Hilbert Spaces from Path Integrals

Fay Dowker, Steven Johnston, Rafael D. Sorkin

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Abstract

It is shown that a Hilbert space can be constructed for a quantum system starting from a framework in which histories are fundamental. The Decoherence Functional provides the inner product on this "History Hilbert space". It is also shown that the History Hilbert space *is* the standard Hilbert space in the case of non-relativistic quantum mechanics.

1 Introduction

It is not yet known how quantum theory and gravity will be reconciled. However, the four-dimensional nature of reality revealed by our best theory of gravity, General Relativity, suggests that unity in physics will only be achieved if quantum theory can be founded on the concept of *history* rather than that of *state*. The same suggestion emerges even more emphatically from the causal set programme, whose characteristic kind of spatio-temporal discreteness militates strongly against any dynamics resting on the idea of Hamiltonian evolution.

A major step toward a histories-based formulation of quantum mechanics was taken by Dirac and Feynman, showing that the quantum-mechanical propagator can be expressed as a *sum over histories* [1, 2, 3], but it remains a challenge to make histories the foundational basis of quantum mechanics. One attempt to do this was made by J. Hartle who set out new, histories-based axioms for *Generalised Quantum Mechanics* (GQM) which do not require the existence of a Hilbert space of states [4, 5]. Closely related in its technical aspects — whilst differing in interpretational aspiration — is *Quantum Measure Theory* (QMT) [6, 7, 8, 9]. Thus far, both these approaches appear in the literature more as formal axiomatic systems than as fully fledged mathematical physics, although some concrete examples going beyond ordinary quantum mechanics have been studied.[10]

In this paper we take a step toward establishing Quantum Measure Theory and Generalised Quantum Mechanics more firmly on their foundations and connecting them up with the more familiar formalism of state-vectors and operators. First we demonstrate in detail the Gel'fand-Naimark-Segal (GNS) type construction given in [10] of a *History Hilbert space* for any quantum measure system (to be defined). It is technically helpful within quantum measure theory that such a construction is available, but the conceptual significance of this fact would be slight, were it not that the constructed Hilbert space provably *is* the usual Hilbert space in the case of certain familiar quantum systems (via an isomorphism that obtains formally in any unitary quantum theory with pure initial state). In this paper we exhibit non-relativistic particle quantum mechanics in d spatial dimensions as a quantum measure system, and we prove that the Hilbert space constructed from the quantum measure is the usual Hilbert space of (equivalence classes of) square integrable complex functions on $\mathbb{R}^d$, given certain conditions on the propagator. The class of systems for which these conditions can be established is large and includes the free particle and the simple harmonic oscillator. Thus, one of the main ingredients of text-book Copenhagen Quantum Mechanics is derivable from the starting point of histories.

2 Quantum Measure Theory: a histories-based framework

We describe here the framework set out in [6, 7, 8, 9]. In Quantum Measure Theory, a physical, quantum system is associated with a *sample space* Ω of possible *histories*, the space over which the integration of the path integral takes place. Each history γ in the sample space represents as complete a description of physical reality as is classically conceivable in the theory. The kind of elements in Ω varies from theory to theory. In n-particle quantum mechanics, a history is a set of n trajectories. In a scalar field theory, a history is a real or complex function on spacetime. The business of discovering the appropriate sample space for a particular theory is part of physics. Even in the seemingly simple case of non-relativistic particle quantum mechanics, we do not yet know what properties the trajectories in Ω should possess, not to mention the knotty problems involved in defining Ω for fermionic field theories for example. We will be able to sidestep these issues in the current work.

2.1 Event Algebra

Once the sample space has been settled upon, any proposition about physical reality is represented by a subset of Ω. For example in the case of the non-relativistic particle, if R is a region of space and T a time, the proposition "the particle is in R at time T" corresponds to the set of all trajectories which pass through R at T. We follow the standard terminology of stochastic processes and refer to such subsets of Ω as *events*.

An *event algebra* on a sample space Ω is a non-empty collection, $\mathfrak{A}$, of subsets of Ω such that

1. For any $\alpha \in \mathfrak{A}$, we have $\Omega \setminus \alpha \in \mathfrak{A}$.
2. For any $\alpha, \beta \in \mathfrak{A}$, we have $\alpha \cup \beta \in \mathfrak{A}$.

An event algebra is then an algebra of sets [11]. It follows immediately that $\varnothing \in \mathfrak{A}$, $\Omega \in \mathfrak{A}$ ($\varnothing$ is the empty set) and $\mathfrak{A}$ is closed under finite unions and intersections.

An event algebra $\mathfrak{A}$ is a Boolean algebra under intersection (logical "and"), union (logical "or") and complement (logical "not") with unit element Ω and zero element $\varnothing$. It is also a (unital) ring with identity element Ω, multiplication as intersection and addition as symmetric difference (logical "xor"):

1. $\alpha \cdot \beta := \alpha \cap \beta$.
2. $\alpha + \beta := (\alpha \setminus \beta) \cup (\beta \setminus \alpha)$.

This ring is Boolean since $\alpha \cdot \alpha = \alpha$. It is also an algebra over $\mathbb{Z}_2$. More discussion of the event algebra is given in [9].

An example of an event algebra is the power set $2^\Omega := \{S : S \subseteq \Omega\}$ of all subsets of Ω. For physical systems with an infinite sample space, however, the event algebra will be strictly contained in the power set of Ω, something which is familiar from classical measure theory[1] where the collection of "measurable sets" is not the whole power set.

If $\mathfrak{A}$ is also closed under countable unions and intersections then $\mathfrak{A}$ is a *σ-algebra*.

2.2 Decoherence Functional

A *decoherence functional* on an event algebra $\mathfrak{A}$ is a map $D : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ such that

1. For all $\alpha, \beta \in \mathfrak{A}$, we have $D(\alpha, \beta) = D(\beta, \alpha)^*$ (*Hermiticity*).
2. For all $\alpha, \beta, \gamma \in \mathfrak{A}$ with $\beta \cap \gamma = \varnothing$, we have $D(\alpha, \beta \cup \gamma) = D(\alpha, \beta) + D(\alpha, \gamma)$ (*Linearity*).
3. $D(\Omega, \Omega) = 1$ (*Normalisation*).
4. For any finite collection of events $\alpha_i \in \mathfrak{A}$ $(i = 1, \ldots, N)$ the $N \times N$ matrix $D(\alpha_i, \alpha_j)$ is positive semidefinite (*Strong positivity*).

A decoherence functional D satisfying the weaker condition $D(\alpha, \alpha) \geq 0$ for all $\alpha \in \mathfrak{A}$ is called *positive*. Note that in Generalised Quantum Mechanics, a decoherence functional is defined to be positive rather than strongly positive [4, 5].

A *quantal measure* on an event algebra $\mathfrak{A}$ is a map $\mu : \mathfrak{A} \to \mathbb{R}$ such that

1. For all $\alpha \in \mathfrak{A}$, we have $\mu(\alpha) \geq 0$ (*Positivity*).
2. For all mutually disjoint $\alpha, \beta, \gamma \in \mathfrak{A}$, we have

   $$\mu(\alpha \cup \beta \cup \gamma) - \mu(\alpha \cup \beta) - \mu(\beta \cup \gamma) - \mu(\alpha \cup \gamma) + \mu(\alpha) + \mu(\beta) + \mu(\gamma) = 0\,.$$

   (*Quantal Sum Rule*)

[1] To contrast with *quantum* measure theory, the usual textbook measure theory (see Halmos, [11]) will be called "classical".

3. $\mu(\Omega) = 1$ (*Normalisation*).

If $D : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ is a decoherence functional then the map $\mu : \mathfrak{A} \to \mathbb{R}$ defined by $\mu(\alpha) := D(\alpha, \alpha)$ is a quantal measure.

A triple, $(\Omega, \mathfrak{A}, D)$, of sample space, event algebra and decoherence functional will be called a *quantum measure system*.

2.3 A Hilbert Space Construction

Given a quantum measure system, $(\Omega, \mathfrak{A}, D)$, we can construct a Hilbert space: a complex vector space with (non-degenerate) Hermitian inner product which is complete with respect to the induced norm. This construction is given in [10] and is essentially that given by V.P. Belavkin in [12, Theorem 3, Part 1] where the decoherence functional is called a "correlation kernel". The construction is akin to the GNS construction of a Hilbert space from a C^*-algebra and is the same as the construction appearing in Kolmogorov's Dilation Theorem [13, Theorem 2.2], [14].

To start, we first construct the free vector space on $\mathfrak{A}$ and use the decoherence functional to define a degenerate inner product on it.

2.3.1 Inner product space: H_1

To define the *free vector space* on an event algebra $\mathfrak{A}$ we start with the set of all complex-valued functions on $\mathfrak{A}$ which are non-zero only on a finite number of events. This set becomes a vector space, H_1, if addition and scalar multiplication are defined by:

1. For all $u, v \in H_1$ and $\alpha \in \mathfrak{A}$, we have $(u + v)(\alpha) := u(\alpha) + v(\alpha)$.
2. For all $u \in H_1$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathfrak{A}$, we have $(\lambda u)(\alpha) := \lambda u(\alpha)$.

We now define an inner product space $(H_1, \langle \cdot, \cdot \rangle_1)$ by defining a degenerate inner product on H_1 using the decoherence functional D. For $u, v \in H_1$ define:

$$\langle u, v \rangle_1 := \sum_{\alpha \in \mathfrak{A}} \sum_{\beta \in \mathfrak{A}} u(\alpha)^* D(\alpha, \beta) v(\beta). \tag{2.1}$$

This sum is well-defined because u and v are non-zero for only a finite number of events. This satisfies the conditions for an inner product. Note that the strong positivity of the decoherence functional is essential for $\langle u, u \rangle_1 \geq 0$.

