Scalar Field Theory on a Causal Set in Histories Form[⋆]

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Abstract

We recast into histories-based form a quantum field theory defined earlier in operator language for a free scalar field on a background causal set. The resulting decoherence functional resembles that of the continuum theory but the counterpart of the d'Alembertian operator $\Box$ is nonlocal and a generalized inverse of the discrete retarded Green function. We comment on the significance of this and we also suggest how to include interactions.

1. Introduction

At this stage in the development of fundamental physics one does not know whether anything like a quantum field will turn out to exist at the Planck scale. Once one abandons the continuum, a vector or a spinor is no longer a particularly natural object, and in the context of causal set theory further difficulties spring from the radical nonlocality inherent in a discrete Lorentzian causal structure. Such difficulties suggest that the type of quantum field theory embodied in the so called standard model will someday come to be seen as no more than a relatively low energy description of underlying "degrees of freedom" of a completely different nature.

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Be that as it may, any attempt to push quantum field theory down to a more fundamental, discrete level seems bound to be illuminating, and as it happens, we now possess a theory that describes a free scalar field propagating in a fixed, background causal set. [1] On the basis of this theory, one can hope to understand better the influence of discreteness on such phenomena as Hawking radiation or the propagation of wave packets over great distances. As formulated so far, however, the theory in question speaks the language of operators and state-vectors. It describes the field by attaching to each element x of the causet C an operator $\phi(x)$ acting in a Hilbert space built up from a "ground state" by the action of creation operators a^*. Although all this is very familiar, one might wish also for a formulation based on "histories" (a history in this case being a function $\phi : C \to \mathbb{R}$). [2]

Indeed, some of us regard histories-based formulations of quantum theories as more basic and more satisfactory than operator formulations, both for the purposes of quantum gravity and for the sake of philosophical understanding. How, for example, might one expect a dynamical theory of causal sets to look? The only way open to a "law of motion" for causets has so far seemed to pass through something like a path integral (or more precisely a decoherence functional), and one might say the same about quantum gravity in general. Even with the causet limited to the background role of "arena", the knowledge of how to write down a path integral for a scalar field on a causet, would allow the concept of "anhomomorphic coevent" to be brought to bear on the question of what sort of reality quantum field theory is in fact describing [7]. In a more practical vein, one encounters the key question of how to generalize from a free field to an interacting one, something which so far has not been possible with the operator formulation. As we will see, a path integral formulation can suggest a very definite answer to this question.

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[1] For some background on causal sets see [1] [2] [3] [4] [5] [6]

[2] Sometimes people say "classical history" to emphasize that $\phi(x)$ is a c-number and not an operator, but I will eschew that usage here in order to avoid any implication that ϕ must obey the classical equations of motion.

Of equal practical importance is the question of the classical limit. Is it possible in this limit to express the equations of motion of a scalar field in such a manner that solutions could be built up iteratively given the values of the field in a sufficiently great initial portion of the causet? If it were, then computer simulations of wave packet propagation through the causet would become possible. On the other hand, if the relevant field equation were to involve reference to the distant future, such simulations would become much more difficult, or even impossible. In fact an equation of motion of exactly the required, retarded sort has been proposed [8], but in what sense can we claim that it represents the classical limit of our discrete quantum field theory?

In this connection, the following difficulty appears to arise [9]. Suppose that the dynamics of the scalar field is to be defined through an action-integral (or sum in the case of the causet), and that the integrand at point x depends on the field in the entire past of x. When one varies $\phi(x)$ in such an expression, the effect will be felt everywhere in the future of x. The resulting Euler-Lagrange equations will thus relate $\phi(x)$ not only to the past but to the future. How then could retarded equations of motion result? We will return to this issue in sections 4 and 5.

2. Review of the continuum theory in operator form

Nothing could be more familiar than a free scalar field in Minkowski spacetime together with its vacuum-state and the concomitant Fock representation of the field operators. But the way in which these things are ordinarily introduced, starting from the Klein-Gordon field equations and the equal-time commutation relations, is not quite suitable for transposition to the context of a discontinuous structure like the causal set which not only does away with continuous time, but also resists any notion of "Cauchy data at a moment of time". [3] Nor does the normally crucial distinction between positive and negative frequency

[3] A causal set does admit the notion of *slice* as maximal antichain Σ, but such a subset unfortunately is not equipped to fill the role of an equal-time hypersurface, or more generally of a Cauchy surface in a globally hyperbolic spacetime, the most important difference being that the great majority of causal *links* will "pass thru" Σ without meeting it at all;

make much sense in a causal set, which insofar as it resembles a spacetime at all, resembles much more a spacetime with curvature than one without it. For such reasons, it has proven more fruitful to rest the derivation of the scalar quantum field theory as much as possible on the commutation relations alone, or more precisely on the retarded Green function, in a way that I will review here. [10]

By way of preparation, let us first review the theory of a "gaussian" noninteracting (massive or massless) real scalar field ϕ in flat spacetime. [4]

the usual story

As this subject is commonly introduced, the input to the theory comprises first of all the operator equations of motion in the form of the Klein-Gordon equation, $(\Box - m^2)\phi(x) = 0$. Supplementing these equations (plural since there is one for each x) are the canonical commutation relations, expressed at a fixed moment of time as $[\phi(x), \partial\phi(y)/\partial t] = i\delta^{(3)}(x-y)$. One then expands ϕ in terms of (four-dimensional) plane waves and identifies the (suitably normalized) coefficients of the positive-frequency waves with annihilation operators $a(k)$. Introducing, finally, the vacuum state $|0\rangle$ via the conditions $a(k)|0\rangle = 0$, one obtains a concrete family of field operators $\phi(x)$ acting in a Hilbert space spanned by states of the form $a^*(k_1)^{n_1} a^*(k_2)^{n_2} a^*(k_3)^{n_3} \cdots |0\rangle$. From an algebraic standpoint, the equations of motion together with the commutation relations define an abstract *-algebra generated by symbols $\phi(x)$, while the choice of vacuum induces an irreducible representation of this algebra in a particular sort of Hilbert space. The full input to the theory is thus the field equations, the commutation relations, and the choice of vacuum (although it is partly

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that is, there will exist pairs of causet elements x, y such that x is to the past of Σ, y to its future, and yet neither Σ nor any other part of the causet contains an element causally intermediate between x and y. In this sense "information can pass through Σ without registering on it", the impossibility of which is precisely what characterizes a Cauchy surface in the continuum.

[4] with a metric η_{ab} of signature $(-+++)$ and with $\Box = \eta^{ab}\partial_a\partial_b$

a matter of opinion whether or not a change of representation or of vacuum should be regarded as changing the theory per se.)

