is antisymmetric and pure imaginary, hence hermitian. We also know that W must be positive: $W \geq 0$. From this, $\overline{W} \geq 0$ follows as well, so that $W - \overline{W} = i\Delta \Rightarrow\ W = i\Delta + \overline{W} \geq i\Delta$. In a certain sense (relative to (14)), our W will be the smallest tensor which is $\geq i\Delta$

This is the basic idea, but the detailed prescription that implements it can be given in three different, but equivalent, forms.[*] L^2-inner product on scalar fields (or more abstractly on the vector-space V) will play an important part:

$$\langle f|g\rangle = \int d^4V(x)\, f(x)^*\, g(x) \tag{14}$$

The W that we will thereby build up has come to be known as that of the "S-J vacuum". As will become clearer in the following section, it provides a kind of "ground state" that continues to be defined even in the presence of curvature and the absence of any Killing vector.

First prescription. By availing ourselves of the inner product that occurs in (14), we can "lower one of the indices" on Δ, and thereby construe it as an operator from $V \otimes \mathbb{C}$ to itself. This in turn makes it possible to form powers and more general functions of Δ, and we use this ability in stating our first prescription for W:

$$W = \mathrm{Pos}(i\Delta) = \frac{i\Delta + \sqrt{-\Delta^2}}{2} \tag{15}$$

Clearly $W \geq 0$, and we have $W = R + i\Delta/2$ with $R = \frac{1}{2}\sqrt{-\Delta^2}$. Hence $W - \overline{W} = i\Delta$ as required.

Remark If we can compute $\sqrt{-\Delta^2}$ directly, then we never need to diagonalize anything!

Problem Devise an algorithm to compute $\sqrt{-\Delta^2}$ without having to diagonalize $i\Delta$. Potential methods: resolvent? iterative? other?

Second prescription. A more explicit expression for W comes from diagonalizing Δ. Partly for variety, but also because it is useful for computer arithmetic, I will describe this in real form, where it amounts to doing a singular-value decomposition of Δ, or equivalently to block-diagonalizing the real skew matrix Δ into 2×2-blocks of the form $\begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$. The resulting form for Δ is

$$\Delta = \sum_j \sigma_j\ u_j \wedge v_j \tag{16}$$

[*] More background on the S-J vacuum can be found in [10] [4] [11] [12] [13] [14]

where the $\sigma_j > 0$ are the so-called singular values, and u_j and v_j are real vectors that together form an orthonormal family:

$$\int_M u_j u_k dV = \delta_{jk}, \quad \int_M u_j v_k dV = 0, \quad \int_M v_j v_k dV = \delta_{jk} \tag{17}$$

Written out fully, (16) reads as

$$\Delta(x,y) = \sum_j \sigma_j \ (u_j(x)\, v_j(y) - v_j(x)\, u_j(y))$$

Substituting (16) into (15) produces successively

$$-\Delta^2 = -\sum \sigma^2 \ (u \wedge v) \cdot (u \wedge v) = \sum \sigma^2 \ (u \otimes u + v \otimes v)$$

$$\sqrt{-\Delta^2} = \sigma \ (u \otimes u + v \otimes v)$$

$$W = \sum \frac{\sigma}{2}(iu \wedge v + u \otimes u + v \otimes v) = \sum \frac{\sigma}{2}(iu \otimes v - iv \otimes u + u \otimes u + v \otimes v) \tag{18}$$

Thus our second prescription furnishes W in the form

$$W = \sum \sigma \ \frac{u - iv}{\sqrt{2}} \otimes \frac{u + iv}{\sqrt{2}} \ , \tag{19}$$

which we can also write as $W = \sum \sigma \ w \otimes \overline{w}$ with $w_j = (u_j - iv_j)/\sqrt{2}$.

This last form is the special case of (8) where we choose the w_k to diagonalize not only W but also (14), and where we also normalize them with respect to (14). If desired, one can of course go on to define operators $\widehat{\phi}(x)$ from (19), just as in (10) or (13).

