Does a Quantum Particle Know its Own Energy?*

Rafael D. Sorkin

Perimeter Institute, 31 Caroline Street North, Waterloo ON, N2L 2Y5 Canada
and
Department of Physics, Syracuse University, Syracuse, NY 13244-1130, U.S.A.

address for email: rsorkin@perimeterinstitute.ca

Abstract

If a wave function does not describe microscopic reality then what does? Reformulating quantum mechanics in path-integral terms leads to a notion of "precluded event" and thence to the proposal that quantal reality differs from classical reality in the same way as a set of worldlines differs from a single worldline. One can then ask, for example, which sets of electron trajectories correspond to a Hydrogen atom in its ground state and how they differ from those of an excited state. We address the analogous questions for simple model that replaces the electron by a particle hopping (in discrete time) on a circular lattice.

Keywords and phrases: quantum foundations, histories, path-integral, co-event formulation, anhomomorphic coevents, quantum logic.

Should we try to form for ourselves an image of the quantum world? Or must our theories find their meaning solely in assertions about laboratory instruments and their readings? In other words, is it permissible to ask the question, *To what 'reality' does the quantum formalism refer?* I believe that this question is not only a legitimate one, but it is one that we must ask. If we try to avoid it, we fall into a vicious circle because instruments are made of atoms and not vice versa. More importantly, quantum gravity and, especially, cosmology

* To appear in *Journal of Physics: Conf. Ser.*

need to deal with parts of nature where one finds neither observers nor instruments, and to which an "operational point of view" therefore seems unsuited.

Beyond the more or less familiar reasons just adduced to support the claim that the problems of quantum gravity and "quantum foundations" are intertwined, there's another connection that could be important via the concept of relativistic causality. If the condition that "physical influences propagate causally" could be given an intrinsic formulation, free of references to external observers, and if the resulting criterion were formulated in terms of histories (as Bell's "local causality" is, for example), then one should be able to decide whether or not quantum mechanics and quantum field theory satisfied this condition. If they did, then it would make sense to require it also of quantum gravity, and this in turn could be the key to constructing a viable quantum dynamics for causal sets.

Perhaps the challenge of "quantum foundations" is not so urgent for quantum computing, which is concerned more with the manipulation of information than with microscopic reality as such. But even there, it seems possible that a more definite picture of the micro-world could someday lead us to widen our conceptions of measurement and of computation.

What I hope to illustrate in this paper is a possible answer to the italicized question posed above, an answer that arose from thinking of quantum mechanics in the language of histories, i.e. the language of the path integral. According to this answer, the micro-world is described by something called a *coevent*, but instead of attempting to define that concept in the general case, I will consider a very simple model of a particle hopping on a lattice, and in that setting, will present a short calculation that I hope will indicate more concretely how the proposal is meant to go. In the particular scheme I will present, reality will be something like a trajectory or worldline of the hopping particle, but instead of being a single trajectory, as in classical physics, it will be a set of trajectories. This particular choice is not necessarily the best one, but it is the simplest and therefore appropriate to illustrate the main idea.

Notice that in a coevent formulation, reality is *not* represented by a wave function ψ. Therefore, although a concept like position will have a straightforward meaning, a concept like momentum or energy will not. But then we have to ask how an electron in a hydrogen atom "knows" for example, whether the atom is in its ground state or in an excited state.

Is there something about the trajectories that carries this information? Hence the title of this paper.

I. A simple unitary model — the n-site hopper

A simple model allowing one to refer to trajectories and ground/excited states is the "n-site hopper", by which I mean a particle residing on an n-site periodic lattice, and at each of a discrete succession of moments either staying where it is or jumping to some other site, the respective amplitudes being those given by the simple "transfer matrix" reproduced below. [1][2] In this "toy world", nothing exists beyond the hopper itself, and since both space and time are discrete, the possible realities or *coevents* can be computed with minimal difficulty.