For our purposes, the most important thing to notice is the need for consistency between the field equations and the commutation relations. In the development we have just called to mind, this is ensured by the fact that the latter are invariant under the time-evolution generated by the former, but we can appreciate better what is going on if we express the commutation relations in a manifestly covariant form, the so called Peierls form. (The Lorentz invariance of the theory then follows trivially.) To that end, let $G(x,y) = G^{ret}(x,y)$ be the retarded Green function belonging to the wave-operator $\Box - m^2$:

$$(\Box - m^2) G^{ret}(x,y) = \delta^{(4)}(x-y) ,$$

where G is required to vanish unless $x \succ y$, meaning that y is within or on the past lightcone of x. Further define

$$\Delta(x,y) = G(x,y) - G(y,x) , \tag{1}$$

which is just the difference between the retarded and advanced Green functions, since (as in fact holds in any globally hyperbolic spacetime) the advanced Green function coincides with the "transpose" of the retarded one. The commutation relations then assume the remarkably simple form,

$$[\phi(x), \phi(y)] = i\Delta(x,y) .$$

If to this equation one applies the wave operator $(\Box - m^2)$ (acting on x), one trivially obtains zero on the left hand side. The required consistency then follows from the fact that the right hand side also vanishes because Δ is the difference of two Green functions for $(\Box - m^2)$. Notice here that the last three equations all make perfect sense, and are valid, in any globally hyperbolic spacetime. The resulting definition of a quantum field theory accordingly goes through as well, except that something is needed to replace the notion of positive frequency if one wishes to define a vacuum state.

Now that we have a field theory in operator form, the next question is how to render it into histories form. For this there is a standard procedure that I won't review here, because we will meet it again in detail when we come to the case of the causal set. Instead, let me just quote the resulting decoherence functional, which by definition attaches a complex amplitude to every *pair* of spacetime histories of the scalar field. (A pair of histories can be called a "Schwinger history" in honor of the so called Schwinger-Keldysh version of the path integral, which operates precisely with such pairs.) To avert confusion between ϕ as a ("classical") history and ϕ as an operator-valued field, let me for now use the symbols ξ and $\overline{\xi}$ to represent histories. The decoherence functional $D(\xi, \overline{\xi})$ is then given formally by the expression

$$\exp\left\{iS(\xi)\right\}\, \exp\left\{iS(\overline{\xi})\right\}^* \; \delta(\xi|\Sigma_T, \overline{\xi}|\Sigma_T) \tag{2}$$

where $S(\xi) = \frac{1}{2}\int(-(\nabla\xi)^2 - m^2\xi^2)$ plus an implicit further term that incorporates the effect of the "initial state", in this case the vacuum. Here also, Σ_T is a spacelike future boundary at which the spacetime has been truncated and the δ-function involving it has the effect of forcing the otherwise independent histories ξ and $\overline{\xi}$ to share the same restriction to this final boundary. From (2), the decoherence functional extended to sets of histories and the corresponding quantum measure can be obtained in the usual way [11].

For future reference let us also define here the so called Wightman or two-point function given by the expectation of $\phi(x)\phi(y)$ operators in the vacuum state:

$$W(x, y) = \langle\phi(x)\phi(y)\rangle \tag{3}$$

and let us note also that in the vacuum

$$W(x) := \langle\phi(x)\rangle = 0\ . \tag{4}$$

Because the vacuum is a "gaussian state" the entire field theory is effectively encapsulated in these two equations, a fact we will use heavily in our construction of the decoherence functional in the causet case.

a different starting point

The logic of the "usual story" that we have just rehearsed is simplest if we present the commutation relations in their Peierls form. We might then summarize in part the steps we followed as: wave operator $\to G^{ret} \to \Delta \to [\phi, \phi]$. But consistency between the wave-operator, $\Box - m^2$, and the consequent commutation relations required, as we saw, that the transpose of the retarded Green function also be a Green function for the same wave operator. Since no such relation is known in the causet case, it turns out that one can get farther by changing the starting point slightly from the equations of motion to the retarded Green function G^{ret}. From it we can derive the commutators in just the same way. Indeed, one can pass freely from G^{ret} to Δ and back by means of equation (1) and the inverse relation $G^{ret}(x, y) = \Delta(x, y)$, which holds when $x \succ y$. (In equation form, $G^{ret}(x, y) = \theta(x, y)\Delta(x, y)$ where θ is the "covariant Heaviside function".)

But how to continue without the aid of any equations of motion? The surprising answer discovered implicitly in [10] is that one can pass directly from the commutator function Δ to the two-point function $W(x, y)$ by thinking of $\Delta(x, y)$ as a real skew matrix Δ^{xy} and forming the positive part of the corresponding Hermitian operator $i\Delta$. Our new logic will therefore go as follows: $G^{ret} \to \Delta \to W$. It would be difficult to imagine anything much simpler!

Notice in this connection that W always is, in fact, a positive (semidefinite) operator, as follows immediately from the fact that $v^* W v \geq 0$ for any complex vector v. Indeed, positivity of the Hilbert space inner product yields for any such v_x , $\sum (v_x)^* W^{xy} v_y = \sum (v_x)^* \langle \phi^x \phi^y \rangle v_y = \langle X^\dagger X \rangle = ||X|0\rangle||^2 \geq 0$, where $X = \sum v_x \phi^x$. It's equally true that $\overline{W}$, the complex conjugate matrix of W, is positive semidefinite and that $i\Delta = W - \overline{W}$, as follows from: $(W - \overline{W})^{xy} = \langle \phi^x \phi^y \rangle - \overline{\langle \phi^x \phi^y \rangle} = \langle \phi^x \phi^y \rangle - \langle \phi^y \phi^x \rangle = \langle [\phi^x, \phi^y] \rangle = \langle i\Delta^{xy} \rangle = i\Delta^{xy}$. Thus, $i\Delta$ is always the difference of two positive matrices W and $\overline{W}$. What our prescription for W is adding to this fact is that these operators have orthogonal support:

$$W\overline{W} = \overline{W}W = 0 \ . \tag{5}$$

One might view this as a kind of "ground-state condition" imposed on W beyond what follows automatically from the fact that it is the two-point function of a selfadjoint operator ϕ^x. From (5) follows immediately a simple matrix equation giving W in terms of Δ (and therefore in terms of G):

$$W = 1/2(i\Delta + \sqrt{-\Delta^2}) \ . \tag{6}$$

In practice one would often compute $\sqrt{-\Delta^2}$ by diagonalizing Δ. Once this was done, one would obtain W just by expanding $i\Delta$ in terms of its eigenvectors and retaining the terms with positive eigenvalues, and one could regard this as the practical meaning of our prescription for obtaining W from Δ.

For future reference let me introduce the notation $R = \frac{1}{2}\sqrt{-\Delta^2}$, a manifestly real and positive matrix. Comparison with (6) then shows that R is just the real part of W, whose decomposition into real and imaginary parts is thus given by

$$W = R + i\Delta/2 \ . \tag{7}$$

It's worth noticing here that the definitions we have given yield a free field theory with a distinguished "ground state" for any globally hyperbolic spacetime or region of spacetime (modulo certain technical questions about convergence that arise when the region's volume is infinite.) Notice also that nowhere do these definitions refer to any notion of positive frequency. In Minkowski spacetime, the new prescription can easily be seen to reproduce the usual vacuum (it more or less had to by Lorentz invariance). In this sense it generalizes the Minkowski vacuum to the case of arbitrary curvature.