To see that the inner product is degenerate consider, for example, the non-zero vector $u \in H_1$ defined by:

$$u(x) := \begin{cases} 1 & \text{if } x = \alpha, \\ 1 & \text{if } x = \beta, \\ -1 & \text{if } x = \alpha \cup \beta, \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

for two nonempty, disjoint events $\alpha, \beta \in \mathfrak{A}$. By applying the properties of the decoherence functional we see that $||u||_1 = 0$.

2.3.2 Hilbert space: H_2

We now quotient and complete the inner product space $(H_1, \langle\cdot,\cdot\rangle_1)$ to form a Hilbert space $(H_2, \langle\cdot,\cdot\rangle_2)$.

For two Cauchy sequences $\{u_n\}, \{v_n\}$ in H_1 we define an equivalence relation

$$\{u_n\} \sim_1 \{v_n\} \iff \lim_{n\to\infty} ||u_n - v_n||_1 = 0. \tag{2.3}$$

We denote the $\sim_1$ equivalence class of a Cauchy sequence $\{u_n\}$ by $[u_n]_1$. The set of these equivalence classes form a Hilbert space, $(H_2, \langle\cdot,\cdot\rangle_2)$, if addition, scalar multiplication and the inner product are defined by:

1. For all $[u_n]_1, [v_n]_1 \in H_1$, we have $[u_n]_1 + [v_n]_1 := [u_n + v_n]_1$.
2. For all $[u_n]_1 \in H_1$ and $\lambda \in \mathbb{C}$, we have $\lambda[u_n]_1 := [\lambda u_n]_1$.
3. For all $[u_n]_1, [v_n]_1 \in H_1$, we have

$$\langle [u_n]_1, [v_n]_1\rangle_2 := \lim_{n\to\infty} \langle u_n, v_n\rangle_1 \tag{2.4}$$

These are all well-defined, independent of which representative is chosen from the equivalence classes.

The construction of a Hilbert space (here $(H_2, \langle\cdot,\cdot\rangle_2)$) from an inner product space (here $(H_1, \langle\cdot,\cdot\rangle_1)$) is a standard operation described in many textbooks (for example, [15, Section 7], [16, p198]).

Whether or not H_2 is separable depends on the particular event algebra and decoherence functional that are used in its construction[2]. In Sections 3, 4 and 4.5 we shall present systems for which the constructed Hilbert space is isomorphic to a separable Hilbert space (the standard Hilbert space for the system). In these examples the constructed Hilbert space is therefore separable.

Note that we did not use the full structure of the quantum measure system: only the event algebra, $\mathfrak{A}$ and the decoherence functional D were used and nowhere did the underlying sample space enter into the game. This will be important in our discussion of particle quantum mechanics where there is an event algebra $\mathfrak{A}$ but we have no precise definition, as yet, of the sample space.

We will refer to the Hilbert space, H_2, constructed from a quantum measure system as the *History Hilbert space*. For quantum systems which have a standard, Copenhagen formulation in terms of unitary evolution on a Hilbert space of states and which can *also* be cast into the form of a quantum measure system, the question arises as to the relationship between the standard Hilbert space and the History Hilbert space. This is the question under study in this paper and it will be shown that in general the answer depends on the initial state and the Schrödinger dynamics for the system since these are what define the decoherence functional. However, we conjecture that *generically* where both Hilbert spaces exist and the decoherence functional encodes a pure initial state,

[2] The dimension of H_1 is equal to the cardinality of $\mathfrak{A}$ but the dimension of H_2, which is less than that of H_1, depends on the $\sim_1$ equivalence relation (which in turn depends on D).

they are isomorphic. Moreover the isomorphism is physically meaningful, so that one can conclude that the History Hilbert space *is* the standard Hilbert space of the system.

We will prove this conjecture for a variety of non-relativistic particle systems and exhibit the isomorphism explicitly. The systems considered include a particle with a finite configuration space, a free non-relativistic particle in d spatial dimensions, and a non-relativistic particle in various backgrounds, including a quadratic potential and an infinite potential barrier. Before turning to these specific cases, we recall the following simple lemma.

Lemma 1. A linear map $f : H_A \to H_B$ from a Hilbert space $(H_A, \langle \cdot, \cdot \rangle_A)$ to a Hilbert space $(H_B, \langle \cdot, \cdot \rangle_B)$ that preserves the inner product, *i.e.*

$$\langle f(u), f(v) \rangle_B = \langle u, v \rangle_A \tag{2.5}$$

for all $u, v \in H_A$, is one-to-one.

Proof. For all $u, v \in H_A$ we have

$$f(v) = f(u) \iff 0 = ||f(u) - f(v)||_B = ||f(u - v)||_B = ||u - v||_A \iff u = v \tag{2.6}$$

□

3 Finite Configuration Space

We analyse the case of a unitary quantum system with finite configuration space as a warm up for the system of main interest, particle quantum mechanics. Consider a system which has a finite configuration space of n possible configurations at any time. We shall only consider the system's configuration at a finite number N of fixed times $t_1 = 0 < t_2 < \ldots < t_N = T$. An example of such a system is a particle with n possible positions at each time which evolves in $N - 1$ discrete time-steps from time $t = 0$ to time $t = T$.

3.1 Standard Hilbert space approach

The Hilbert space for the system is $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ and states of the system at a particular time are represented by vectors in $\mathbb{C}^n$. For a state $\psi \in \mathbb{C}^n$ the i^{th} component, ψ_i, is the amplitude that the system is in configuration i. For all $\psi, \phi \in \mathbb{C}^n$ the non-degenerate inner product is given by

$$\langle \psi, \phi \rangle := \sum_{i=1}^{n} \psi_i^* \phi_i. \tag{3.1}$$

There exists a time evolution operator, $U(t', t)$, a unitary transformation which evolves states at time t to states at time t' and which satisfies the folding property

$$U(t'', t')U(t', t) = U(t'', t) \,. \tag{3.2}$$

3.2 A Quantum Measure System

Each history, γ, of the system is represented by an N-tuple of integers $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ (with $1 \leq \gamma_a \leq n$ for all $a = 1, \ldots, N$) where each integer γ_a denotes the configuration of the system at time $t = t_a$. The system's sample space, Ω, is the (finite) collection of these n^N possible histories. The event algebra, $\mathfrak{A}$, is the power set of Ω: $\mathfrak{A} := 2^\Omega = \{S : S \subseteq \Omega\}$.

To define the decoherence functional we assume there is an initial state $\psi \in \mathbb{C}^n$ of unit norm. This can be thought of as a vector in $\mathbb{C}^n$ or simply as an n-tuple of amplitudes weighting each initial configuration at time $t = 0$. The decoherence functional for singleton events is,

$$D(\{\gamma\}, \{\overline{\gamma}\}) := \psi(\gamma_1)^* U^*_{\gamma_2\gamma_1} U^*_{\gamma_3\gamma_2} \ldots U^*_{\gamma_N\gamma_{N-1}} \tag{3.3}$$

$$\delta_{\gamma_N\overline{\gamma}_N} U_{\overline{\gamma}_N\overline{\gamma}_{N-1}} \ldots U_{\overline{\gamma}_2\overline{\gamma}_1} \psi(\overline{\gamma}_1) \tag{3.4}$$

where $\gamma, \overline{\gamma} \in \Omega$, $\psi(\gamma_1)$ is the γ_1-th component of ψ and $U_{\gamma_2\gamma_1}$ is short hand for $U(t_2, t_1)_{\gamma_2\gamma_1}$, the amplitude to go from γ_1 at t_1 to γ_2 at t_2. D has "Schwinger-Kel'dysh" form, equalling the complex conjugated amplitude of γ times the amplitude of $\overline{\gamma}$ when the two histories end at the same final position, and zero otherwise. The decoherence functional of events $\alpha, \beta \in \mathfrak{A}$ is then fixed by the bi-additivity property:

$$D(\alpha, \beta) := \sum_{\gamma\in\alpha} \sum_{\overline{\gamma}\in\beta} D(\{\gamma\}, \{\overline{\gamma}\}) \, . \tag{3.5}$$

We define the *restricted evolution* of the initial state $\psi \in \mathbb{C}^n$ with respect to a history γ to be the state $\psi_\gamma \in \mathbb{C}^n$ given by:

$$\psi_\gamma := P^{\gamma_N} U(t_N, t_{N-1}) P^{\gamma_{N-1}} \cdots P^{\gamma_3} U(t_3, t_2) P^{\gamma_2} U(t_2, t_1) P^{\gamma_1} \psi \tag{3.6}$$

where P^i is the projection operator in $\mathbb{C}^n$ that projects onto the state which is non-zero only on the ith configuration. [Thus ψ_γ is just the configuration γ_N weighted by the amplitude $U_{\gamma_N\gamma_{N-1}} \ldots U_{\gamma_2\gamma_1} \psi(\gamma_1)$.] Restricted evolution of the initial state with respect to an event α is then defined to be the state $\psi_\alpha \in \mathbb{C}^n$

$$\psi_\alpha := \sum_{\gamma\in\alpha} \psi_\gamma. \tag{3.7}$$

Note that $\psi_\gamma = \psi_{\{\gamma\}}$, so we can use either notation when an event is a singleton. It is easy to see that the decoherence functional for two events $\alpha, \beta \in \mathfrak{A}$ is equal to the inner product between the two restricted evolution states, ψ_α and ψ_β:

$$D(\alpha, \beta) := \langle \psi_\alpha, \psi_\beta \rangle. \tag{3.8}$$

3.3 Isomorphism

We now look at conditions on the initial state and evolution of the system that ensure the History Hilbert space $(H_2, \langle\cdot,\cdot\rangle_2)$ is isomorphic to $(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$.