In order to describe the hopping amplitudes, let us identify the nodes of the lattice with the elements of $\mathbb{Z}_n$, the integers modulo n. Further let $x \in \mathbb{Z}_n$ be the location of the particle at some moment and let x' be its location at the next moment, and write for brevity $\exp(2\pi i z) \equiv \mathbf{1}^z$. The amplitude to go from x to x' in a single step is then

$$\frac{1}{\sqrt{n}}\,\mathbf{1}^{(x-x')^2/n} \tag{1a}$$

for n odd, and

$$\frac{1}{\sqrt{n}}\,\mathbf{1}^{(x-x')^2/2n} \tag{1b}$$

for n even. For example, for $n = 6$ and with $q = \mathbf{1}^{1/12}$, the (un-normalized) amplitudes to hop by 0, 1, 2 or 3 sites respectively are $q^0 = 1$, $q^1 = q$, q^4, and $q^9 = -i$.

It is not difficult to verify that these amplitudes (more precisely the matrix they comprise) are unitary. Interestingly, they take precisely the form of the propagator of a non-relativistic free particle in one-dimension, suggesting that in a suitable $n \to \infty$ limit, this hopper model could provide a fully self-consistent regularization of the path-integral for such a particle.

For the 2- and 3-site hoppers, the amplitudes are particularly simple, yielding for $n = 3$ the matrix

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \omega & \omega \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{pmatrix} \qquad\qquad (\omega = \mathbf{1}^{1/3})$$

and for $n = 2$ the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} .$$

II. Review of anhomomorphic coevents and the "Multiplicative Scheme"

In the formulation I am advocating, "nature" is represented in terms of *histories*, which for present purposes means trajectories of the hopping particle. [3][4][5] Physical reality — a "possible world" — is then described by a *coevent* which specifies which events happen and which don't, where an *event* is by definition a set of histories.* More formally, a coevent will be a function ϕ that assigns either 0 or 1 to each event, according as the event doesn't or does happen in the world described by ϕ.

From this perspective, our description of the physical world would be complete if we were able to specify fully the actual coevent ϕ. The role of "dynamics", then, is to help us toward a fuller such specification by placing conditions on ϕ that narrow down the range of possibilities that need to be considered. I will assume that the input to this dynamics takes the form of a path integral (or in the case of the hopper, a path sum). Wave functions ψ, insofar as they play a role at all, will provide provisional initial amplitudes that go into the computation of the path-sum (as an approximate summary of past).

But is it possible to base a dynamics on a path-sum alone? If we wish to do so then, plainly, we must construe the latter as something more than a technical device to compute transition amplitudes between some initial wave-function and some final one. Instead we will interpret it as providing for any event A, the *quantal measure* $\mu(A)$ of that event.† Once again, I will omit the formal definition [6][7][8], since we will see very soon how concretely to compute $\mu(A)$ in the case at hand.

* not to be confused with the word "event" used to denote a point of spacetime. The definition here follows the usage in probability theory, which in turn is more in tune with the everyday meaning of the word.

† $\mu(A)$ is also known as the diagonal element of the decoherence functional $D(A, A)$.

The preclusion principle

It is not hard to convince oneself that in one special case $\mu(A)$ has the meaning of an ordinary probability, namely when the event A can be described as a possible outcome ("instrument reading") of a given laboratory experiment. In such a case, the analyses of numerous gedankenexperiments over the years have made it plausible (albeit people don't always express it this way) that $\mu(A)$ coincides with the Born-rule probability of the outcome A. But if this is accepted, then it follows immediately that a zero value of $\mu(A)$ implies that the corresponding instrument-event almost surely does not occur — it is *precluded*.[♭] Extending this conclusion to the case of an arbitrary — macroscopic or microscopic — event, we arrive at a dynamical principle of general applicability: *If* $\mu(A) = 0$ *then the event* A *cannot happen.*

Preclusion and primitivity

The preclusion principle requires of any dynamically viable coevent ϕ that it deny every event whose quantal measure vanishes: $\phi(A) = 0$ if $\mu(A) = 0$. We have seen that this principle flows naturally from the path-integral, but by itself it is still rather weak, in the sense that a vast number of coevents can satisfy it even when preclusions abound. In our hopper example, for instance, we will have 27 histories, and consequently $2^{2^{27}} = 2^{134217728}$ coevents in toto. The number of precluded events is also large (2017807), but even so, there remain $2^{132199921}$ coevents which are *preclusive* in the sense that they satisfy our condition. On the other hand, $2^{132199921}$ is only a tiny fraction of $2^{134217728}$, so one might feel on the contrary that the preclusion principle is rather strong.