Perhaps also it's also worth mentioning a possible further generalization that would tie W less closely to G. In order to reproduce correctly the commutation relations, it is necessary only that the imaginary part of W coincide with $\Delta/2$. It is also necessary for mathematical consistency that W be positive. Perhaps other conditions are needed as well, but it is clearly not necessary that W simply be the positive part of $i\Delta$. One could thus consider relaxing condition (5) as well as the assumption that $\langle\phi\rangle = 0$. As will be seen, neither generalization would require major changes to our discussion below of the decoherence functional on a causal set.

3. Scalar field theory on a causet in operator form

Let us now leave the continuum behind and turn to the case of a causal set. Henceforth our discussion will be in the context of a fixed, finite causal set C, and it will assume further that a fixed "retarded Green function" has been adopted, which I'll write as either $G(x, y)$ or G^{xy} , where x and y vary over the elements of C. In some of the most important cases a convincing candidate for G is known and has been tested in practice to a greater or lesser extent. These cases include zero mass in a causet intended to approximate an arbitrary 2-dimensional spacetime, as well as arbitrary mass in a causet intended to approximate a flat spacetime of dimension either 2 or 4. (See [12].) In addition, the "retarded d'Alembertians" of [8] and [13] could be inverted to produce candidate Green functions in more general causets, but this type of prescription has not yet been put to a test.

Starting from G and following the same steps as delineated above yields then a scalar field theory for the given causet, complete with operators $\phi(x)$, vacuum-state $|0\rangle$, and Wightman function $W(x, y) = \langle\phi(x)\phi(y)\rangle$, where all expectations are taken with respect to the state $|0\rangle$.

As we have already noted, the fact that this theory-cum-state is in some sense gaussian means that all its consequences can be recovered starting from W alone. It might be interesting to tease out better in exactly what sense the word "gaussian" can be applied to our causet theory. Having no Cauchy surfaces, it also has no straightforward Schrödinger representation, and therefore no vacuum wave function whose exponential form could be held up as the meaning of the word. On the other hand, the fact that the ϕ^x are linear combinations of raising and lowering operators implies that Wick's theorem will work in the expected manner, and this is all that we will need. Indeed, we will need only the following consequence of Wick's theorem, where Φ stands for any linear combination of the field operators $\phi(x)$.

$$\langle \exp\{\Phi\} \rangle = \exp\left\{ \frac{\langle \Phi\Phi \rangle}{2} \right\} \tag{8}$$

In view of equation (21) of the appendix, the following calculation demonstrates this identity.

$$\langle \exp\{\Phi\} \rangle = \sum_n \frac{\langle \Phi^n \rangle}{n!} = \sum_n \frac{\langle \Phi^{2n} \rangle}{(2n)!} = \sum_n \frac{(2n-1)!!\langle \Phi\Phi \rangle^n}{(2n)!} = \sum_n \frac{1}{n!} \frac{\langle \Phi\Phi \rangle^n}{2^n}$$

This seems a good place to mention a way of thinking about W that is more geometrical in nature than the characterizations we have encountered so far and that involves only real, as opposed to complex, vector spaces. Namely W can be represented geometrically as a family of mutually orthogonal, "axially weighted 2-planes" in the real vector space $\mathbb{R}^N$, where N is the cardinality of the underlying causet, these being a family of pairs (ϖ, λ) where ϖ is an oriented 2-plane in $\mathbb{R}^N$ and λ is a strictly positive real number corresponding to one of the eigenvalues of $i\Delta$. This one can see by splitting the eigenvectors of $i\Delta$ into their real and imaginary parts, or equivalently by decomposing Δ as a sum of terms $a \wedge b$ with $a, b \in \mathbb{R}^N$ being orthogonal vectors. The latter vectors then provide the basis of a singular-value decomposition of the real matrix Δ. Thanks to these relationships, one can construct W directly from Δ by forming the singular value decomposition of the latter, thereby economizing — in computer simulations — on memory and CPU-time, since only real numbers need be involved.

We are almost ready to turn our efforts to producing a histories version of our theory, but first, perhaps a simple example of the scheme $G \to \Delta \to W$ would be in order. Consider then a causet of only two elements, $e_0 \prec e_1$, making up a 2-chain. (Given the interpretation of W as a collection of weighted 2-planes, we could in this very simple case pass immediately to a unique W without bothering with G and Δ, but that would not illustrate the general situation.) That G is retarded means precisely that $G^{01} = 0$, but since only off-diagonal terms survive the antisymmetrization in (1), we might as well suppose that the diagonal of G vanishes as well. Up to sign and normalization, G is then unique:

$$G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,$$

from which there result immediately $\Delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$i\Delta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2$$

(the Pauli matrix). Substituting this into (6) then furnishes W as

$$W = \frac{1}{2}(1+\sigma_2) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} ,$$

i.e. $2R$ in (7) is simply the identity matrix. We could also have arrived at the same result by diagonalizing $i\Delta$ explicitly as $\sigma_2 = uu^\dagger - \overline{uu}^\dagger$, where $u = 1/\sqrt{2}\begin{pmatrix} 1 \\ i \end{pmatrix}$, and then discarding the negative term. Notice finally that W correctly satisfies the "ground-state condition" (5). Given that W is hermitian, (5) is just the assertion that $W\widetilde{W} = 0$, tilde denoting the matrix transpose. Alternatively expressed, this says that the dot-product (the bilinear one, not the hermitian one) of any row (resp. column) with itself or with any other row must vanish, and this is true by inspection.

4. Scalar field theory on a causet in histories form

Henceforth it will be convenient to redefine $G = G^{jk}$ to be the retarded Green function *with its diagonal set to zero.* Since Δ is G minus its transpose, it is unaffected by this modification, and since W was defined through Δ, it is not affected either. Equivalently, we could take W as our starting point and then simply define G to be twice the retarded half of the imaginary part of W. Either way its clear that for purposes of this paper, there's no harm in dropping diag(G). But in a larger context this change would need to be borne in mind. For example, if we had obtained G by inverting some retarded D'alembertian operator for the causet, then we could not recover the latter from G without restoring its diagonal.

By a histories-based formulation of a field theory, one means (for a real scalar field) a formulation that works directly with field configurations $\phi : C \to \mathbb{R}$ ("histories"), and avoids reference to field operators or state-vectors, except possibly as auxiliary technical devices [14]. In practical terms, this means a path-integral formulation, although one might hope that more general frameworks will one day be available. [5] In order that a framework based on path-integrals be self-contained, moreover, it seems necessary to assign amplitudes to Schwinger histories rather than to individual histories, as the more familiar path-integral does. That is to say that one needs to express the dynamics in terms of a *decoherence functional* D, from which the corresponding quantal measure [15] can then be computed. Since the definitions of these things have been presented many times in the literature, I'll not repeat them here. Rather, I'll begin by writing down the expression (an expectation of a product of projectors) that one needs to evaluate in order to recover the decoherence functional from a theory expressed in the operator language.