Notice here that in diagonalizing W, we have also diagonalized the hermitian tensor $i\Delta$, because in

$$i\Delta = W - \overline{W} = \sum_j \sigma_j \ w_j \otimes \overline{w}_j - \sigma_j \ \overline{w}_j \otimes w_j \tag{20}$$

the w and $\overline{w}$ are orthogonal, thanks to (17). The w_j [resp. $\overline{w}_j$] are thus the eigenvectors of $i\Delta$ with positive [resp. negative] eigenvalue. We could of course have started with this form, rather than with (16).

Remark In 4D with M compact, one can show that $i\Delta$ is self-adjoint, whence "diagonalizable". [13]

Third prescription. From the first prescription, we have

$$4\, W\, \overline{W} = (\sqrt{-\Delta^2} + i\Delta)(\sqrt{-\Delta^2} - i\Delta) = (\sqrt{-\Delta^2})^2 + \Delta^2 = -\Delta^2 + \Delta^2 = 0$$

Since $W = W^*$, this is equivalent to $W\widetilde{W} = 0$, which in turn says that the rows (and columns) of W square to zero. Still another way to say the same thing is $W \perp \overline{W}$, meaning that W and $\overline{W}$ have disjoint supports. (Notice that $W\overline{W} = 0 \Rightarrow \overline{W}W = 0 \Rightarrow W \natural \overline{W}$, where '$\natural$' denotes "commutes with".) These relationships make it possible to characterize W in a third way, as follows.

$$(\alpha)\ W - \overline{W} = i\Delta$$

$$(\beta)\ W\,\overline{W} = 0$$

$$(\gamma)\ W \geq 0$$

Let us prove that $(\alpha) - (\gamma)$ determine W uniquely, given Δ.

$(\alpha) \Rightarrow W = R + i\Delta/2$ where $R = \overline{R}$

$(\beta) \Rightarrow (R + i\Delta/2)(R - i\Delta/2) = 0$

$\quad\Rightarrow R^2 + \Delta^2/4 + i/2[\Delta, R]) = 0$

$\quad\Rightarrow R^2 = -\Delta^2/4 \qquad \& \qquad \Delta \natural R$

$(\gamma) \Rightarrow R = \mathrm{Re}\, W \geq 0 \Rightarrow R = \sqrt{-\Delta^2/4} \Rightarrow W = R + i\Delta/2 = (\sqrt{-\Delta^2} + i\Delta)/2$,

as required by (15).

In this third characterization of W, the crucial new condition is (β). We might therefore term it "the ground state condition". In light of an earlier remark, it would also make sense to regard it as a "purity condition", but it seems best to reserve that exact phrase for a more general condition we will encounter later, of which (β) is a special case.

Remark Our first two conditions, being equivalent to the quadratic equations, $R^2 = -\Delta^2/4$ and $R\Delta = \Delta R$, are relatively simple algebraic requirements on the real matrices R and Δ. It seems probable that they by themselves determine W up to some sign ambiguities that (γ) would then resolve as its only significant input to the prescription. This suggests yet another possible way to avoid having to diagonalize any matrices in building W.

In the previous section we learned that W fully determines our gaussian theory, and in particular that it furnishes an essentially unique field of operators $\widehat{\phi}(x)$. To round out our story, let us prove that this field satisfies the standard equation of motion, $(\Box - m^2)\widehat{\phi} = 0$.

This follows first of all from $\ker(\Box - m^2) = \mathrm{im}\,\Delta$, which is a lemma proven in Section 2. Combining it with (10), which shows that $\widehat{\phi}(x)$ is assembled from vectors $w(x)$ in the image of W, and with (15), which shows that the image of W is included in that of Δ, we conclude, as desired, that $\widehat{\phi}$ is assembled entirely from solutions of the Klein-Gordon equation.