For this system both the sample space and event algebra are finite so the inner product space $(H_1, \langle \cdot, \cdot \rangle_1)$ is finite dimensional and therefore complete but with a degenerate inner product. In this case there is no need to consider Cauchy sequences of elements of H_1 — indeed all Cauchy sequences are eventually constant. Instead, we define the equivalence relation directly on H_1: $u \sim_1 v$ if $||u - v||_1 = 0$. And H_2 is defined as $H_2 := H_1 / \sim_1$ the space of equivalence classes, $[u]_1$ under $\sim_1$. For all $u, v \in H_1$, we have by (3.8)

$$\langle [u]_1, [v]_1 \rangle_2 := \langle u, v \rangle_1. \tag{3.9}$$

It will prove useful to define a map $f_0 : H_1 \to \mathbb{C}^n$ given by

$$f_0(u) := \sum_{\alpha \in \mathfrak{A}} u(\alpha) \psi_\alpha, \tag{3.10}$$

for all $u \in H_1$. This sum is well-defined since $u(\alpha)$ is non-zero for only a finite number of $\alpha \in \mathfrak{A}$. This f_0 is linear and, for all $u, v \in H_1$, we have

$$\langle f_0(u), f_0(v) \rangle = \langle u, v \rangle_1, \tag{3.11}$$

which ensures

$$[u]_1 = [v]_1 \implies f_0(u) = f_0(v). \tag{3.12}$$

Using the map f_0 we define the candidate isomorphism $f : H_2 \to \mathbb{C}^n$ by

$$f([u]_1) := f_0(u), \tag{3.13}$$

for all $[u]_1 \in H_2$. By (3.12), f is well-defined, independent of the equivalence class representative chosen. The map f is linear and (3.9) and (3.11) ensure that for all $[u]_1, [v]_1 \in H_2$, we have:

$$\langle f([u]_1), f([v]_1) \rangle = \langle [u]_1, [v]_1 \rangle_2. \tag{3.14}$$

By Lemma 1, since f is linear and satisfies (3.14), it is one-to-one. If we can find a condition on the initial state and dynamics that ensures the map f is onto then it is the isomorphism we seek.

Theorem 1 (Onto)**.** Let the evolution operators $U(t', t)$ and initial state $\psi \in \mathbb{C}^n$ be such that, for each configuration $j = 1, \ldots, n$ at the final time, there exists a history ending at j, $\gamma^j = (\gamma^j_1, \gamma^j_2, \ldots, \gamma^j_{N-1}, j) \in \Omega$, with non-zero amplitude. In other words, the j-th component of the restricted evolution of the initial state with respect to history γ^j is non-zero: $(\psi_{\gamma^j})_j \neq 0$. Then the map f is onto.

Proof. For each j choose a history $\gamma^j \in \mathfrak{A}$ such that $(\psi_{\gamma^j})_j \neq 0$ (note that ψ_{γ^j} is only non-zero in the j-th component). Let $\phi \in \mathbb{C}^n$ be a vector we wish to map to.

Define $u \in H_1$ by

$$u(x) := \begin{cases} \phi_j / (\psi_{\gamma^j})_j & \text{if } x = \{\gamma^j\} \text{ for } j = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

This is a well-defined vector in H_1 and satisfies $f([u]_1) = \phi$. Hence f is onto. $\square$

An example of a case in which H_2 is not isomorphic to $\mathbb{C}^n$ is if the initial state has support only on a single configuration, k, and the evolution is trivial, $U(t,t')=1$. Then the only configuration at the final time with nonzero amplitude is k and the History Hilbert space is one dimensional, not $\mathbb{C}^n$. Another example is if the evolution is "local" on the lattice, so that after the first time step, only k and $k\pm 1$ say have nonzero amplitude. Then the dimension of the History Hilbert space will depend on the number of time steps and will grow with N until it reaches n after which it will be constant.

4 Particle in d dimensions

We turn now to a less trivial system, that of a non-relativistic particle moving in d dimensions.

4.1 Hilbert space approach

We recall some basic technology in order to fix our notation. The Hilbert space for the system is $(L^2(\mathbb{R}^d), \langle\cdot,\cdot\rangle)$. In order to define this, we first define the inner product space $(\mathcal{L}^2(\mathbb{R}^d), \langle\cdot,\cdot\rangle_0)$, the space of square integrable functions $\psi:\mathbb{R}^d\to\mathbb{C}$. For all $\psi,\phi\in\mathcal{L}^2(\mathbb{R}^d)$ a degenerate inner product is given by

$$\langle\psi,\phi\rangle_0 := \int_{\mathbb{R}^d}\psi^*(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}. \tag{4.1}$$

To see that the inner product is degenerate consider any vector $\psi\in\mathcal{L}^2(\mathbb{R}^d)$ which is non-zero only on a set of measure zero. Although $\psi\neq 0$, we have $||\psi||_0=0$.

For two vectors $\psi,\phi\in\mathcal{L}^2(\mathbb{R}^d)$ define the equivalence relation $\sim$ by

$$\psi\sim\phi \iff ||\psi-\phi||_0=0. \tag{4.2}$$

The $\sim$ equivalence class of $\psi\in\mathcal{L}^2(\mathbb{R}^d)$ will be denoted by $[\psi]$. The set of all equivalence classes forms the Hilbert space $(L^2(\mathbb{R}^d),\langle\cdot,\cdot\rangle)$ where, for all $[\psi],[\phi]\in L^2(\mathbb{R}^d)$, $\langle[\psi],[\phi]\rangle := \langle\psi,\phi\rangle_0$. State vectors for the particle at a fixed time are vectors in $L^2(\mathbb{R}^d)$.

4.2 Quantum Measure System

The sample space of the system, Ω, is the set of all continuous[3] maps $\gamma:[0,T]\to\mathbb{R}^d$. These maps represent the trajectory of the particle from an initial time $t=0$ to a final "truncation time" $t=T$.

[3]We choose continuous maps for definiteness but recognise that the correct sample space may have more refined continuity conditions or even be something more general. The results of our work will remain applicable so long as the actual event algebra contains a subalgebra isomorphic to the $\mathfrak{A}$ we define here and on which the measure is defined by the propagator in the same — standard — way.

Introducing a truncation time T seems necessary for the construction undertaken below, which produces the quantal measure for the corresponding subalgebra $\mathfrak{A}_T \subseteq \mathfrak{A}$. This limitation to a subalgebra of the full event algebra is only apparent, however, because $\mathfrak{A}$ is the union of the $\mathfrak{A}_T$, and the measure of an event $A \in \mathfrak{A}$ does not depend on which subalgebra we refer it to. In section 4.6 we explain this in detail for the case of unitary theories such as we are concerned with in the present paper.

4.2.1 Event algebra

The event algebra $\mathfrak{A}$ we now define is strictly contained in the power set 2^Ω. Let N be any positive integer, $N \geq 2$. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be any N-tuple of real numbers with $0 = t_1 < t_2 < \ldots < t_N = T$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ any N-tuple of subsets of $\mathbb{R}^d$ such that, for each $k = 1, \ldots, N$, either α_k or its complement α_k^c is a bounded Lebesgue measurable set. A subset $\alpha \subseteq \Omega$ is called a *homogeneous event*[4] [17] if there exists an integer N and a pair $(\mathbf{t}, \boldsymbol{\alpha})$ such that

$$\alpha = \{\gamma \in \Omega : \gamma(t_k) \in \alpha_k, k = 1, \ldots, N\}. \tag{4.3}$$

Each α_k can be thought of as a condition on the system, a restriction on the position of the particle, at time t_k. We represent a homogeneous event by the pair $(\mathbf{t}, \boldsymbol{\alpha})$. This representation is non-unique because, for example, the same homogeneous event α is represented by the pairs

$$\mathbf{t} := (t_1, t_2, t_3), \quad \boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3), \tag{4.4}$$

and

$$\mathbf{t}' := (t_1, t_2, t', t_3), \quad \boldsymbol{\alpha}' := (\alpha_1, \alpha_2, \mathbb{R}^d, \alpha_3). \tag{4.5}$$

The event algebra $\mathfrak{A}$ is defined to be the collection of all finite unions of homogeneous events. Any event $\alpha \in \mathfrak{A}$ which is not a homogeneous event will be called *inhomogeneous*.

We can better understand the structure of the event algebra if we consider a few set operations in it. Abusing notation slightly we'll represent a homogeneous event α (with representation $(\boldsymbol{t}, \boldsymbol{\alpha})$) by its ordered collection of sets: $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$. The complement of α is then a finite union of $2^N - 1$ disjoint homogeneous events. For example for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ we have

$$\begin{aligned} \alpha^c = (\alpha_1^c, \alpha_2, \alpha_3) \cup (\alpha_1, \alpha_2^c, \alpha_3) \cup (\alpha_1, \alpha_2, \alpha_3^c) \\ \cup (\alpha_1^c, \alpha_2^c, \alpha_3) \cup (\alpha_1^c, \alpha_2, \alpha_3^c) \cup (\alpha_1, \alpha_2^c, \alpha_3^c) \cup (\alpha_1^c, \alpha_2^c, \alpha_3^c) \end{aligned} \tag{4.6}$$

where ^{c} denotes set-complement.

The intersection of two homogeneous events $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N), \beta = (\beta_1, \beta_2, \ldots, \beta_N)$ (which, by adding extra copies of $\mathbb{R}^d$ as needed, can be assumed to have the same time-sequence $\boldsymbol{t}$) is the homogeneous event $\alpha \cap \beta = (\alpha_1 \cap \beta_1, \alpha_2 \cap \beta_2, \ldots, \alpha_N \cap \beta_N)$.