In line with the definitions laid down above, let ξ and $\overline{\xi}$ be two (completely independent) histories, each being specified by a list of real numbers ξ^x, one for each element x of the causet. The complex number $D(\xi, \overline{\xi})$ is then given by the equation

$$D(\xi, \overline{\xi}) = \Big\langle \delta(\phi^1 - \overline{\xi}^1)\delta(\phi^2 - \overline{\xi}^2) \cdots \delta(\phi^N - \overline{\xi}^N)\delta(\phi^N - \xi^N) \cdots \delta(\phi^2 - \xi^2)\delta(\phi^1 - \xi^1) \Big\rangle \quad (9)$$

Here $\phi^j = \phi(x_j)$, $\xi^j = \xi(x_j)$, etc, where $j = 1 \cdots N$ is any *natural labeling* of C. In other words the elements $x \in C$ of the causet must be labeled so that no element with a smaller label ever temporally follows an element with a higher label: $x_j \prec x_k \Rightarrow j < k$. Such a labeling always exists, and any two natural labelings are guaranteed to produce the same result for $D(\xi, \overline{\xi})$ because $\Delta(x, y)$, and hence $[\phi(x), \phi(y)]$, automatically vanishes when x and y are causally unrelated ("spacelike"). Notice that D is normalized such that $\int d^N\xi d^N\overline{\xi} D(\xi, \overline{\xi}) = 1$.

[5] perhaps obtained by freeing the concept of preclusion from its dependence on the notion of quantum measure and measure-zero, or perhaps even by freeing anhomomorphic inference from its dependence on preclusion, cf. [7]

Re-expressing the δ-functions as integrals puts (9) into the form

$$\int_{-\infty}^{\infty} \frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi}\cdots\frac{d\overline{\lambda}_2}{2\pi}\frac{d\overline{\lambda}_1}{2\pi} e^{-i\overline{\lambda}_1\overline{\xi}^1 - i\overline{\lambda}_2\overline{\xi}^2 \cdots - i\lambda_2\xi^2 - i\lambda_1\xi^1} \left\langle e^{i\overline{\lambda}_1\phi^1} e^{i\overline{\lambda}_2\phi^2} \cdots e^{i\lambda_2\phi^2} e^{i\lambda_1\phi^1} \right\rangle , \quad (10)$$

which represents $D(\xi,\overline{\xi})$ as the Fourier transform of what one might call its "non-commutative characteristic function" in analogy with the concept of characteristic function that figures in ordinary probability theory. (Here again the real numbers λ_j and $\overline{\lambda}_j$ are independent parameters of the Fourier transform. The bar is not being used to denote complex conjugation.)

Before plunging into the evaluation of this expression for the decoherence functional, let us evaluate the simpler expression which results from integrating out the $\overline{\xi}$ and all but one of the variables $\xi = \xi(x)$ in $D(\xi,\overline{\xi})$, which mathematically is analogous to a marginal probability density for the remaining variable $\phi(x)$. After doing the integrals, we are left simply with

$$\int \frac{d\lambda}{2\pi} \exp\{-i\lambda\xi\} \left\langle \exp\{i\lambda\phi(x)\} \right\rangle ,$$

which in light of the identity (8) turns into

$$\int \frac{d\lambda}{2\pi} \exp\left\{-w\lambda^2/2 - i\lambda\xi\right\} = \frac{1}{\sqrt{2\pi w}} e^{\frac{-\xi^2}{2w}} ,$$

where I have written ξ for $\xi(x)$ and w for $\langle\phi(x)\phi(x)\rangle$. Unsurprisingly for a free field, we just obtain a gaussian.

The next simplest case, which already illustrates most of the complications, results from integrating out all but two of the field-values, and yields an integral of the form

$$\int \frac{d\alpha}{2\pi}\frac{d\beta}{2\pi} \exp\{-i\alpha\xi - i\beta\eta\} \left\langle \exp\{i\alpha\phi(x)\} \exp\{i\beta\phi(y)\} \right\rangle .$$

We can again avail ourselves of (8), but only after bringing to bear the well known identity

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} , \quad (11)$$

which holds whenever the commutator $[A,B]$ is a c-number. In virtue of (11) we find

$$\exp\{i\alpha\phi(x)\}\exp\{i\beta\phi(y)\} = \exp\left\{i\alpha\phi(x) + i\beta\phi(y) - \frac{1}{2}\alpha\beta[\phi(x),\phi(y)]\right\}$$

which can also be written as $\exp\left\{i\alpha\phi(x) + i\beta\phi(y) - \frac{1}{2}\alpha\beta(W^{xy} - W^{yx})\right\}$ because

$$[\phi(x),\phi(y)] = \phi(x)\phi(y) - \phi(y)\phi(x) = \langle\phi(x)\phi(y) - \phi(y)\phi(x)\rangle = W^{xy} - W^{yx} \ ,$$

the original commutator being a c-number. Hence,

$$\begin{aligned}
&\langle\exp\{i\alpha\phi(x)\}\exp\{i\beta\phi(y)\}\rangle \\
&= \exp\left\{-\frac{1}{2}\left(\left\langle(\alpha\phi(x)+\beta\phi(y))^2\right\rangle + \alpha\beta(W^{xy}-W^{yx})\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\alpha^2W^{xx} + \beta^2W^{yy} + \alpha\beta W^{xy} + \alpha\beta W^{yx} + \alpha\beta W^{xy} - \alpha\beta W^{yx}\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\alpha^2W^{xx} + \beta^2W^{yy} + 2\alpha\beta W^{xy}\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\begin{pmatrix}\alpha & \beta\end{pmatrix}\begin{pmatrix}W^{xx} & W^{xy} \\ W^{xy} & W^{yy}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}\right)\right\}
\end{aligned}$$

The important thing to notice here is that the order of the indices in W^{xy} copies the order of x and y in the original factors.

Following this pattern, it is straightforward to evaluate the quantity within angle brackets in (10). The result is

$$\exp\left\{-\frac{1}{2}\left(\overrightarrow{W}^{jk}\overline{\lambda}_j\overline{\lambda}_k + \overleftarrow{W}^{jk}\lambda_j\lambda_k + 2W^{jk}\overline{\lambda}_j\lambda_k\right)\right\} \tag{12}$$

where $\overrightarrow{W}^{jk}$ denotes W^{jk} if $j \le k$ and W^{kj} if $k \le j$, for example $\overrightarrow{W}^{23} = \overrightarrow{W}^{32} = W^{23}$; and $\overleftarrow{W}$ follows the reversed convention. (That the arrows in the first and second terms point in opposite directions merely reflects the opposite ordering of the causet elements in the two halves of the Schwinger history.) Notice in connection with (12) that the matrices $\overleftarrow{W}$ and $\overrightarrow{W}$ are complex conjugates of each other, and recall also that W itself is positive semidefinite.

Let us observe in passing that integrating out the $\overline{\xi}$ variables from $D(\xi,\overline{\xi})$ would have the effect of setting $\overline{\lambda}=0$ in (12), leaving in the exponent only $-\frac{1}{2}\overleftarrow{W}^{jk}\lambda_j\lambda_k$, which involves only the time-ordered Wightman function, or equivalently the "Feynman Green function".

Now since $D(\xi,\overline{\xi})$ is nothing more than the Fourier transform of (12), its evaluation would be relatively carefree, were it not for the fact that the quadratic form in (12) is not invertible. This will prevent $D(\xi,\overline{\xi})$ from assuming the pure gaussian form that it would otherwise have had, and that one might naively have expected from a free field theory. Instead, it will be the product of a gaussian with a δ-function that will enforce certain "constraints" on its arguments.