Alternatively, without invoking that lemma, we can simply note, as seen in (10) and (20), that $\widehat{\phi}$ is built exclusively from eigenfunctions of Δ with strictly positive eigenvalues. Since such an eigenfunction f is by definition a multiple of Δf, it is annihilated by the wave operator, because Δ itself is.

5. Stationary spacetimes and the harmonic oscillator

Regarded as a $0+1$-dimensional scalar field, the simple harmonic oscillator is the special case of our considerations given by $m^2 = \omega^2 > 0$ and $M = \mathbb{M}^1$, or $M = [0, T] \subseteq \mathbb{M}^1$ (where $\mathbb{M}^D$ is Minkowski spacetime of D dimensions.) Let us compute the S-J vacuum for this case.

In the present context the equation for the retarded Green function can be written as

$$(\Box - \omega^2)G = (-\frac{\partial^2}{\partial t^2} - \omega^2)G = \delta(t)$$

Evidently $G(t)$ will be a linear combination of $\sin \omega t$ and $\cos \omega t$, and it turns out that

$$G(t) = \frac{-1}{\omega} \sin \omega t \; \theta(t) \ ,$$

Proof $\dot{G} = -\cos \omega t \theta(t) - (1/\omega) \sin \omega t \delta(t)$, the second term of which vanishes inasmuch as $t\,\delta(t) = 0\,\delta(t) = 0$. Differentiating again yields $\ddot{G} = \omega \sin(\omega t)\theta(t) - \cos(\omega t)\delta(t) = \omega \sin(\omega t)\theta(t) - \delta(t)$, whereupon $\ddot{G} + \omega^2 G = 0 - \delta(t)$ as required.
From this follows immediately

$$\Delta(t) = G(t) - G(-t) = \frac{-\sin(\omega t)}{\omega} \ ,$$

the unique odd extension of $G(t)$.

We can now proceed in either "real" or "complex" mode. Let's do the latter. † To that end, we need to cast $i\Delta(t, t') = i\Delta(t - t')$ into the mould of (20). When $M = (-\infty, \infty)$

† In real mode we'd deduce $u(t)$ and $v(t)$ directly from the decomposition $\sin \omega(t - t') = \sin \omega t \ \cos \omega t' - \cos \omega t \ \sin \omega t'$

we do this almost unthinkingly, simply by writing sine as a difference of exponentials, producing thereby

$$i\Delta(t) = \frac{1}{2\omega}(e^{-i\omega t} - e^{i\omega t})$$

$$i\Delta(t,t') = \frac{1}{2\omega}(e^{-i\omega t}e^{i\omega t'} - e^{i\omega t}e^{-i\omega t'}) \tag{21}$$

which has the form (20) with

$$w(t) = \frac{1}{\sqrt{2\omega}}e^{-i\omega t}$$

Moreover, on $M = (-\infty, \infty)$, this does in fact diagonalize $i\Delta$, because $\langle w|\overline{w}\rangle = 0$. Thus the positive part of $i\Delta$ demanded by (15) is seen to be

$$W(t,t') = \frac{1}{2\omega}e^{-i\omega t}e^{i\omega t'} = \frac{e^{-i\omega(t-t')}}{2\omega} \tag{22}$$

Notice that everything in this formula but the factor of 1/2 follows from the general properties of W. (That w cannot be normalized because it is not square integrable clearly does not affect W itself, which one obtains from (21) simply by crossing out the negative term.)

Incidentally, does (22) satisfy our ground-state condition, which required that the columns of W all "square to zero"? Well, in this case, there is one such column-vector for each t', and all them are proportional to the single function $f(t) = e^{-i\omega t}$. But for this function,

$$\int_{\infty}^{\infty} f(t)^2 dt = \int_{\infty}^{\infty} e^{-2i\omega t} dt = 2\pi\delta(2\omega) = 0$$

since $\omega \neq 0$. The answer is therefore "yes" when we work on all of $\mathbb{M}^1$, but notice that on $M = [0, T]$ we would have had instead,

$$\int_0^T e^{-2i\omega t} dt \sim 1/\omega T \tag{23}$$

which does not vanish unless ωT happens to be a multiple of π. (Notice here, in passing, that even on all of $\mathbb{M}^1$, something like $\sin \omega t$ or $\cos \omega t$ would not have worked as a "row" or "column" because its square would have integrated to infinity. Notice also that the "squaring" in question is purely algebraic, involving no complex conjugation: the integrand was $f(t)f(t)$, not $f^*(t)f(t)$)