Be that as it may, I think the weightiest reason why the preclusion principle cannot stand alone is that it seems incapable of yielding the classical conception of reality-as-a-single-history when the measure μ is classical (and possibly "deterministic"). Thus, preclusion alone would not exclude coevents for which an experiment had more than one macroscopic outcome, or for which no definite outcome at all happened. In other words it would not resolve the "measurement problem".

[♭] To this extent, $\mu(A)$ is a sort of "propensity" for the event A to happen. One cannot go further and interpret it as a genuine probability, because, thanks to quantal interference, it is not additive on disjoint events.

To complete the dynamical story, then, we will supplement preclusion with a further principle of "minimality" or *primitivity* designed to remedy the deficiencies just cited. How properly to frame such a supplementary condition is a question not yet settled, but the simplest proposal is that of the so-called *multiplicative scheme*, and this is the one I will adopt for the present analysis.

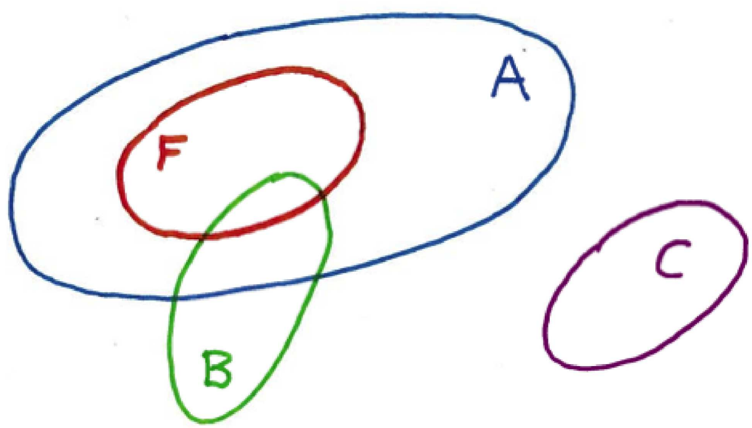

Figure 1. Three events and a multiplicative coevent. The three events are three sets of histories, A, B, C, while the coevent ϕ corresponds to a further set of histories F called its *support*. In the "reality" described by ϕ, A happens, while B and C do not happen. In formulas, $\phi = F^*$, $\phi(A) = 1$, and $\phi(B) = \phi(C) = 0$.

the multiplicative scheme

Classical physics identified reality with a single history, but that no longer seems possible quantum mechanically because the characteristic phenomenon of *interference* produces non-classical patterns of preclusion which seem to demand a modified conception of reality.

In place of a single history, the multiplicative scheme describes the physical world by a coevent of the form $\phi = F^*$, where F is now a *set* of histories that reduces to a singleton set only in very special (effectively classical) circumstances. As illustrated in figure 1, the coevent F^* assigns 1 ('true') to an event A if and only if A is a superset of F. Thus in the diagram, if $\phi = F^*$ then $\phi(A) = 1$ while $\phi(B) = \phi(C) = 0$. When $\phi = F^*$ I will refer to F

as the *support* of ϕ. Within the multiplicative scheme, a coevent is thus fully determined by its support.*

Now when is a coevent $\phi = F^*$ "dynamically viable" within the multiplicative scheme? By assumption it must be preclusive, and this means precisely that its support F must not fall wholly within any precluded event. Beyond this, we require further that F be as small as possible consistent with the preclusivity condition just stated. A coevent (or its support) that fulfills all these conditions I will call *primitive preclusive*, or for short just *primitive*. A primitive coevent thus describes a "possible world", where "possible" is to be understood relative to the given set of preclusions.