The further manipulations to be done will become more transparent if at this point we switch to a matrix notation, and I'll continue to use a tilde to denote transpose. Then for example, we can re-express (12) as $\exp\left\{-\frac{1}{2}Q\right\}$ where

$$Q=\lambda\overleftarrow{W}\,\lambda+\overline{\lambda}\,\overrightarrow{W}\,\overline{\lambda}+2\overline{\lambda}\,W\,\lambda\ .$$

Let us now express Q directly in terms of G and R using that, according to its definition, $W=R+i\Delta/2=R+(i/2)(G-\widetilde{G})$, and noting also that $\overleftarrow{R}=\overrightarrow{R}=R$ and $\overleftarrow{\Delta}=-\overrightarrow{\Delta}=G+\widetilde{G}$, whence

$$\overleftarrow{W}=\overleftarrow{R+i\Delta/2}=R+i\overleftarrow{\Delta}/2=R+(i/2)(G+\widetilde{G})$$

whence similarly $\overrightarrow{W}=R-(i/2)(G+\widetilde{G})$. Expanding $\overleftarrow{W}$ this way yields

$$\lambda\overleftarrow{W}\,\lambda=\lambda(R+(i/2)(G+\widetilde{G}))\lambda=\lambda R\lambda+i\lambda G\lambda$$

with similar expansions for $\overline{\lambda}\overrightarrow{W}\,\overline{\lambda}$ and $2\overline{\lambda}W\lambda$. Adding these together then reveals that Q is expressed most simply in terms of sum and difference variables for λ and $\overline{\lambda}$.

$$Q=(\lambda+\overline{\lambda})R(\lambda+\overline{\lambda})+i(\lambda+\overline{\lambda})G(\lambda-\overline{\lambda})\ .$$

Its fourier transform can then be expressed in terms of the corresponding sum and difference variables for ξ and $\overline{\xi}$. To that end, let us define

$$K=\lambda+\overline{\lambda}\ ,\ k=\frac{1}{2}(\lambda-\overline{\lambda})\ ,\quad \phi=\frac{1}{2}(\xi+\overline{\xi})\ ,\ \varphi=\xi-\overline{\xi}$$

Here ϕ, the mean of the two halves of the Schwinger history, is what is sometimes called "the classical field" because $\xi = \overline{\xi}$ in the classical limit, while φ represents in some sense the deviation from classicality.[6]

With these definitions, $\lambda\xi + \overline{\lambda\xi} = K\phi + k\varphi$, and the fourier transform yielding $D(\xi, \overline{\xi})$ turns into

$$\int d^N K \, d^N k \, \exp\left\{-iK\phi - ik\varphi - \frac{1}{2}KRK - iKGk\right\} \tag{13}$$

where I've left out an overall normalization which needn't be carried along because it can be recovered at the end from the condition $\int d^N\xi \, d^N\overline{\xi} \, D(\xi, \overline{\xi}) = D(\Omega, \Omega) = 1$.

Evidently the conjugate variable k occurs only linearly in the exponent of (13), in the combination $k\varphi + KGk = k(\widetilde{G}K + \varphi)$; the dk integral therefore produces the δ-function I mentioned earlier, which constrains φ to be in the image of $\widetilde{G}$, the "advanced Green function". Equivalently, it constrains φ to be orthogonal to $\ker(G) = \{K \mid GK = 0\}$. We now have

$$D(\xi, \overline{\xi}) = (const) \int d^N K \; \delta(\widetilde{G}K + \varphi) \, \exp\left\{-\frac{1}{2}KRK - iK\phi\right\} \; . \tag{14}$$

When φ is in the image of $\widetilde{G}$, as required for D to be nonzero, (14) has the form of a gaussian integral over an affine subspace of $\mathbb{R}^N$, namely the space of solutions to the equation $\widetilde{G}K + \varphi = 0$. To convert (14) into an integral over a full vector space, let $I : V \to \mathbb{R}^N$ be the inclusion map of $V = \ker \widetilde{G}$ into $\mathbb{R}^N$ and let K_0 be any solution of the equation $\widetilde{G}K_0 = -\varphi$, so that the space over which we have to integrate is $K_0 + V$. (Concretely we can represent I by a matrix $I_j{}^\alpha$, where the vectors $I_j{}^\alpha$ for fixed α furnish

[6] Please don't confuse ϕ as defined here with the same symbol used earlier for the field *operator*.

a basis for V.) The substitution of K_0+K for K then yields our integral in the form

$$
\begin{aligned}
&\int_{K\in K_0+V} d^{N'}K \,\exp\left\{-\frac{1}{2}KRK - iK\phi\right\} \\
&= \int_{K\in V} d^{N'}K \,\exp\left\{-\frac{1}{2}(K+K_0)R(K+K_0) - i(K+K_0)\phi\right\} \\
&= \exp\left\{-\frac{1}{2}K_0RK_0 - iK_0\phi\right\} \int_{K\in V} d^{N'}K \,\exp\left\{-\frac{1}{2}KRK - K(RK_0+i\phi)\right\}
\end{aligned}
$$

where $N'=\dim\ker\widetilde{G}$ is the dimensionality of V. Then with the aid of the matrix I we obtain an integral over $\mathbb{R}^{N'}$ by putting Iv in place of K :

$$
\exp\left\{-\frac{1}{2}K_0RK_0 - iK_0\phi\right\} \int d^{N'}v \,\exp\left\{-\frac{1}{2}v\widetilde{I}RIv - (K_0R+i\phi)Iv\right\} \ . \tag{15}
$$

The result is now an ordinary gaussian integral over $\mathbb{R}^{N'}$ of the form $\int \exp\left\{-\frac{1}{2}vAv + Bv\right\}$ with $A=\widetilde{I}RI$ and $B=(K_0R+i\phi)I$. Under the assumption that A is invertible, this integral evaluates to (a constant multiple of) $\exp\left\{-\frac{1}{2}B(1/A)B\right\}$. When expanded out, the exponent here consists of three terms:

$$
\phi\, I(\widetilde{I}RI)^{-1}\widetilde{I}\,\phi \ , \tag{16 − 1}
$$

$$
-iK_0\,(1 - RI(\widetilde{I}RI)^{-1}\widetilde{I})\,\phi \ , \tag{16 − 2}
$$

$$
-\frac{1}{2}K_0\,(R - RI(\widetilde{I}RI)^{-1}\widetilde{I}R)\,K_0 \ . \tag{16 − 3}
$$

We now have the decoherence functional in hand, except that (16) is expressed in terms of the ambiguous constant K_0 rather than φ. For consistency, this apparent dependence on K_0 must be illusory, i.e. (16) must be independent of K_0 for fixed φ.