[4]Alternative names include elementary event, regular event or cylinder set.

These two properties say that the homogeneous events form a *semiring* and ensure that for two homogeneous events α and β the event $\alpha \setminus \beta = \alpha \cap \beta^c$ is a finite union of disjoint homogeneous events. This means that a finite union of homogeneous events can be re-expressed as a finite union of *disjoint* homogeneous events. As an example consider the event $\alpha = \alpha_H^1 \cup \alpha_H^2 \cup \alpha_H^3$ for three homogeneous events α_H^A $(A = 1, 2, 3)$. We can define three disjoint events $\bar{\alpha}^A$ by

$$\bar{\alpha}^1 = \alpha_H^1, \quad \bar{\alpha}^2 := \alpha_H^2 \setminus \alpha_H^1, \quad \bar{\alpha}^3 := (\alpha_H^3 \setminus \alpha_H^1) \cap (\alpha_H^3 \setminus \alpha_H^2). \tag{4.7}$$

Now, from the remarks above,

$$\alpha_H^2 \setminus \alpha_H^1 = \bigcup_{i=1}^{N_1} \beta_1^i, \quad \alpha_H^3 \setminus \alpha_H^1 = \bigcup_{j=1}^{N_2} \beta_2^j, \quad \alpha_H^3 \setminus \alpha_H^2 = \bigcup_{k=1}^{N_3} \beta_3^k, \tag{4.8}$$

where $\beta_1^i, \beta_2^j, \beta_3^k$ are homogeneous events such that $\beta_A^i \cap \beta_A^{i'} = \varnothing$ if $i \neq i'$ (for $A = 1, 2, 3$ and $i, i' = 1, \ldots, N_A$).

We therefore have

$$\begin{aligned} \alpha = \bar{\alpha}^1 \cup \bar{\alpha}^2 \cup \bar{\alpha}^3 &= \alpha_H^1 \cup \left(\bigcup_{i=1}^{N_1} \beta_1^i \right) \cup \left(\left(\bigcup_{j=1}^{N_2} \beta_2^j \right) \cap \left(\bigcup_{k=1}^{N_3} \beta_3^k \right) \right) \\ &= \alpha_H^1 \cup \left(\bigcup_{i=1}^{N_1} \beta_1^i \right) \cup \left(\bigcup_{j=1}^{N_2} \bigcup_{k=1}^{N_3} \beta_2^j \cap \beta_3^k \right) \end{aligned} \tag{4.9}$$

which expresses α as a finite union of *mutually disjoint* homogeneous events—namely α_H^1, β_1^i $(i = 1, \ldots, N_1)$ and $\beta_2^j \cap \beta_3^k$ $(j = 1, \ldots, N_2, k = 1, \ldots, N_3)$. The procedure followed in this example extends without difficulty to $M > 3$ homogeneous events but with an associated proliferation of notation.

(For representing such relationships, the Boolean-algebraic notation can be quite expressive. For example, the essence of (4.7)-(4.9) is the disjoint decomposition, for any three events, $\alpha \cup \beta \cup \gamma = \alpha + (1+\alpha)\beta + (1+\alpha)(1+\beta)\gamma$. Notice here that $1 + \alpha$ is the complement of α, as is clearly visible in the calculation, $\alpha \cap (1 + \alpha) \equiv \alpha(1 + \alpha) = \alpha + \alpha^2 = \alpha + \alpha = 0$.)

The event algebra $\mathfrak{A}$ defined here is an algebra but not a σ-algebra. We allow only a finite number of times when defining a homogeneous event which means $\mathfrak{A}$ is closed under finite unions but not under countable unions. In Section 4.2.2, a decoherence functional will be defined on $\mathfrak{A}$. It is not clear whether this definition can be *extended* to define a decoherence functional on the full σ-algebra (of subsets of Ω) generated by $\mathfrak{A}$. For this to be done it would require a "fundamental theorem of quantum measure theory" analogous to the Carathéodory-Kolmogorov Extension Theorem for classical measures (Theorem A, p. 54 of [11])

4.2.2 Decoherence functional

Let $\psi \in \mathcal{L}^2(\mathbb{R}^d)$ be the normalised initial state, then the decoherence functional for singleton events is given formally by

$$D(\{\gamma\},\{\overline{\gamma}\}) := \psi(\gamma(0))^* e^{-iS[\gamma]}\delta(\gamma(T)-\overline{\gamma}(T))e^{iS[\overline{\gamma}]}\psi(\overline{\gamma}(0))\,. \tag{4.10}$$

By bi-additivity, the decoherence functional for events $\alpha, \beta \in \mathfrak{A}$ is then given by the double path integral:

$$D(\alpha,\beta) := \int_{\gamma\in\alpha}[d\gamma]\int_{\overline{\gamma}\in\beta}[d\overline{\gamma}]D(\{\gamma\},\{\overline{\gamma}\})\,. \tag{4.11}$$

All these formulae are, as yet, only formal. We do not know rigorously what Ω is, whether the singleton subsets of Ω are measureable, or how to define the integration-measure $[d\gamma]$. Indeed, one might anticipate that, as with Wiener measure, neither $e^{iS[\gamma]}$ nor $[d\gamma]$ can be defined separately, and only their combination in (4.11) will exist mathematically.

Nonetheless, we can make sense of the decoherence functional (4.11) on $\mathfrak{A}$ because the form of the events — unions of homogeneous events — allows us to equate the path integrals in (4.11) to well-defined expressions involving the *propagator* à la Feynman. The propagator is a function[5] $K(\mathbf{x}', t'|\mathbf{x}, t)$ that encodes the dynamics of the particle. We assume that the dynamics of the system is unitary. We define the restricted evolution of $\psi \in \mathcal{L}^2(\mathbb{R}^d)$ according to a homogeneous event $\alpha \in \mathfrak{A}$ (with representation $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$) to be ψ_α given by

$$\begin{aligned}\psi_\alpha(\mathbf{x}_T, T) := \chi_{\alpha_N}(\mathbf{x}_T)\int_{\alpha_{N-1}} d\mathbf{x}_{N-1}\int_{\alpha_{N-2}} d\mathbf{x}_{N-2}\cdots\int_{\alpha_2} d\mathbf{x}_2\int_{\alpha_1} d\mathbf{x}_1 \\ K(\mathbf{x}_T, T|\mathbf{x}_{N-1}, t_{N-1})\cdots K(\mathbf{x}_2, t_2|\mathbf{x}_1, 0)\psi(\mathbf{x}_1),\end{aligned} \tag{4.12}$$

where

$$\chi_A(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases} \tag{4.13}$$

is the characteristic function of $A \subset \mathbb{R}^d$.

The convergence of the integrals (and therefore the existence of ψ_α) in (4.12) depends on the propagator for the system and the type of α_k subsets allowed. For the examples we shall consider[6] in Section 4.4 the integrals converge if all the α_k subsets are bounded and, it turns out, in the isomorphism proof in Section 4.3 we will only require such events. In fact we will deal only with two-time homogeneous events with bounded measurable sets at the initial and final times.

Nevertheless we must still define the decoherence functional on the entire event algebra $\mathfrak{A}$ and to do this we must define restricted evolution according to

[5]The propagator may in general be a distribution, as in the case of a simple harmonic oscillator example in Section 4.4.

[6]These examples include the free particle and the simple harmonic oscillator.

a homogeneous event α when some of the α_k subsets are unbounded (which, for the event algebra we are considering, only happens if the α_k are *complements* of bounded measurable sets). In general (and certainly for the examples we shall look at) the propagator is oscillatory in position and if α_k, say, is unbounded the $d\mathbf{x}_k$ integral in (4.12) does not converge absolutely.

We deal with this non-convergence in the standard way (see *e.g.* [2, footnote 13]) by introducing a convergence factor. For each unbounded α_k we replace the non-convergent $d\mathbf{x}_k$ integral

$$\int_{\alpha_k} K(\mathbf{x}_{k+1}, t_{k+1}|\mathbf{x}_k, t_k) K(\mathbf{x}_k, t_k|\mathbf{x}_{k-1}, t_{k-1}) d\mathbf{x}_k, \tag{4.14}$$

in (4.12) by

$$\lim_{\epsilon\to 0^+} \int_{\alpha_k} K(\mathbf{x}_{k+1}, t_{k+1}|\mathbf{x}_k, t_k) K(\mathbf{x}_k, t_k|\mathbf{x}_{k-1}, t_{k-1}) \exp\left(-\epsilon \mathbf{x}_k^2\right) d\mathbf{x}_k. \tag{4.15}$$

For the propagators we consider this integral converges and the $\epsilon \to 0^+$ limit exists. By using these convergence factors we can define ψ_α for all homogeneous events $\alpha \in \mathfrak{A}$.

For the propagators we will consider, the following composition property holds:

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d} K(\mathbf{x}_{k+1}, t_{k+1}|\mathbf{x}_k, t_k) K(\mathbf{x}_k, t_k|\mathbf{x}_{k-1}, t_{k-1}) \exp\left(-\epsilon \mathbf{x}_k^2\right) d\mathbf{x}_{t_k} \tag{4.16}$$

$$= K(\mathbf{x}_{k+1}, t_{k+1}|\mathbf{x}_{k-1}, t_{k-1}).$$

This property is the analogue of the Einstein-Smoluchowski-Chapman-Kolmogorov equation in the theory of Brownian motion. This property is essential if ψ_α is to depend only on the homogeneous event α and not its representation in terms of the pair $(\mathbf{t}, \boldsymbol{\alpha})$ and we assume it holds for all propagators henceforth.