When $M = [0, t]$ is compact, it thus takes a bit more work to diagonalize W. Instead of a single exponential, we will need $w(t) = \alpha e^{-i\omega t} + \beta e^{+i\omega t}$, where $\beta/\alpha = O(1/\omega T)$ as follows from (23). For $\omega T \gg 1$, the resulting W will therefore contain a small admixture of negative frequencies, that is, the S-J vacuum in this case will differ from the Minkowski vacuum by a small Bogoliubov transformation. The oscillator will be slightly excited.

The same thing will happen in higher dimensions when M is compact, but now it will happen "mode by mode", with the consequence that W will in general not have the so-called Hadamard form.[♭] Since this form enjoys a special status in curved-space quantum field theory, one might wonder whether the S-J prescription could be modified to accommodate it.

I believe that the answer is yes, and that one can in fact do this rather easily, at least when ∂M is everywhere spacelike, by modifying the spacetime volume-element, $dV = \sqrt{-g}\, d^4x$, in such a way as to "soften the boundary of M":

$$dV \;\rightarrow\; \widetilde{dV} = \rho(x)\, \sqrt{-g}\, d^4x \ , \tag{24}$$

where $\rho(x) \rightarrow 0$ smoothly at ∂M (see figure 7).

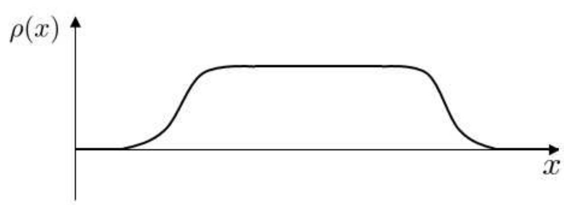

Figure 7. A function ρ that could be used to "soften the boundary"

For the harmonic oscillator, this will enhance the $O(1/\omega)$ falloff exhibited in (23) to be faster than any power of the frequency (an essentially exponential falloff), and the same

[♭] See [13]. Although the "occupation probability" of any given "mode" will die out like $1/k^2$, as we just saw, the number of modes grows so rapidly in $D = 4$ that the "net occupation number" will correspond to the divergent integral $\int k^2 dk/k^2$.

should happen mode by mode in more general spacetimes (not necessarily stationary). Indeed, a related technique (which however refers to an ambient spacetime in which M is embedded) is developed in [15], and the resulting W proven to be Hadamard in a range of cases.

If the method based on (24) also works, as I believe it will, it will have the advantage of being "self-contained". Moreover, one can easily make a "universal" choice of ρ, by taking it to be a fixed function of some convenient "distance to the boundary". For example, letting $V^{\pm}$ denote the volume of $J^{\pm}(x)$, one could take $\rho(x) = f(1/(1/V^{+} + 1/V^{-}))$, where $f(v) \geq 0$ is a fixed function, chosen once and for all, that goes smoothly to 0 at $v = 0$ and smoothly to 1 at some v_0 which sets the scale of the softening. Once such an f has been chosen, (24) provides for any compact M, a distinguished vacuum whose UV behaviour is Hadamard and uniquely determined, if this is thought to be important.

Returning now to the simple harmonic oscillator, and to the unbounded case where $M = (-\infty, \infty)$, we recognize that (22) corresponds exactly to the minimum-energy "vacuum". That in itself would not be surprising if our S-J prescriptions had made some appeal to energy or to positive frequency, but no amount of re-reading of the three characterizations given in the previous section will reveal any such appeal, either overt or covert. Why then did the S-J vacuum of the oscillator turn out to coincide with its minimum energy state?