In considering (16), let us observe first of all that one would expect I to be "small" since one would expect $\dim\ker\widetilde{G}$ to be small compared to N, the number of elements in our causet. It can't vanish entirely, given that G is strictly retarded — now that we've removed its diagonal — but an analogy with the continuum might suggest that it would receive contributions only from the "boundary region" of the causet. To the extent that I actually can be neglected, (14) could have been evaluated without doing any integrations at all, as the

δ-function therein sets K to $-\widetilde{G}^{-1}\varphi$, yielding $D = \exp\left\{-\frac{1}{2}\varphi(G^{-1}R\widetilde{G}^{-1})\varphi + i\varphi G^{-1}\phi\right\}$. Now a natural symbol to represent G^{-1} would be $\Box$, since in the continuum G is in some sense the inverse of $\Box$. In the present setting we can also appreciate that G^{-1} (if it actually existed) would be retarded too because G itself is retarded. One might then refer to $\Box = G^{-1}$ as a "retarded d'Alembertian" and its transpose $\widetilde{\Box}$ as an "advanced d'Alembertian". With this nomenclature we would have $D = \exp\left\{-\frac{1}{2}\varphi(\Box\, R\widetilde{\Box})\varphi + i\varphi\,\Box\,\phi\right\}$. Moreover, in the continuum, $\Box\, R = \mathrm{Re}(\Box\, W) = 0$, therefore one might also expect $-\frac{1}{2}\varphi(\Box\, R\widetilde{\Box})\varphi$ to be small in the causal set. In that case the whole of the decoherence functional would simplify to $\exp\{i\varphi\,\Box\,\phi\}$. Remarkably, this is exactly the decoherence functional of the continuum theory, as an integration by parts reveals, given that $\varphi = 0$ on the future boundary. More precisely, the continuum decoherence functional is this expression together with the δ-function in (2) that mandates the just-mentioned equality between ξ and $\overline{\xi}$ on the future boundary, cf. [11]. Such a δ-function is present implicitly in (14) and explicitly in (9), but I don't know how to extract it as such from a well-defined continuum limit of the discrete theory. We'll return to the constraints on φ shortly.

It is useful at this point to distinguish conceptually between the space $\mathbb{R}^N$ of field-configurations and the space $\mathbb{R}^N$ of configurations of the dual variables K, λ, etc. Since we have already employed the symbol Ω for the former, we might as well denote the latter by Ω^* , using a star to denote dual space. I'll also write V for $\mathbb{R}^{N'}$ considered as $\ker\widetilde{G}$ considered as a separate vector space. Then $I : V \to \Omega^*$, $\widetilde{I} : \Omega \to V^*$, $G : \Omega^* \to \Omega$, $\widetilde{G} : \Omega^* \to \Omega$, etc.

Let us return now to the full decoherence functional as given by (16). Its second member can be written as $-iK_0(1-\Pi)\phi$ if we define Π to be the operator

$$\Pi = RI(\widetilde{I}RI)^{-1}\widetilde{I} \ .$$

Here, of course, we continue to assume that $(\widetilde{I}RI)^{-1}$ exists. From its definition it is trivial to check that Π, though not hermitian, is a projection in the sense that $\Pi^2 = \Pi$; and also that

$$\widetilde{I}\,\Pi = \widetilde{I} \ , \quad \widetilde{\Pi}\, I = I \ . \tag{17}$$

Now the term by which $K_0 = -\widetilde{G}^{-1}\varphi$ is ambiguous is by definition Iv for arbitrary $v \in V$. Hence the ambiguity in $-K_0(1-\Pi)\phi$ is precisely $(Iv)(1-\Pi)\phi = v\widetilde{I}(1-\Pi)\phi = v(\widetilde{I} - \widetilde{I}\Pi)\phi$, which vanishes by (17). Consequently, it is justified to write (16-2) simply as $i(\widetilde{G}^{-1}\varphi)(1-\Pi)\phi = i\varphi G^{-1}(1-\Pi)\phi$. In view of these considerations, it is natural to define

$$\Box = G^{-1}(1-\Pi) \ , \tag{18}$$

meaning that $\Box$ solves the equation $G\Box = 1 - \Pi$. We have just seen that $\Box\,\phi$ is unambiguous when contracted with any $\varphi \in \mathrm{im}(\widetilde{G})$. Thus $\Box$ itself is uniquely defined if we construe it as a map from Ω to $(\mathrm{im}\,\widetilde{G})^*$, the dual vectorspace to $(\mathrm{im}\,\widetilde{G})$: in symbols, $\Box : \Omega \to (\mathrm{im}\,\widetilde{G})^* \subseteq \Omega^*$. Regarded however as mapping Ω to all of Ω^*, $\Box$ is ambiguous by the addition of any linear operator with image in $(\mathrm{im}\,\widetilde{G})^0 = \ker G$. A nice idea at first sight would be to take advantage of this ambiguity to render $\Box$ fully retarded, but unfortunately that is not possible. Reasoning similar to the above also proves that $G\,\Box\,G = G$, that $\mathrm{im}\,\widetilde{\Pi} = \mathrm{im}\,I = \ker\widetilde{G}$, that $\ker\Pi = \mathrm{im}\,G$, and therefore that $\mathrm{im}(1-\Pi) = \mathrm{im}\,G$. Also that $\Pi R = R\widetilde{\Pi} = RI(\widetilde{I}RI)^{-1}\widetilde{I}R$.

So far, we have been occupied primarily with the analysis of (16-2), but we also need to deal briefly with (16-1) and (16-3). The former is complete as it stands, but the latter requires to be expressed in terms of φ, just as we did for (16-2). To that end, rewrite (16-3) as $-\frac{1}{2}K_0(1-\Pi)RK_0 = -\frac{1}{2}K_0(1-\Pi)^2RK_0 = -\frac{1}{2}K_0(1-\Pi)R(1-\widetilde{\Pi})K_0$ and observe that the ambiguity of adding a vector Iv to K_0 drops out just as before because $(1-\widetilde{\Pi})I = 0$. Continuing on, and replacing K_0 by $-\widetilde{G}^{-1}\varphi$, we obtain with the aid of equation (18), $-\frac{1}{2}\varphi G^{-1}(1-\Pi)R(1-\widetilde{\Pi})\widetilde{G}^{-1}\varphi = -\frac{1}{2}\varphi\,\Box(1-\Pi)R\widetilde{\Box}\varphi$, which can also be written as $-\frac{1}{2}(\widetilde{\Box}\varphi)R(\widetilde{\Box}\varphi)$. Collecting our three terms then yields in total

$$D(\xi,\overline{\xi}) = (constant)\,\exp\left\{i\varphi\,\Box\,\phi - \frac{1}{2}(\widetilde{I}\phi)(\widetilde{I}RI)^{-1}(\widetilde{I}\phi) - \frac{1}{2}(\widetilde{\Box}\varphi)R(\widetilde{\Box}\varphi)\right\} \tag{19}$$

In interpreting this formula, one should bear in mind that it includes an implicit delta-function [7] that restricts φ to lie in the image of $\widetilde{G}$. That is, $D(\xi,\overline{\xi})$ vanishes unless ξ and

[7] Since $\mathrm{im}\,\widetilde{G} = \perp\ker G$, we could make the delta-function explicit by selecting a basis of (co)vectors v for the kernel of G and forming the product over these v of $\delta(v\cdot\varphi)$

$\overline{\xi}$ differ by an element of $\mathrm{im}\,\widetilde{G}$. If we remember also that with our definitions, the diagonal elements of G all vanish, then we see immediately that every function in $\mathrm{im}\,\widetilde{G}$ vanishes on the maximal elements of our causal set C, which implies in particular that (just as in (2)) ξ and $\overline{\xi}$ must agree on these elements. In general though, there will arise additional linear relations among the φ^j that lack any obvious continuum analog.