One sees immediately that in the absence of any preclusions (other than the empty event itself) a primitive preclusive support will consist solely of a single history, and the same holds whenever the pattern of preclusions is of the type that occurs in either classical deterministic, or classically stochastic theories. Much more than this could be said about the multiplicative scheme and its consequences, [9][10][11] but now I want to focus on a very concrete example and on our specific question: Does the particle know its own energy?

III. Histories and amplitudes for the 3-site hopper

In simple cases, preclusion is decided by whether the sum of the amplitudes vanishes. This will be true for our example, and it will make it easy to find the primitive supports.

To simplify as much as possible, let the hopper take three steps and then stop. And let there be nothing else in the world beside this hopper. Our space of histories then comprises exactly 81 trajectories, depending on where the hopper starts from and where it lands at each of the three subsequent moments. In fact, however, we only need to consider the 27 histories shown in figure 2, because one can prove, as a general feature of the multiplicative scheme, that any primitive support must correspond to a sharp final position. That is, one of the three events, "the hopper terminates at 0", "the hopper terminates at 1", or "the hopper terminates at 2", must happen. Without loss of generality we can suppose it is the first of these events, and this is what the figure illustrates.

* A little thought should convince you that the rule that $\phi(A) = 1$ iff $F \subseteq A$ also holds classically, with F being the set that contains the "actual history" as its sole element.

With the final position fixed at $x = 0$, it is also very easy to decide whether a given set of histories is precluded: this occurs iff the amplitudes of the constituent histories sum to zero. To work out the pattern of preclusions, it thus suffices to know all the amplitudes.

Now given a history γ, its net amplitude is the amplitude it inherits from its starting location, multiplied by the amplitudes of the individual hops, the former being given by what one might call the "initial wave-function" $\psi_{initial}$. For $\psi_{initial}$ let us consider two possible choices, ψ_0 and ψ_+, which we may call by way of analogy "ground state" and "traveling wave". The amplitudes for these are respectively $(\psi(0), \psi(1), \psi(2)) = (1, 1, 1)$ and $(\psi(0), \psi(1), \psi(2)) = (1, \omega, \omega^2)$, where $\omega = \mathbf{1}^{1/3}$ as before. Clearly both ψ_0 and ψ_+ are eigenvectors of the "transfer matrix" defined earlier. (The overall normalization of $\psi_{initial}$ is immaterial since it has no effect on preclusion.)

It is now straightforward to compute the amplitudes of our 27 histories in each case. The figure exhibits them for the traveling wave, and those for the ground-state are similar (but even simpler to compute since the dependence on the starting position is absent.) Curiously, the amplitudes per se (though not of course the way they are distributed among the histories) turn out to be exactly the same for both cases. Counting them up, one obtains the following multiset:

$$\overline{\omega}^{\{12\}} \qquad 1^{\{9\}} \qquad \omega^{\{6\}} \ ,$$

meaning 12 histories with amplitude $\omega^2 = \overline{\omega}$, 9 with amplitude 1, and 6 with amplitude ω.