Evidently, our "ground-state condition" must have been responsible, but how? In (0+1)-dimensions, W will take the form

$$W(t,t') = f(t)\ \overline{f(t')}$$

where, as we saw, f will solve the oscillator equation of motion. Since $f(t) = \xi \sin \omega t + \eta \cos \omega t$ is the general solution, we will have for the integeral of f^2,

$$\int_{-\infty}^{\infty} f^2 = \int_{-\infty}^{\infty} \xi^2 \sin^2 + \eta^2 \cos^2 + 2\xi\eta \sin\cos\ ,$$

which will vanish iff $\xi^2 + \eta^2 = 0$, given that the third term oscillates and integrates to 0. This in turn implies that $\xi = \pm i\eta$, whence (up to normalization)

$$f(t) = e^{\pm i\omega t}$$

In this way words, our ground-state condition singles out from all the solutions, the two of pure frequency. Although that almost determines W, it still leaves open the choice between between purely positive and purely negative frequency. To grasp what resolves this final ambiguity, recall that W was defined to be the *positive* part of $i\Delta$. This implies that $\langle f|i\Delta|f\rangle > 0$, which in turn rejects the negative frequency solution, leaving only $f(t) = e^{-i\omega t}$. Perhaps then, it would be fair to summarize the explanation by saying that "purity of $W \Rightarrow$ purity of frequency", while "positivity of $W \Rightarrow$ positivity of frequency".

The explanation we have just gone through makes it clear that the S-J vacuum will coincide with the minimum-energy vacuum in any spacetime which is stationary in the sense that it admits a Killing vector that is timelike everywhere. In particular the usual and S-J vacua will coincide in $\mathbb{M}^4$. (It is of course necessary that the minimum-energy vacuum exist itself! For example, if ϕ is massless, one must exclude spacetimes like the static cylinder, $\mathbb{R} \times S^3$, or for that matter $\mathbb{M}^2$.)

Question What happens when the spacetime M contains an ergoregion or horizon (in other words when the Killing vector ξ is spacelike in places)?

Remark From its definition, it is obvious that the S-J vacuum — like any vacuum — is inherently global in nature. Perhaps the simplest way to make obvious that this could not have been avoided is to invoke a spacetime which includes two static regions separated in time by a generic dynamical region where a certain amount of "particle creation" will occur. No W could then look like the locally defined vacuum in both static regions. One knows after all, that "particle" is a nonlocal concept — at least for the kind of particle whose absence the word "vacuum" normally signifies!

Remark When M is infinite, our prescription for W need not converge. Consider for example, the harmonic oscillator with $\omega = 0$, more commonly known as the free particle. To what could W possibly converge? If the limit existed and were unique then W would have to be time-translation invariant, so $W(t,t') = f(t-t')$, which seems impossible, given our other conditions. It might be interesting to see exactly what goes wrong in this case when one takes T to infinity in $M = [-T, T]$.

6. For which W does the entropy vanish?

My final lecture at the Workshop was intended to cover entanglement entropy, but time remained only for some brief remarks that mostly summarized material already available in [16] [17]. In addition however, they included a necessary and sufficient algebraic condition for the entropy associated with W to vanish. I will devote the present, final section to this "purity criterion" for W.

To any Wightman-tensor W^{xy} there corresponds an entropy $S(W)$ defined by the sum,

$$S = \sum_{\lambda} \lambda \log |\lambda| \tag{25}$$

where λ runs over the solutions of the generalized eigenvalue equation,

$$W f = i \lambda \, \Delta \, f \tag{26}$$

In the sum (25), each eigenvalue λ must of course be given its correct multiplicity, which can be done by treating as equal any two covectors f that differ by an element of $\ker R$, where R is the real (equivalently the symmetric) part of $W = R + i\Delta/2$.