In view of the rather intricate derivation we've just been through it seems worth recording here the much simpler expression that results from integrating out the $\overline{\xi}$ variables in $D(\xi, \overline{\xi})$ or simply from returning to (9) and deleting the factors involving $\overline{\xi}$. One sees immediately from (12) that — provided that $\overleftarrow{W}$ is invertible — the result will be

$$D(\xi, \Omega) = \int d^N \overline{\xi}\, D(\xi, \overline{\xi}) = (const)\, \exp\left\{ \frac{i}{2} \xi \underset{F}{\Box}\, \xi \right\} \tag{20}$$

where I've written $\Box_F$ for $i\overleftarrow{W}^{-1}$, a notation that seems natural because $-i\overleftarrow{W}$ corresponds to the Feynman Green function of the continuum theory.

5. Consequences and questions

Perhaps the most important conclusion from the above work is that the equations of motion for the "classical" field ϕ are essentially a retarded non-local version of the continuum equations $\Box\,\phi = 0$. Equations of precisely this sort were proposed in [8], but until now there has seemed to be no sound reason for preferring retarded equations over, for example, the time-reverse or some combination of the two. Granted that retardation greatly facilitates computer solution, but that would be more an opportunistic reason than one of principle. Retardation also agrees better with the notion that physics in the classical regime, even if it were to be non-local, would still be "causal", but here we have reached a similar conclusion in a far more convincing manner!

But why should one refer to $\Box\,\phi = 0$ as the classical equations of motion? Expressed through the decoherence functional, classicality amounts to the vanishing of φ, i.e. to the equality of ξ with $\overline{\xi}$ (in which case both coincide with ϕ). But if we vary φ in (19) and then set $\varphi = 0$, the result is precisely $\Box\,\phi = 0$. Further support for this point of view comes from the fact, already alluded to, that the second and third terms in (19) seem likely to be relatively negligible, having no further significance than either initial-conditions or small corrections. In the case of the second term specifically (the term in ϕ), its character as a kind of initial condition or "state" can be perceived fairly clearly, because it depends on ϕ only through $\widetilde{I}\phi$, which is a combination of contractions of ϕ with vectors in $\ker \widetilde{G}$. But since $\widetilde{G}$ is "future-looking" it will annihilate any function with support on the minimal level of C. Plausibly then, its entire kernel is comprised of functions supported near this bottom level, whence $\widetilde{I}\phi$ would depend only on such initial values of ϕ.

Of course if we vary all the arguments in (19), we obtain more equations, and taken together they tell us more than just $\Box\,\phi = 0$. In fact they constrain ϕ itself to be zero (along with φ). This however only reflects the fact that our action (or rather decoherence functional) implicitly includes a specification of "initial conditions", something like an "initial state wave-function" in the continuum theory. In fact exactly the same thing happens in that case: a unique "classical" solution is picked out. In the present context it's unremarkable that the corresponding solution is $\phi = 0$, because our construction of the decoherence functional assumed early on that (4) was true.

We might also ask how an apparently time-symmetric scheme has given rise to a very time-asymmetric set of classical equations. To the extent that we can regard $\Box$ as retarded, the combination $i\varphi\,\Box\,\phi$ is entirely retarded in the "classical component" ϕ and entirely advanced in the "non-classical component" φ. Just how this has come about seems mysterious, but logically the asymmetry had to originate in the order chosen for the factors in (9). Perhaps a careful tracing through of the steps leading from there to (19) would shed more light on the question.

Of course the operator $\Box$ is not fully retarded but only approximately so (if our plausibility arguments are valid). Should this worry us? I'm not sure, but one reason for equanimity at this point springs from the observation that a dynamics taking place in a frozen causal set can only be a partial reflection of full quantum gravity. Really the causet itself needs to be included in the decoherence functional. Not having done this we have restricted its future development in an unphysical manner, and thereby we have imposed a sort of future boundary condition. It would be no surprise if such a condition were to be reflected to some extent in the dynamics of the variables (namely the scalar field) that we have left free. (Whether we can be equally happy with an anti-causal equation for φ is another question however.)

A question might be raised here about our assumption that the matrix $\widetilde{I}RI$ is invertible. How well does this hold up in practice? Quite well if the few numerical experiments that I've done are indicative, but so far they have been limited in scope. An interesting exception, however, springs from the existence of so called non-Hegelian elements or pairs within the causal set C, viz. pairs of incomparable elements x and y which share the same relation to every third element $z \in C$. Such elements cannot be distinguished from each other by the causal structure, and in consequence the "difference function" defined by $f(x) = +1$, $f(y) = -1$, and $f(z) = 0$ otherwise, will be in the kernel of G, Δ, etc. In particular it will be in the kernel of $\widetilde{I}RI$, rendering the latter non-invertible. But the problem this might seem to raise solves itself. Indeed x and y are in some sense not two distinct elements at all, and the theory seems to know this, as one can see by returning to equation (13) or (14) and noticing that f is a degenerate direction for the quadratic form involved. Consequently the resulting decoherence functional will vanish unless $\xi(x) = \xi(y)$ and $\overline{\xi}(x) = \overline{\xi}(y)$. The theory thus lives in effect on the quotient causet C' obtained by identifying x with y, and one might just as well eliminate y (say) at the outset and continue in this manner until no further non-Hegelian elements remain.

Many more questions could be raised at this point, but let me just conclude with two of a more general nature, the first being this. All our work herein has presupposed a fixed, background causal set C with a finite number of elements. But in reality, one expects the causet to be growing and infinite, or at least potentially infinite toward the future. We cannot actually handle dynamical growth without setting up a full quantum gravity theory, but we can certainly imagine C being infinite toward the future. In that case our theory herein would correspond to only some initial portion of C (a so called partial stem) and this partial theory should be obtained from the full one by restriction.

Conversely, if the theory is formulated initially on the stems, then the need arises for consistency between the partial theories belonging to the different stems of C. (From a histories point of view, almost all quantum theories are built up this way from partial theories which provide the measures of so-called cylinder events. See for example [16] or [17].) It is therefore an important question whether the theory herein can be, or can be adapted to be, a component of a coherent theory defined consistently on an untruncated causal set.

The other general question concerns interactions. The theory we have been working with herein is that of the causet counterpart of a free field, as manifested in the quadratic nature of (19). In its original operator form, however, the theory does not readily suggest a generalization to include interactions. A potential advantage of the histories formulation, then, is that it does suggest such a generalization. To incorporate a ϕ^4 interaction, for example, one need only include a multiple of $i\xi^4 - i\overline{\xi}^4$ into the exponent of (19). For the sake of consistency, one should check that the decoherence functional, thus modified, remains positive semidefinite, but this can be done. It would be interesting to compare the consequences of this generalized theory with those of its continuum counterpart.