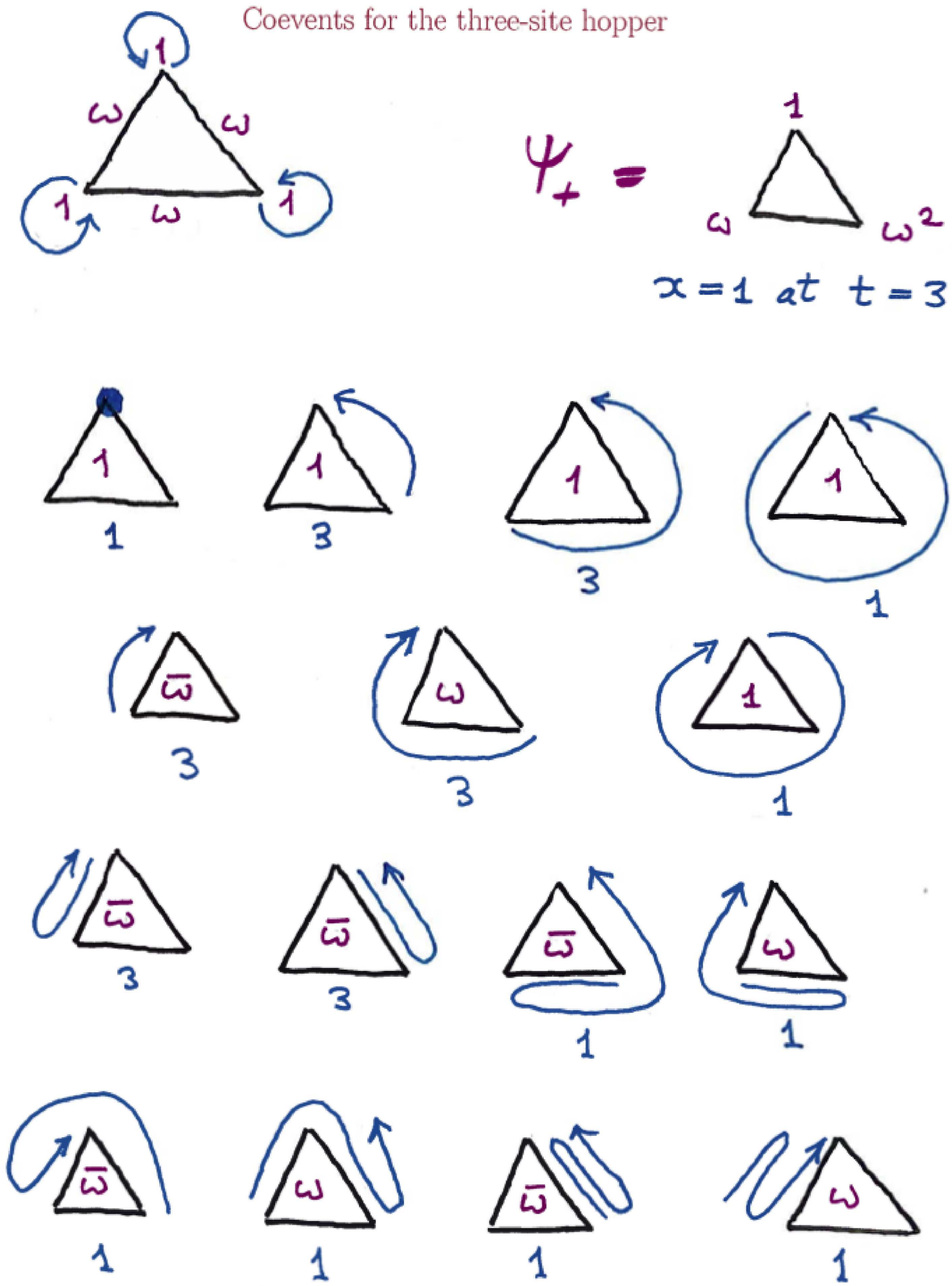

Figure 2. The 27 histories, and their amplitudes for the case of the "traveling wave". Each path indicated by an arrow represents from 1 to 3 possible histories differing from each other by the moments at which the hopper chooses to rest. The resulting multiplicity is shown under the triangle, while the number inside the triangle is the amplitude itself.

IV. Primitive coevents for the ground state and the traveling wave

Which combinations of the histories illustrated in figure 2 yield primitive coevents? If we think in terms of the amplitudes abstractly, it is easy to answer this question, and the same answer will then apply unchanged to both cases of traveling wave and ground state. An event E will correspond to a set of amplitudes (strictly speaking a multiset), and E will be precluded precisely when the amplitudes sum to zero. Because the only real-linear relation among the complex numbers 1, ω, $\overline{\omega}$ is $1+\omega+\overline{\omega}=0$, this will occur when, and only when, the three amplitudes occur in equal numbers within E. It is then easy to see what are the maximal precluded multisets of amplitudes. They are those consisting of 6 copies each of 1, ω, and $\overline{\omega}$.