Having defined restricted evolution according to a homogeneous event we now define it for all events in $\mathfrak{A}$. Let α be an event given by

$$\alpha = \bigcup_{k=1}^{M} \alpha_H^k, \tag{4.17}$$

with the α_H^k $(k = 1, \ldots, M)$ a finite collection of mutually disjoint homogeneous events. We define ψ_α as the sum

$$\psi_\alpha := \sum_{k=1}^{M} \psi_{\alpha_H^k}. \tag{4.18}$$

If the propagator satisfies the composition property (4.16) this doesn't depend on the representation of α as a union of homogeneous events.

We can now define the decoherence functional for the system. For two events $\alpha, \beta \in \mathfrak{A}$ and an initial normalised vector $\psi \in \mathcal{L}^2(\mathbb{R}^d)$ we define the decoherence functional $D : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ by

$$D(\alpha, \beta) := \langle \psi_\alpha, \psi_\beta \rangle_0 \,. \tag{4.19}$$

This can be shown to be equal to (4.11) using the familiar expression for the propagator K as a path integral

$$K(\mathbf{x}_2, t_2 | \mathbf{x}_1, t_1) = \int [d\gamma] e^{iS[\gamma]} \tag{4.20}$$

where the integral is over all paths γ which begin at $\mathbf{x}_1$ at t_1 and end at $\mathbf{x}_2$ at t_2.

4.3 Isomorphism

Henceforth we assume the initial state $\psi \in \mathcal{L}^2(\mathbb{R}^d)$ has unit norm, the decoherence functional for events in $\mathfrak{A}$ is given by the propagator K as described in section 4.2.2, the spaces H_1, H_2 are defined as in sections 2.3.1 and 2.3.2, and the maps f_0 and f are defined as in section 4.3. We will find conditions on the initial state and propagator that ensure the History Hilbert space $(H_2, \langle \cdot, \cdot \rangle_2)$ is isomorphic to $(L^2(\mathbb{R}^d), \langle \cdot, \cdot \rangle)$.

It will prove useful to define a map $f_0 : H_1 \to L^2(\mathbb{R}^d)$ given by

$$f_0(u) := \sum_{\alpha \in \mathfrak{A}} u(\alpha) \, [\psi_\alpha] \,, \tag{4.21}$$

for all $u \in H_1$. The sum is well-defined since $u(\alpha)$ is only non-zero for a finite number of events $\alpha \in \mathfrak{A}$. This map f_0 is linear and for all $u, v \in H_1$, we have

$$\langle f_0(u), f_0(v) \rangle = \langle u, v \rangle_1. \tag{4.22}$$

Since the map f_0 is linear and preserves the inner products in H_1 and $L^2(\mathbb{R}^d)$ it maps a Cauchy sequence, $\{u_n\}$ of elements of H_1 to a Cauchy sequence in $L^2(\mathbb{R}^d)$. Since $L^2(\mathbb{R}^d)$ is complete this sequence has a limit and it is this limit we assign as the image of our candidate isomorphism, $f : H_2 \to L^2(\mathbb{R}^d)$ defined by:

$$f([u_n]_1) := \lim_{n \to \infty} f_0(u_n). \tag{4.23}$$

The map f is linear and well-defined, independent of which representative, $\{u_n\}$ of the $[u_n]_1$ equivalence class is used in the definition above.

Using (4.22) and the continuity of the $\langle \cdot, \cdot \rangle$ inner product [20, Lemma 3.2-2] we have

$$\begin{aligned} \langle f([u_n]_1), f([v_n]_1) \rangle &:= \langle \lim_{n \to \infty} f_0(u_n), \lim_{m \to \infty} f_0(v_m) \rangle \\ = \lim_{n \to \infty} \langle f_0(u_n), f_0(v_n) \rangle &= \lim_{n \to \infty} \langle u_n, v_n \rangle_1 =: \langle [u_n]_1, [v_n]_1 \rangle_2. \end{aligned} \tag{4.24}$$

By Lemma 1, since f is linear and satisfies (4.24), it is one-to-one. We can now state our main theorem:

Theorem 2 (Onto)**.** Let the propagator $K(\mathbf{x}_T, T|\mathbf{x}_0, 0)$ be continuous as a function of $(\mathbf{x}_T, \mathbf{x}_0) \in \mathbb{R}^{2d}$ and such that for each $\mathbf{x}_T$, $\exists \mathbf{x}_0$ with $K(\mathbf{x}_T, T|\mathbf{x}_0, 0)$ non-zero. Then the map f defined by (4.23) is onto.

To prove Theorem 2 we follow a strategy suggested by the proof of Theorem 1: we want to show, roughly, that every final position can be reached by a history of nonzero amplitude. The implementation of the strategy is more complicated than in the finite case and will proceed by establishing a series of Lemmas.

Lemma 2. Let the propagator $K(\mathbf{x}_T, T|\mathbf{x}_0, 0)$ be continuous as a function of $(\mathbf{x}_T, \mathbf{x}_0) \in \mathbb{R}^{2d}$. Let $\psi \in \mathcal{L}^2(\mathbb{R}^d)$ be the initial state. Let $A \subset \mathbb{R}^d$ be a compact measurable set and α be the homogeneous event represented by $\mathbf{t} = (0, T), \boldsymbol{\alpha} = (A, \mathbb{R}^d)$. Then

$$\psi_\alpha(\mathbf{x}_T, T) := \int_A K(\mathbf{x}_T, T|\mathbf{x}_0, 0)\psi(\mathbf{x}_0) d\mathbf{x}_0, \tag{4.25}$$

is continuous as a function of $\mathbf{x}_T \in \mathbb{R}^d$.

Proof. Fix a position $\mathbf{x}_T \in \mathbb{R}^d$ at the final time. Let C be the closed unit ball centred at $\mathbf{x}_T$. By assumption, $K(\mathbf{x}_T, T|\mathbf{x}_0, 0)$ is continuous (as a function of $(\mathbf{x}_T, \mathbf{x}_0) \in \mathbb{R}^{2d}$) so, by the Heine-Cantor theorem, it is uniformly continuous (as a function of $(\mathbf{x}_T, \mathbf{x}_0)$) on the compact set $C \times A \subset \mathbb{R}^{2d}$. This means for any $\epsilon > 0$ there exists $\delta > 0$ such that for $(\mathbf{x}_T, \mathbf{x}_0), (\mathbf{x}'_T, \mathbf{x}'_0) \in C \times A$ we have

$$\sqrt{|\mathbf{x}_T - \mathbf{x}'_T|^2 + |\mathbf{x}_0 - \mathbf{x}'_0|^2} < \delta \implies |K(\mathbf{x}_T, T|\mathbf{x}_0, 0) - K(\mathbf{x}'_T, T|\mathbf{x}'_0, 0)| < \epsilon. \tag{4.26}$$

In particular if $\mathbf{x}_0 = \mathbf{x}'_0$ and $|\mathbf{x}_T - \mathbf{x}'_T| < \delta < 1$ then

$$\Big| K(\mathbf{x}_T, T|\mathbf{x}_0, 0) - K(\mathbf{x}'_T, T|\mathbf{x}_0, 0) \Big| < \epsilon \qquad \forall \mathbf{x}_0 \in A\,. \tag{4.27}$$

So for $|\mathbf{x}_T - \mathbf{x}'_T| < \delta < 1$ we have

$$\begin{aligned}
&|\psi_\alpha(\mathbf{x}_T, T) - \psi_\alpha(\mathbf{x}'_T, T)| \\
&\quad := \left| \int_A \Big(K(\mathbf{x}_T, T|\mathbf{x}_0, 0) - K(\mathbf{x}'_T, T|\mathbf{x}_0, 0) \Big) \psi(\mathbf{x}_0) d\mathbf{x}_0 \right| \\
&\quad \leq \left(\int_A \Big| K(\mathbf{x}_T, T|\mathbf{x}_0, 0) - K(\mathbf{x}'_T, T|\mathbf{x}_0, 0) \Big|^2 d\mathbf{x}_0 \right)^{\frac{1}{2}} \left(\int_A |\psi(\mathbf{x}_0)|^2 d\mathbf{x}_0 \right)^{\frac{1}{2}} \\
&\quad < \epsilon |A|
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and the normalisation of ψ and $|A|$ is the Lebesgue measure of A. $|A|$ is finite so, since ϵ is arbitrary, $\psi_\alpha(\mathbf{x}_T, T)$ is continuous at $\mathbf{x}_T$. This holds for any $\mathbf{x}_T \in \mathbb{R}^d$. □

Lemma 3. Let the propagator $K(\mathbf{x}_T, T|\mathbf{x}_0, 0)$ be continuous as a function of $(\mathbf{x}_T, \mathbf{x}_0)$ and be such that for each $\mathbf{x}_T$, $\exists \mathbf{x}_0$ s.t. $K(\mathbf{x}_T, T|\mathbf{x}_0, 0)$ is non-zero. Then for any point $\mathbf{x}_T \in \mathbb{R}^d$ at the truncation time $t = T$ there exists a compact measurable set $A \subset \mathbb{R}^d$ (depending on $\mathbf{x}_T$) such that the homogeneous event α represented by $\mathbf{t} = (0, T), \boldsymbol{\alpha} = (A, \mathbb{R}^d)$ satisfies

$$\psi_\alpha(\mathbf{x}_T, T) := \int_A K(\mathbf{x}_T, T|\mathbf{x}_0, 0)\psi(\mathbf{x}_0)d\mathbf{x}_0 \neq 0. \tag{4.28}$$

Proof. The proof relies on Lebesgue's Differentiation Theorem [19, p100] which states that if $G : \mathbb{R}^d \to \mathbb{C}$ is an integrable function then

$$G(\mathbf{x}) = \lim_{B \to \mathbf{x}} \frac{\int_B G(\mathbf{x}')d\mathbf{x}'}{|B|}, \tag{4.29}$$

for almost all $\mathbf{x} \in \mathbb{R}^d$. Here B is an d-dimensional ball centred on $\mathbf{x}$ which contracts to $\mathbf{x}$ in the limit and $|B|$ is its Lebesgue measure.