Appendix. Proof of an identity used in the text

In proving equation (8) we used the following identity (21), which is a special case of Wick's theorem. Indeed, the coefficient $(n-1)!! = (n-1)(n-3)\cdots 5 \cdot 3 \cdot 1$ in (21) is easily seen to count the number of ways of pairing up the $2n$ factors of Φ in the expression $\langle \Phi^n \rangle = \langle \Phi\Phi\Phi \cdots \Phi \rangle$. Here we prove this special case directly by a straightforward method.

$$\left\langle \Phi^{2n} \right\rangle = (2n-1)!! \, \langle \Phi\Phi \rangle^n \tag{21}$$

Suppose first that $\Phi = a + a^*$ with a normalized as usual so that $[a, a^*] = 1$. Also let $|0\rangle$ be the "vacuum" relative to a with $\langle \cdot \rangle = \langle 0| \cdot |0\rangle$. Noticing that $a|0\rangle = 0$ and $\langle 0|a^* = 0$, and that $[a, \Phi] = [a, a + a^*] = [a, a^*] = 1$, we have then

$$\langle \Phi^n \rangle = \langle (a + a^*)\Phi^{n-1} \rangle = \langle a\Phi^{n-1} \rangle = \langle [a, \Phi^{n-1}] \rangle = \langle (n-1)\Phi^{n-2} \rangle = (n-1)\langle \Phi^{n-2} \rangle \ ,$$

where we used the fact that $[a, \cdot]$ is a derivation (it obeys the Leibniz rule). Since $\langle\Phi^0\rangle = 1$, it follows by induction that for even n, $\langle\Phi^n\rangle = (n-1)!!$. When n is odd $\langle\Phi^n\rangle$ of course vanishes, since an odd number of applications of a or a^* to the vacuum can never bring one back to the vacuum.

Finally, consider the more general situation where Φ is some linear combination of field operators $\phi(x)$. Because $\Phi = \Phi^*$ we can always write it in the form $\phi = A + A^*$, where A contains only the terms with lowering operators a. The proof then proceeds as before except that in place of $[a, a^*] = 1$ we have instead $[A, A^*] = \langle\Phi\Phi\rangle$, as we see from $\langle\Phi\Phi\rangle = \langle(A + A^*)(A + A^*)\rangle = \langle AA^*\rangle = \langle[A, A^*]\rangle = [A, A^*]$.

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References

[1] Luca Bombelli, Joohan Lee, David Meyer and Rafael D. Sorkin, "Spacetime as a Causal Set", *Phys. Rev. Lett.* **59:** 521-524 (1987)

[2] Rafael D. Sorkin, "Causal Sets: Discrete Gravity (Notes for the Valdivia Summer School)", in *Lectures on Quantum Gravity* (Series of the Centro De Estudios Científicos), proceedings of the Valdivia Summer School, held January 2002 in Valdivia, Chile, edited by Andrés Gomberoff and Don Marolf (Plenum, 2005)
`gr-qc/0309009`
`http://www.pitp.ca/personal/rsorkin/some.papers/`

[3] Joe Henson, "The causal set approach to quantum gravity" `gr-qc/0601121` This is an extended version of a review published in in *Approaches to Quantum Gravity – Towards a new understanding of space and time*, edited by Daniele Oriti (Cambridge University Press 2009) (ISBN: 978-0-521-86045-1), pages 26-43,

[4] Fay Dowker, "Causal sets and the deep structure of Spacetime", in *100 Years of Relativity - Space-time Structure: Einstein and Beyond*" ed Abhay Ashtekar (World Scientific 2005) gr-qc/0508109

[5] Petros Wallden, "Causal Sets: Quantum gravity from a fundamentally discrete spacetime", *J. Phys. Conf. Ser.* **222 :** 012053 (2010) arXiv:1001.4041v1 [gr-qc]

[6] Sumati Surya, "Directions in Causal Set Quantum Gravity", arXiv:1103.6272v1 [gr-qc]

[7] Rafael D. Sorkin, "Logic is to the quantum as geometry is to gravity" in G.F.R. Ellis, J. Murugan and A. Weltman (eds), *Foundations of Space and Time* (Cambridge University Press, to appear) http://arXiv.org/abs/arXiv:1004.1226 [quant-ph] http://www.pitp.ca/personal/rsorkin/some.papers/

Rafael D. Sorkin, "An exercise in 'anhomomorphic logic' ", *Journal of Physics: Conference Series (JPCS)* **67 :** 012018 (2007), a special volume edited by L. Diosi, H-T Elze, and G. Vitiello, and devoted to the Proceedings of the DICE-2006 meeting, held September 2006, in Piombino, Italia. arxiv quant-ph/0703276 , http://www.pitp.ca/personal/rsorkin/some.papers/

[8] Rafael D. Sorkin, "Does Locality Fail at Intermediate Length-Scales?" in *Approaches to Quantum Gravity – Towards a new understanding of space and time*, edited by Daniele Oriti (Cambridge University Press 2009) (ISBN: 978-0-521-86045-1), pages 26-43, gr-qc/0703099, http://www.pitp.ca/personal/rsorkin/some.papers/

[9] Don Marolf, private communication

[10] Steven Johnston, "Feynman Propagator for a Free Scalar Field on a Causal Set" *Phys. Rev. Lett* **103 :** 180401 (2009) arXiv:0909.0944 [hep-th]

[11] Xavier Martin, Denjoe O'Connor and Rafael D. Sorkin, "The Random Walk in Generalized Quantum Theory" *Phys. Rev. D* **71 :** 024029 (2005) gr-qc/0403085 http://www.pitp.ca/personal/rsorkin/some.papers/

[12] Steven Johnston, "Particle propagators on discrete spacetime" *Class.Quant.Grav.* **25 :** 202001 (2008) http://arXiv.org/abs/0806.3083 [gr-qc]

[13] Dionigi M.T. Benincasa and Fay Dowker, "The Scalar Curvature of a Causal Set" arXiv:1001.2725 *Phys.Rev.Lett.* **104 :** 181301 (2010)

[14] Rafael D. Sorkin, "Quantum dynamics without the wave function" *J. Phys. A: Math. Theor.* **40 :** 3207-3221 (2007) (http://stacks.iop.org/1751-8121/40/3207) quant-ph/0610204 http://www.pitp.ca/personal/rsorkin/some.papers/

[15] Rafael D. Sorkin, "Quantum Mechanics as Quantum Measure Theory", *Mod. Phys. Lett. A* **9** (No. 33) **:** 3119-3127 (1994) gr-qc/9401003 http://www.pitp.ca/personal/rsorkin/some.papers/

[16] S. Gudder and Rafael D. Sorkin, "Two-site quantum random walk", *General Relativity and Gravitation* (to appear) http://arXiv.org/abs/1105.0705 http://www.pitp.ca/personal/rsorkin/some.papers/

[17] Rafael D. Sorkin, "Toward a 'fundamental theorem of quantal measure theory'", *Mathematical Structures in Computer Science* (to appear)
http://arxiv.org/abs/1104.0997
http://www.pitp.ca/personal/rsorkin/some.papers/141.fthqmt.pdf