Now let F be the support of a preclusive coevent. In order to be preclusive, F must not fall wholly within any precluded event, and this means that it must not be possible to adjoin further amplitudes to those of F such that the resulting multiset sums to zero. Plainly F will be protected in this way iff it contains at least 7 copies of 1 or 7 copies of $\overline{\omega}$. On the other hand, we also want F to be primitive, meaning minimal among the preclusive supports. Again it is easy to see what this means: it must comprise *precisely* 7 copies of 1 or 7 copies of $\overline{\omega}$.

In this way, we find a total of $\binom{12}{7}+\binom{9}{7}=828$ primitive coevents of the multiplicative form $\phi=F^*$, each made up of a total of 7 histories.

In the case of the traveling wave, for example, one such set of histories corresponds to the last three patterns in the first row of figure 2. Interestingly, all seven of these trajectories move in the positive (counterclockwise) direction and none of them in the contrary direction. Thus the event, $P=$ "The particle circulates exclusively in the positive sense", happens in the reality described by this coevent.

Happily enough, this sense of circulation corresponds perfectly with the "phase velocity" of ψ_+, but we cannot assert that all of the 828 traveling-wave coevents also affirm this same event P. In order to quantify the tendency toward counterclockwise motion, then, let us associate a "net circulation" with each coevent, as the total number of "forward" hops less the total number of "backward" ones. (So for example the above coevent has a net circulation of $3\times1+3\times2+1\times3=12$.) Averaging this quantity over all 828 primitive coevents yields an average net circulation of 7/23. A tendency toward counterclockwise motion is therefore present but not extremely pronounced – just as one might expect since a lattice of only $n = 3$ positions can lend no more than a very rudimentary meaning to a derivative like "d(phase)/d(angle)".

The primitive preclusive coevents for the ground-state are again supported on sets of seven histories, as we have already remarked. By symmetry we cannot expect a favoured sense of circulation in this case, but it is also interesting to ask how "restless" the particle proves to be. Here, it is natural to compare with "Bohmian" or "pilot wave" conceptions of reality, since they also give meaning to the notion of particle-trajectory. As far as I know, the Bohmian "guidance equation" is limited to continuum space and time, where the nearest analog to our hopper ground-state might be the ground-state of a particle in a box. Since the phase of the Schrödinger wave-function is independent of position in that case, the Bohmian particle does not move at all. Rather than explore its surroundings, it just stays put, wherever it happens to find itself.[†]

It turns out that the coevents of the multiplicative scheme paint a very different picture. The event, "The hopper never moves" is of course denied by all 828 coevents, while the contrary event that "The hopper never rests" is affirmed by eight of them. (For such a coevent, all seven of the histories in its support are in constant motion.) Of the other 820 primitive coevents, 28 of them affirm that the hopper either never rests or never moves (6 histories vs. 1), while the remaining 792 primitive supports consist entirely of histories such that the hopper rests once and hops twice. Moreover, for none of the coevents does the event "the hopper avoids some lattice-site" happen.[♭] All in all, very peripatetic indeed.

[†] The first excited state, and indeed every standing wave of pure frequency, also leads to static trajectories. In this sense a Bohmian particle in a box with reflecting walls would be about as far as possible from knowing its own energy. However, the strictly stationary wave-functions are very special, and their superpositions might generically lead to different typical trajectories. [12]

[♭] Which of course doesn't mean that the complementary event, "the hopper visits all three lattice sites" ever happens either. That's why the coevents are "anhomomorphic": with $\phi = F^*$ defined as above, it can easily happen that two events A and B are complementary subsets of the space of histories, but both $\phi(A)$ and $\phi(B)$ vanish.

V. Does the hopper know its energy?

If the multiplicative scheme (or any other coevent scheme) is a kind of "equation of motion for coevents", then a primitive preclusive coevent is a kind of "solution of the equations of motion". In celestial mechanics such a solution would be the orbit of a planet. One can deduce the binding energy of the planet from a knowledge of its orbit, but the converse is impossible since the energy is only one among a number of orbital parameters. Now consider some microscopic counterpart of this problem, like a hydrogen atom. By analogy one should not expect to deduce a unique coevent ϕ from a knowledge of the total energy, but one might wonder whether, conversely, the coevent determines the energy unambiguously.