Aiming for a contradiction we assume that

$$\int_A K(\mathbf{x}_T, T|\mathbf{x}', 0)\psi(\mathbf{x}')d\mathbf{x}' = 0, \tag{4.30}$$

for all compact measurable sets $A \subset \mathbb{R}^d$.

Taking A to be a sequence of closed balls contracting to an arbitrary point $\mathbf{x} \in \mathbb{R}^d$ at the initial time then (4.29) gives

$$K(\mathbf{x}_T, T|\mathbf{x}, 0)\psi(\mathbf{x}) = \lim_{A \to \mathbf{x}} \frac{\int_A K(\mathbf{x}_T, T|\mathbf{x}', 0)\psi(\mathbf{x}')d\mathbf{x}'}{|A|} = 0, \tag{4.31}$$

for almost all $\mathbf{x} \in \mathbb{R}^d$. This is a contradiction since K is continuous and $\exists \mathbf{x}_0$ with $K(\mathbf{x}_T, T|\mathbf{x}_0, 0) \neq 0$ so there is a compact set containing $\mathbf{x}_0$ on which $K(\mathbf{x}_T, T|\mathbf{x}, 0) \neq 0$.

□

Lemma 4. Let the propagator $K(\mathbf{x}_T, T|\mathbf{x}_0, 0)$ satisfy the conditions of Lemma 3. Then, for any point $\mathbf{x}_T \in \mathbb{R}^d$ at the truncation time there exists a homogeneous event α represented by $\mathbf{t} = (0, T), \boldsymbol{\alpha} = (A, B)$ (with $A \subset \mathbb{R}^d$ a compact measurable set and $B \subset \mathbb{R}^d$ an open ball centred on $\mathbf{x}_T$) and a strictly positive real number P such that ψ_α is uniformly continuous in B and $|\psi_\alpha(\mathbf{x}, T)| > P$ for all $\mathbf{x} \in B$.

Proof. By Lemmas 2 and 3, there exists a compact measurable set $A \subset \mathbb{R}^d$ such that, for the homogeneous event β represented by $\boldsymbol{\beta} = (A, \mathbb{R}^d)$ the function $\psi_\beta(\mathbf{x}, T)$ is continuous for all $\mathbf{x} \in \mathbb{R}^d$ and satisfies $\psi_\beta(\mathbf{x}_T, T) \neq 0$.

This implies there exists $\delta > 0$ such that

$$|\mathbf{x} - \mathbf{x}_T| < \delta \implies |\psi_\beta(\mathbf{x}, T) - \psi_\beta(\mathbf{x}_T, T)| < \frac{|\psi_\beta(\mathbf{x}_T, T)|}{2}. \tag{4.32}$$

Let B be the open ball of radius δ centred on $\mathbf{x}_T$. Setting $P = |\psi_\beta(\mathbf{x}_T, T)|/2 > 0$ we see that $\mathbf{x} \in B$ implies $|\psi_\beta(\mathbf{x}, T)| > P > 0$.

Since $\psi_\beta(\mathbf{x}, T)$ is continuous it is uniformly continuous in any compact set and therefore any subset of a compact set. It is thus uniformly continuous in B. For $\boldsymbol{\alpha} := (A, B)$ we then have $\psi_\alpha = \chi_B \psi_\beta$ (χ_B is the characteristic function of B) and the result follows. □

The next lemma is the heart of the proof.

Lemma 5. Let the propagator $K(\mathbf{x}_T, T|\mathbf{x}_0, 0)$ satisfy the conditions of Lemmas 2 and 3. Let I be a compact d-interval with positive measure $|I| > 0$ at the truncation time. Then for any $\epsilon > 0$ there exists a vector $u \in H_1$ such that

$$||[\chi_I] - f_0(u)|| < \epsilon\,. \tag{4.33}$$

Proof. Let $\epsilon > 0$. For any $\mathbf{x} \in I$ there exists, by Lemma 4, a homogeneous event $\alpha_{\mathbf{x}}$ represented by $\boldsymbol{\alpha}_{\mathbf{x}} = (A_{\mathbf{x}}, B_{\mathbf{x}})$ (with $B_{\mathbf{x}}$ an open ball centred on $\mathbf{x}$) and a real number $P_{\mathbf{x}} > 0$ such that $\psi_{\alpha_{\mathbf{x}}}$ is uniformly continuous in $B_{\mathbf{x}}$ and $|\psi_{\alpha_{\mathbf{x}}}(\mathbf{x}', T)| > P_{\mathbf{x}}$ for all $\mathbf{x}' \in B$.

The collection of $B_{\mathbf{x}}$, taken for all $\mathbf{x} \in I$, form an open cover of I, which, since I is compact, admits a finite subcover labelled by $\{\mathbf{x}_i \in I \,|\, i = 1 \ldots N\}$. Define $A_i := A_{\mathbf{x}_i}$, $B_i := B_{\mathbf{x}_i}$, $\alpha_i := \alpha_{\mathbf{x}_i}$ and $P_i := P_{\mathbf{x}_i}$.

Each B_i will now be "cut up" into finitely many disjoint sets, D_{ij}, over which the ψ_{α_i} functions vary by only "small amounts". The first step toward this is to form a finite number of N mutually disjoint sets $C_i \subseteq B_i$ given by

$$C_1 := B_1 \cap I, \quad C_i := (B_i \cap I) \setminus \bigcup_{j=1}^{i-1} C_j \quad (i = 2, \ldots, N), \tag{4.34}$$

and such that

$$I = \bigcup_{i=1}^{N} C_i. \tag{4.35}$$

Without loss of generality we assume the C_i are non-empty.

Each function ψ_{α_i} is uniformly continuous in C_i and satisfies $|\psi_{\alpha_i}(\mathbf{x}, T)| > P_i$ for all $\mathbf{x} \in C_i$ for some strictly positive $P_i \in \mathbb{R}$.

Let $P > 0$ be the minimum value of the P_i and let $\delta_i > 0$ $(i = 1, \ldots, N)$ be chosen such that

$$|\mathbf{x} - \mathbf{y}| < \delta_i \implies |\psi_{\alpha_i}(\mathbf{x}, T) - \psi_{\alpha_i}(\mathbf{y}, T)| < \frac{\epsilon P}{\sqrt{|I|}}, \tag{4.36}$$

for all $\mathbf{x}, \mathbf{y} \in C_i$.

Letting $\delta > 0$ be the minimum of the δ_i now subdivide each C_i into a finite number, M_i, of non-empty disjoint sets D_{ij} $(i = 1, \ldots, N; j = 1, \ldots, M_i)$ such that

$$\bigcup_{j=1}^{M_i} D_{ij} = C_i \quad \text{and} \quad \mathbf{x}, \mathbf{y} \in D_{ij} \implies |\mathbf{x} - \mathbf{y}| < \delta. \tag{4.37}$$

If we arbitrarily choose points $\mathbf{x}_{ij} \in D_{ij}$ the D_{ij} sets are "small enough" that, by (4.36), $|\psi_{\alpha_i}(\mathbf{x}_{ij}, T) - \psi_{\alpha_i}(\mathbf{x}, T)| < \epsilon P/\sqrt{|I|}$ for all $\mathbf{x} \in D_{ij}$. Defining homogeneous events α_{ij} to be represented by $\boldsymbol{\alpha_{ij}} := (A_i, D_{ij})$ therefore gives

$$\left|1 - \frac{\psi_{\alpha_{ij}}(\mathbf{x}, T)}{\psi_{\alpha_{ij}}(\mathbf{x}_{ij}, T)}\right| = \frac{|\psi_{\alpha_{ij}}(\mathbf{x}_{ij}, T) - \psi_{\alpha_{ij}}(\mathbf{x}, T)|}{|\psi_{\alpha_{ij}}(\mathbf{x}_{ij}, T)|} < \frac{1}{|\psi_{\alpha_{ij}}(\mathbf{x}_{ij}, T)|}\frac{\epsilon P}{\sqrt{|I|}} < \frac{\epsilon}{\sqrt{|I|}}, \tag{4.38}$$

for all $\mathbf{x} \in D_{ij}$ where we note $|\psi_{\alpha_{ij}}(\mathbf{x}_{ij})| > P > 0$.

Now define a H_1 vector by

$$u(x) := \begin{cases} 1/\psi_{\alpha_{ij}}(\mathbf{x}_{ij}, T) & \text{if } x = \alpha_{ij} \text{ for } i = 1, \ldots, N; j = 1, \ldots, M_i \\ 0 & \text{otherwise.} \end{cases} \tag{4.39}$$

This is a well-defined vector in H_1 since there are only a finite number of events $x = \alpha_{ij}$ on which $u(x)$ is non-zero. We now compute

$$||[\chi_I] - f_0(u)||^2 = \int_{\mathbb{R}^d} \left| \chi_I(\mathbf{x}) - \sum_{i=1}^{N} \sum_{j=1}^{M_i} \frac{\psi_{\alpha_{ij}}(\mathbf{x}, T)}{\psi_{\alpha_{ij}}(\mathbf{x}_{ij}, T)} \right|^2 d\mathbf{x} \tag{4.40}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{M_i} \int_{D_{ij}} \left| 1 - \frac{\psi_{\alpha_{ij}}(\mathbf{x}, T)}{\psi_{\alpha_{ij}}(\mathbf{x}_{ij}, T)} \right|^2 d\mathbf{x} < \sum_{i=1}^{N} \sum_{j=1}^{M_i} \int_{D_{ij}} \frac{\epsilon^2}{|I|} d\mathbf{x} = \epsilon^2,$$

where we have used (4.38), the disjointness of the D_{ij} and

$$\sum_{i=1}^{N} \sum_{j=1}^{M_i} \int_{D_{ij}} d\mathbf{x} = |I|. \tag{4.41}$$

We have thus constructed $u \in H_1$ such that

$$||[\chi_I] - f_0(u)|| < \epsilon. \tag{4.42}$$

□

We can now prove Theorem 2.