[Thus, one treats W, R, and Δ as linear maps from $V^*/\ker R$ to V, this being consistent because $W \geq 0 \Rightarrow \ker R \subseteq \ker \Delta$, whence also $\ker R \subseteq \ker W$. The multiplicity of a given eigenvalue λ is then defined to be the dimension of the subspace of $V^*/\ker R$ comprising the solutions f of (26). Here, V^* is the dual space to V, and I have not bothered to distinguish between V and its complexification, $V \otimes \mathbb{C}$, in the formulas just written. Notice also that $\ker R \subseteq \ker \Delta \Rightarrow \operatorname{im} \Delta \subseteq \operatorname{im} R$, so that we can further regard W as mapping $V^*/\ker R$ to $\operatorname{im} R = R[V^*]$. In terms of a matrix representing W, this amounts to the following more "practical" prescription: diagonalize W over the reals into 2×2 and 1×1 blocks and discard the zero blocks. Notice finally, that although the positivity of W entails $\ker R \subseteq \ker \Delta$, the opposite inclusion can in priniciple fail. For such an f, namely one such that $\Delta f = 0$ but $Rf \neq 0$, one has $Wf = (R + i/2\Delta) f \neq 0$, and only $\lambda = \infty$ could satisfy (26). This seems to imply an infinite contribution to (25), and in fact an f of this type, if present, would correspond to a "purely classical component" of W, which would indeed contribute an infinite entropy, the entropy of a classical gaussian probability distribution being infinite unless one introduces some sort of cutoff. In a matrix representation, such an f corresponds to a 1×1 block which is non-vanishing. Before leaving the question of multiplicity, perhaps it's worthwhile to mention yet another equivalent definition. One can construe WR^{-1} as an operator T on $\operatorname{im} R$, and having done so, one sees by suitably rearranging equation (26), that the eigenvalues of T take the form $\lambda/(\lambda - 1/2)$.]

Now let us regard W as *pure* when $S(W) = 0$. It is clear from (25) that this will be the case if and only if the eigenvalues λ are all either 0 or 1, which formally is expressed by the equality, $(\Delta^{-1}W)^2 = i(\Delta^{-1}W)$, or

$$W\,\Delta^{-1}\,W = iW \tag{27}$$

When R and Δ have equal kernels, this equation is actually well defined, even when (as will usually be the case) Δ is uninvertible. It could therefore be taken as our criterion of purity. More generally, though, one can take instead of (27) the rearrangement

$$W\,R^{-1}\,W = 2\,W \tag{28}$$

which is well defined whenever $W \geq 0$ (which it always is). Both (27) and (28) possess equivalent forms involving solely R and Δ, which respectively are

$$R\,\Delta^{-1}\,R = -\Delta/4 \quad \text{and} \quad \Delta\,R^{-1}\,\Delta = -4R \tag{29 a,b}$$

It may be that (27)-(29) have not already appeared in the literature in exactly this form.*

Problem Verify the claims about when (27)-(29) are well defined, and verify that when defined, they are equivalent. Prove more generally that (28) [or (29b)] implies the other three.

Problem Show that the "ground-state condition", $W\overline{W} = 0$, implies the purity condition, $\Delta R^{-1}\Delta = -4R$. The S-J vacuum is therefore always pure.†

In concluding, I would like to express my gratitude to Niayesh Afshordi, Siavash Aslanbeigi, Michel Buck, Fay Dowker, David Rideout, Mehdi Saravani, and Yasaman Yazdi for

* Section 2.2 of [15] defines purity by saturation of a certain inequality of Cauchy-Schwarz type. Very likely, it is provably equivalent to (28).

† Mehdi Saravani has pointed out that the converse seems to hold as well: Every pure W has the S-J form with respect to some metric on V. In this sense, our ground-state condition is actually equivalent to purity.

contributing to the viewpoint put forward herein. Special thanks go to A.P. Balachandran and other members of the audience for asking how the Fock representation of Section 3 could be, paradoxically, both irreducible and reducible at the same time. And special thanks also to Marco Laudato for preparing the diagrams.

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