Quantum mechanically, energies are determinate only for eigenfunctions of the Hamiltonian operator, and the nearest analogs for these in our hopper-world are the initial-amplitude sets, ψ_0 and ψ_+ (plus, of course, the parity-reversed set ψ_-, together with linear combinations like the "standing wave", $\psi_+ + \psi_-$). We are thus led to ask whether the primitive coevents pertaining to ψ_0 overlap with those pertaining to ψ_+. To the extent that the answer is negative one can indeed deduce the hopper's energy from a knowledge of the coevent that describes its motion. Based on the above analysis of the primitive coevents, the required calculation is straightforward, and the result is that the degree of overlap is zero. No coevent is common to both ψ_0 and ψ_+.

Moreover it is possible to distinguish ψ_0 from ψ_+ in terms of relatively elementary consequences. For example if the hopper never rests (meaning $\phi(E) = 1$, where E is the event consisting of the fourth, seventh, and tenth through fifteenth histories shown in figure 2), then ϕ pertains to $\psi = \psi_0$. Similarly, if the hopper moves only counterclockwise, or if it rests exactly once, then $\psi = \psi_+$.

In this sense, we can say that the hopper does know its own energy. We can also say that it "knows its own angular momentum", since a similar comparison reveals that the primitive coevents pertaining to ψ_+ are disjoint from those pertaining to ψ_-.

Of course the example we've studied is exceptionally simple, both with respect to the hopping amplitudes and the amplitudes of the "initial states" $\psi_{initial}$ which we have considered. It would be good to analyze also the case of the standing wave, and more generally to extend the analysis to include longer times, and larger lattice sizes n. (The case of shorter times is also instructive. For only two time-steps, one finds that there do exist primitive coevents common to ψ_+ and ψ_0. This strengthens the impression that with the passage of time the hopper would "know" more and more about $\psi_{initial}$.)

Our hopper-world is exceptionally simple in another way too, that relates to the radical inseparability (or "interconnectedness") that seems to show up in the quantum world. Were we to include a second system or process in our idealized world, say for example a four-site hopper, the coevents would change. Obviously the global coevents would change, but even those induced for the three-site hopper would in general be different. This at least is a feature of the multiplicative scheme, and it is likely true more generally. It is therefore important to study extensions of our model of this type.

Extensions of our model in any of the directions just mentioned would of course be interesting. But even without them, I hope the examples studied in this paper suffice to illustrate how one can start to think about the quantum world without invoking the ideas of either evolving wave-functions or external observers.

If these examples are not misleading us, then we can already draw some conclusions of more general validity, concerning first of all the relation between wave-functions and descriptions of reality. A histories-based or "path-integral" formulation of the sort we have been working with has no use for the Schrödinger equation at a fundamental level. The basic concept of precluded event has a "spacetime character" and refers directly to the histories and their amplitudes, not to any evolving wave-function. On the other hand, something like a wave-function does enter into the "initial conditions" needed in setting up the path-integral. In principle one should probably replace these initial amplitudes with cosmological boundary conditions imposed directly on the histories, but in our hopper-world there is no moment earlier than $t = 0$, and we have assumed instead that in a more complete model, we would be able to condense the effect of the true boundary conditions into a set of three initial amplitudes for the hopper at $t = 0$. This is the wave-function $\psi_{initial}$ in its role as "effective summary of the past".

What our hopper model teaches us about $\psi_{initial}$ is that it is far from furnishing a detailed description of physical reality. Rather, the coevent which by definition does furnish such a description is determined by $\psi_{initial}$ only to a very limited extent. Reality possesses far more "internal structure" than is reflected in a wave-function.

This observation in turn resolves an old paradox that has recently been emphasized in a cosmological setting by Daniel Sudarsky [13]. In its terrestrial form the paradox asks how the spherically symmetric wave-function resulting from the decay of a spinless nucleus can be compatible with the fact that the daughter nuclei will be found to be localized in angle if one sets up detectors. How does the symmetry break? But in terms of coevents, there's no problem. Just because the ψ-function is symmetric, that doesn't imply the same of the individual coevents. Indeed, we see exactly such a phenomenon in our hopper model, where $\psi_{initial}$ exhibits perfect rotational symmetry while the individual coevents are completely asymmetric. No single coevent shares the symmetry of $\psi_{initial}$, but only the ensemble of all 3×828 of them taken together.