Proof. (of Theorem 2)

A *step function* on $\mathbb{R}^d$ is a function $S : \mathbb{R}^d \to \mathbb{C}$ that is a finite linear combination of characteristic functions of compact d-intervals.

Let $[\phi] \in L^2(\mathbb{R}^d)$ be the element we wish to map to. We assume $[\phi] \neq 0$ for otherwise the zero vector in H_2 would satisfy $f(0) = [\phi]$. Let $\{[S_n]\}$ be a sequence of $L^2(\mathbb{R}^d)$ vectors, where the S_n are step functions that are not identically zero, such that

$$||[\phi] - [S_n]|| < \frac{1}{2n}, \tag{4.43}$$

for each positive integer n. Such a sequence $\{[S_n]\}$ exists since the step functions are dense in $\mathcal{L}^2(\mathbb{R}^d)$ [19, p133].

For each step function S_n we can (non-uniquely) decompose it as

$$S_n = \sum_{i=1}^{N_n} s_{n,i} \chi_{I_{n,i}}, \tag{4.44}$$

for a finite collection of $N_n \geq 1$ non-zero complex numbers $s_{n,i}$ and mutually disjoint compact d-intervals $I_{n,i}$. Define $M_n > 0$ to be the maximum value of $|s_{n,i}|$ $(i = 1, \ldots, N_n)$.

By Lemma 5, for each $n = 1, 2, \ldots$ and each $i = 1, \ldots, N_n$ there exists a vector $u_{n,i} \in H_1$ such that

$$||[\chi_{I_{n,i}}] - f_0(u_{n,i})|| < \frac{1}{2nN_nM_n}. \tag{4.45}$$

Defining $u_n \in H_1$ by

$$u_n := \sum_{i=1}^{N_n} s_{n,i}u_{n,i}, \tag{4.46}$$

we see that

$$||[S_n] - f_0(u_n)|| \leq \sum_{i=1}^{N_n} |s_{n,i}|||[\chi_{I_{n,i}}] - f_0(u_{n,i})|| < \sum_{i=1}^{N_n} \frac{|s_{n,i}|}{2nN_nM_n} < \sum_{i=1}^{N_n} \frac{1}{2nN_n} = \frac{1}{2n}. \tag{4.47}$$

This, together with (4.43), implies

$$||[\phi] - f_0(u_n)|| < \frac{1}{n}, \tag{4.48}$$

i.e. $f_0(u_n)$ is a Cauchy sequence converging to $[\phi]$. Since f_0 preserves the inner product this means $\{u_n\}$ is a Cauchy sequence of H_1 elements such that $f([u_n]_1) = [\phi]$. $[\phi] \in L^2(\mathbb{R}^d)$ was arbitrary so the map f is onto. □

Theorem 2 gives sufficient conditions on the propagator for the History Hilbert space to be isomorphic to $L^2(\mathbb{R}^d)$ *for any initial state*. If the initial state itself satisfies certain conditions, then the conditions on the propagator can be relaxed. For example, if the initial state, ψ, is everywhere nonzero, then even a trivial evolution with a delta-function propagator will suffice to make the History Hilbert space isomorphic to $L^2(\mathbb{R}^d)$.

4.4 Examples

We now look at examples for which the propagator is known explicitly. The expressions for the propagators are taken from [21].

For a free particle of mass m in d dimensions the Lagrangian is

$$L = \frac{m}{2}\dot{\mathbf{x}}^2. \tag{4.49}$$

The propagator is given by

$$K(\mathbf{x}', t'|\mathbf{x}, t) = \left(\frac{m}{2\pi i\hbar(t'-t)}\right)^{d/2} \exp\left[\frac{im}{2\hbar(t'-t)}(\mathbf{x}'-\mathbf{x})^2\right]. \tag{4.50}$$

For a charged particle (with mass m and charge e) in a constant vector potential $\mathbf{A}$ the Lagrangian is

$$L = \frac{m}{2}\dot{\mathbf{x}}^2 + e\mathbf{A}\cdot\dot{\mathbf{x}}. \tag{4.51}$$

The propagator is given by

$$K(\mathbf{x}', t'|\mathbf{x}, t) = \left(\frac{m}{2\pi i\hbar(t'-t)}\right)^{d/2} \exp\left[\frac{im}{2\hbar(t'-t)}(\mathbf{x}'-\mathbf{x})^2 + \frac{ie\mathbf{A}}{\hbar}\cdot(\mathbf{x}'-\mathbf{x})\right]. \tag{4.52}$$

Both of these propagators satisfy the conditions for Theorem 2. Since the system with constant vector potential is gauge equivalent to the free particle, the theorem is bound to hold for both or neither.

A particle of mass m in a simple harmonic oscillator potential of period $2\pi/\omega$ in one spatial dimension has Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2\,. \tag{4.53}$$

Defining, $\Delta t := t' - t$, the propagator is

$$K(x', t'|x, t) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega\Delta t)}\right)^{1/2} \exp\left[-\frac{m\omega}{2i\hbar}\left[(x'^2 + x^2)\cot(\omega\Delta t) - 2\frac{xx'}{\sin(\omega\Delta t)}\right]\right], \tag{4.54}$$

if $\Delta t \neq M\pi/\omega$ for integer M. If $\Delta t = M\pi/\omega$ for integer M we have

$$K(x', t'|x, t) = e^{(-iM\pi/2)}\delta(x' - (-1)^M x). \tag{4.55}$$

Clearly the propagator fulfils the conditions for Theorem 2 if the truncation time T is not equal to $M\pi/\omega$ for integer M. More care is needed if the truncation time is an integer multiple of π/ω.

If $T = M\pi/\omega$ for integer M then the propagator does not fulfil the conditions for Theorem 2. In this case we cannot use only two-time events to demonstrate the isomorphism. The Hilbert spaces are still isomorphic, however, as can be seen by using three-time homogeneous events α represented by $\boldsymbol{\alpha} = (\mathbb{R}, \alpha_{t_2}, \alpha_T)$ in which the set at time $t_1 = 0$ is $\mathbb{R}$ and such that $T - t_2$ is not an integer multiple of π/ω. Evolving the initial state according to these events is equivalent to unrestrictedly evolving the initial state from $t_1 = 0$ to $t_2 > 0$. The state at time t_2 can then be viewed as the "initial state" for two-time homogeneous events represented by (α_{t_2}, α_T). The conditions for Theorem 2 are met by $K(x_T, T|x_{t_2}, t_2)$ so the theorem can be applied and the isomorphism demonstrated. These ideas can similarly be applied to the simple harmonic oscillator in d dimensions.

4.5 Particle with an infinite potential barrier

Consider a physical system of a non-relativistic particle in one dimension restricted to the positive halfline $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ by an infinite potential barrier.

The Hilbert space for this system is $L^2(\mathbb{R}^+)$ which we define as a vector subspace of $L^2(\mathbb{R})$:

$$L^2(\mathbb{R}^+) := \{[\psi] \in L^2(\mathbb{R}) : \psi(x) = 0 \text{ for } x \leq 0\}. \tag{4.56}$$

The sample space, Ω, and event algebra, $\mathfrak{A}$, for this system will be the same as for a particle in 1 dimension. The difference is that, when defining the decoherence functional we now use an initial vector $\psi \in \mathcal{L}^2(\mathbb{R})$ such that $\psi(x) = 0$ for $x \leq 0$ and a propagator defined by [22, p40]:

$$\begin{aligned} K(x', t'|x, t) = \chi_{\mathbb{R}^+}(x')\chi_{\mathbb{R}^+}(x) \left(\frac{m}{2\pi i\hbar(t'-t)} \right)^{1/2} & \qquad (4.57) \\ \times \left[\exp\left[\frac{im(x'-x)^2}{2\hbar(t'-t)} \right] - \exp\left[\frac{im(x'+x)^2}{2\hbar(t'-t)} \right] \right], & \end{aligned}$$

where m is the mass of the particle.

This propagator does not satisfy the conditions of Theorem 2—it is continuous as a function of $(x, x') \in \mathbb{R}^2$ but is zero for $x \leq 0$ or $x' \leq 0$. It is not surprising therefore that the map $f : H_2 \to L^2(\mathbb{R})$ defined by (4.23) is not an isomorphism with this event algebra and decoherence functional, namely because f only gives vectors $[\psi] \in L^2(\mathbb{R}^+)$ as expected.

It is possible to show, by using the same methods used in the isomorphism proof for a particle in d dimensions, that the History Hilbert space for this event algebra and decoherence functional is isomorphic to $L^2(\mathbb{R}^+)$.

4.6 Infinite times

In the preceding sections we assumed a finite time interval both in the finite configuration space case and the quantum mechanics case. We can extend the analysis to cover all times to the future of the initial time, $t \in [0, \infty)$. We will describe how to do this in the quantum mechanics case; the extension can be applied, *mutatis mutandis*, to the finite configuration space case.