For discussions of these questions over a long period, I would like to thank Fay Dowker, Cohl Furey, Yousef Ghazi-Tabatabai, Stan Gudder, Joe Henson, David Rideout, Sumati Surya, and Petros Wallden. I also thank the Foundational Questions Institute (FQXI) for a grant, and the maintainers of Steel Belt Common Lisp (SBCL), whose software I used at several points in the analysis reported above. Finally, I thank the members of the Raman Research Institute (RRI) for their warm hospitality while parts of this paper were being written.

This research was supported in part by NSERC through grant RGPIN-418709-2012. Research at Perimeter Institute for Theoretical Physics is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI.

References

[1] Rafael D. Sorkin, "Toward a 'fundamental theorem of quantal measure theory' " *Mathematical Structures in Computer Science* **22(5) :** 816-852 (2012) `http://arxiv.org/abs/1104.0997` `http://www.pitp.ca/personal/rsorkin/some.papers/141.fthqmt.pdf`.

[2] S. Gudder and Rafael D. Sorkin, "Two-site quantum random walk" *General Relativity and Gravitation* **43 :** 3451-3475 (2011) `http://arXiv.org/abs/1105.0705` `http://www.pitp.ca/personal/rsorkin/some.papers/`

[3] Rafael D. Sorkin, "Quantum dynamics without the wave function" *J. Phys. A: Math. Theor.* **40 :** 3207-3221 (2007) (http://stacks.iop.org/1751-8121/40/3207) `quant-ph/0610204` `http://www.pitp.ca/personal/rsorkin/some.papers/`

[4] Petros Wallden, "The coevent formulation of quantum theory", http://arxiv.org/abs/1301.5704

[5] Rafael D. Sorkin, "Logic is to the quantum as geometry is to gravity" in G.F.R. Ellis, J. Murugan and A. Weltman (eds), *Foundations of Space and Time: Reflections on Quantum Gravity* (Cambridge University Press, 2012) http://arXiv.org/abs/arXiv:1004.1226 [quant-ph] http://www.pitp.ca/personal/rsorkin/some.papers/

[6] Rafael D. Sorkin, "Quantum Mechanics as Quantum Measure Theory", *Mod. Phys. Lett. A* **9** (No. 33)**:** 3119-3127 (1994) gr-qc/9401003 http://www.pitp.ca/personal/rsorkin/some.papers/80.qmqmt.pdf

[7] Roberto B. Salgado, "Some Identities for the Quantum Measure and its Generalizations", *Mod. Phys. Lett.* **A17:** 711-728 (2002) gr-qc/9903015

[8] Sumati Surya and Petros Wallden, "Quantum Covers in Quantum Measure Theory", http://arXiv.org/abs/0809.1951 *Foundations of Physics* **40:** 585-606 (2010)

[9] Yousef Ghazi-Tabatabai and Petros Wallden, "Dynamics & Predictions in the Co-Event Interpretation", *J. Phys. A: Math. Theor.* **42:** 235303 (2009) http://arXiv.org/abs/0901.3675

[10] Yousef Ghazi-Tabatabai and Petros Wallden, "The emergence of probabilities in anhomomorphic logic", *Journal of Physics: Conf. Ser.* **174:** 012054 (2009) http://arXiv.org/abs/0907.0754 (quant-ph)

[11] Fay Dowker and Yousef Ghazi-Tabatabai, "The Kochen-Specker Theorem Revisited in Quantum Measure Theory", *J.Phys.A* **41:** 105301 (2008) http://arXiv.org/abs/0711.0894 (quant-ph)

[12] Lucien Hardy, remark at a conference.

[13] Daniel Sudarsky, "Shortcomings in the Understanding of Why Cosmological Perturbations Look Classical" *Int. J. Mod. Phys.* **D20:** 509-552 (2011) arXiv:0906.0315 [gr-qc]