The sample space Ω is now the set of continuous real functions on $[0, \infty)$. The homogeneous events, α, are defined as before as represented by a positive integer $N \geq 1$, an N-tuple of times $\mathbf{t} = (t_1 = 0, t_2, \dots t_N)$ and an N-tuple of measurable subsets of $\mathbb{R}^d$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ such that either α_k or α_k^c is bounded. Now, however, there is no truncation time and therefore no restriction on the times t_k, they can be arbitrarily large. The event algebra, $\mathfrak{A}$, is the set of finite unions of the homogeneous events. Since there is no common truncation time T, the restricted evolution of the initial state with respect to a homogeneous event, as defined by 4.12, results in a state defined at a time, t_N, that depends on the event. Such states cannot be added together to define the restricted evolution of the initial state with respect to a event which is a union of disjoint homogeneous events which have different last times. Instead, we evolve the restricted state back to the initial time $t = 0$, *i.e.* we define

$$\psi_\alpha(\mathbf{x}_0, 0) := \int_{\mathbb{R}^d} d\mathbf{x}_N K(\mathbf{x}_0, 0|\mathbf{x}_N, t_N)\psi_\alpha(\mathbf{x}_N, t_N) \tag{4.58}$$

for each homogeneous event α.

We can now work at the initial time. The restricted state at $t = 0$ of an inhomogeneous event is the sum of the restricted states at $t = 0$ of its constituent disjoint homogeneous events (as in (4.18)). The decoherence functional is defined as the inner product of the restricted states at $t = 0$.

The event algebra, $\mathfrak{A}_\infty$, in the infinite time case contains a subalgebra, $\mathfrak{A}_\infty|_T$, which is canonically isomorphic to the event algebra, $\mathfrak{A}_T$ with a truncation time because each history with a truncation time corresponds to an event in the infinite time case: the event is the set of infinite time histories which match the truncated history. The decoherence functionals on $\mathfrak{A}_\infty|_T$ and $\mathfrak{A}_T$ agree because the unitary evolution back to the initial time preserves the inner product. Theorem 2 therefore also applies to the semi-infinite time case: if the History Hilbert space is isomorphic to $L^2(\mathbb{R}^d)$ with a truncation time, it is isomorphic without. In the former case it is convenient to construct the History Hilbert space at the truncation time as we did and in the latter it is convenient to consider the History Hilbert space associated with the initial time, but for the unitary systems we are considering this is not a real distinction, being akin to working in the Schrödinger or Heisenberg Picture.

4.7 Mixed states

The conjecture made at the end of section 2 was that where both standard and History Hilbert spaces exist, generically they are isomorphic if the decoherence functional encodes a pure initial state. If in contrast the initial state is a statistical mixture then the decoherence functional is a convex combination of decoherence functionals, and then the History Hilbert space can be bigger than the standard Hilbert space. Indeed, we make note of the following expectations for the case of a finite configuration space. If the initial state is a density matrix of rank r_i then the History Hilbert space is generically the direct sum of r_i copies of the standard Hilbert space $\mathbb{C}^n$. Even more generally, if there is also a final density matrix of rank r_f then the History Hilbert space is the direct sum of r_i copies of $\mathbb{C}^{r_f}$. [28]

5 Discussion

If histories-based formulations of quantum mechanics are nearer to the truth than state- and operator-based formulations, and in particular if something like Quantum Measure Theory is the right framework for a theory of quantum gravity, then there is no particular reason why one should expect Hilbert spaces to be part of physics at a fundamental level.

Indeed, in histories formulations which assume only plain, "weak" positivity (and in which, therefore, no Hilbert space arises), certain kinds of devices can in principle exist that are not possible within ordinary quantum mechanics, and this could be regarded as desirable. For example non-signaling correlations of the "PR box" type become possible [23].

Nor does reference to a Hilbert space seem to be needed for interpretive reasons. On the contrary, attempts to overcome the "operationalist" bias of the so called Copenhagen interpretation tend to lead in the opposite direction, away from state-vectors and toward histories and the associated events [7].

Thus, it seems hard to argue on principle that a Hilbert space is needed. On the other hand, there do exist good reasons to regard strong positivity as more natural than weak positivity. First, it is mathematically much simpler than weak positivity, whence more amenable to being verified and worked with [10]. (Not that its definition is any simpler, but that it comprises, apparently, far fewer independent conditions.) Second, strong positivity is preserved under composition of subsystems, whereas the obvious "product measure" of two weakly positive quantal measures is not in general positive at all. And third — at a technical level — the histories hilbert space to which strong positivity leads has already proven to be useful in certain applications [23, 25], while there are also indications that the map taking events $\alpha \in \mathfrak{A}$ to vectors in H could be of aid in the effort to extend the decoherence functional from $\mathfrak{A}$ to a larger fragment of the σ-algebra it generates.

It thus seems appropriate to add strong positivity to the axioms defining a decoherence functional (as we have done in this paper), and from a strongly positive decoherence functional a histories hilbert space H automatically arises. Once we have it, we can ask whether the histories hilbert space helps us to make contact with the quantal formalism of standard textbooks. This is something that any proposed formulation has to be able to do, and it is the principal question animating the present paper. The positive answer we have obtained is that for the systems we have studied, the histories hilbert spaces that pertain to them can be directly identified with the corresponding state-spaces of the ordinary quantum description. (The two are "naturally isomorphic".) Thereby an important part of the mathematical apparatus of ordinary quantum mechanics is recovered quite simply. This result can be seen as an advance for both Generalised Quantum Mechanics and Quantum Measure Theory because the basic underlying structures — histories and decoherence functionals — are common to both approaches. (Strong positivity has not normally been assumed in Generalised Quantum Mechanics, but there is no reason why it could not be.)

Beyond state-vectors, the other main ingredients of the standard quantum machinery are the operators representing position, momentum, field values, "observables", and the like. How might they be derived from histories? In the specialized context of unitary, Hamiltonian evolution and the Schrödinger equation, time-ordered operators can be obtained from functions ("functionals") on the sample space Ω (see [2]), but whether such a relationship exists in the same generality as the histories hilbert space itself (that is for any quantum measure theory) remains to be seen. An interesting generalization where one does seem able to recover field operators from the decoherence functional is that of quantum field theory on a causal set [26].

The context of this paper has been that of non-relativistic quantum mechanics, yet people have not yet completely laid the rigorous mathematical foundations of a histories framework for this theory. Nevertheless the decoherence functional limited to $\mathfrak{A} \times \mathfrak{A}$ is known (we haven't yet defined calculus but we can calculate the volume of a pyramid, see footnote 10, page 371 of [2]), and this sufficed to demonstrate our main result, that the History Hilbert space H is the standard Hilbert space. A key question for the future that will also be of interpretational significance is what the sample space of histories is. Is it the set of all continuous trajectories and if so, exactly how continuous are they? This is closely related to the question, can the decoherence functional — and hence the quantal measure — be extended to a larger collection of sets than $\mathfrak{A}$? Is that larger collection the whole σ-algebra generated by $\mathfrak{A}$ or something smaller? These questions have been explored by Geroch [27]. To the extent that they find satisfactory answers, we will be able to say that Quantum Mechanics as Quantum Measure Theory is as well-defined mathematically as the Wiener process.

Be that as it may, neither Brownian motion nor the quantum mechanics of nonrelativistic point-particles can lay claim to fundamental status in present-day physics. Relativistic quantum field theory comes closer, but in that context, neither formulation — neither path-integrals/histories nor state-vectors-cum-operators — enjoys a mathematically rigorous existence. Instead we have the divergences and other pathologies whose resolution is commonly anticipated from the side of quantum gravity. If this expectation is borne out, the decoherence functional of quantum gravity might actually be easier to place on a sound mathematical footing than that of the Hydrogen atom, because in place of a path-integral over an infinite dimensional function-space, we will have something more finitary in nature, like a summation over a discrete space of histories.[7]

In that case the trek back to nonrelativistic quantum mechanics will be longer, but we expect that the histories hilbert space defined above will still be an important milestone along the way.

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[7] One can already observe such a trend in the theory formulated in [26] of a free scalar field on a causal set C corresponding to a bounded spacetime region. The decoherence functional of the theory can be computed and is again given by a double integral of the type of (4.11). Now however, the domain of integration is just $\mathbb{R}^n$ rather than some infinite-dimensional path-space. Moreover, the integrand contains, besides the expected oscillating phases, damping terms that lessen by half the need for integrating-factors like those in (4.15).

References

[1] P.A.M. Dirac, *The Lagrangian in Quantum Mechanics*, Phys. Z. Sowjetunion, (1933)

[2] R. P. Feynman, *Space-Time Approach to Non-Relativistic Quantum Mechanics*, Rev. Mod. Phys, 20, 367 (1948)

[3] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw Hill 1965.

[4] J. B. Hartle, *The quantum mechanics of cosmology* Dec 1989, Lectures at Winter School on Quantum Cosmology and Baby Universes, Jerusalem, Israel, Dec 27, 1989 - Jan 4, 1990.

[5] J. B. Hartle, *Spacetime quantum mechanics and the quantum mechanics of spacetime* in Proceedings of the Les Houches Summer School on Gravitation and Quantizations, Les Houches, France, 6 Jul - 1 Aug 1992, eds J. Zinn-Justin, and B. Julia, North-Holland (1995) arXiv:gr-qc/9304006

[6] R.D. Sorkin, *Quantum Mechanics as Quantum Measure Theory*, Mod.Phys.Lett. A9 (1994) 3119-3128, arXiv:gr-qc/9401003

[7] R.D. Sorkin *Quantum Dynamics without the Wave Function*, J. Phys. A: Math. Theor. 40 (2007) 3207-3221, arXiv:quant-ph/0610204v1 http://www.perimeterinstitute.ca/personal/rsorkin/some.papers/

[8] R.D. Sorkin, *Quantum Measure Theory and Its Interpretation*, in Quantum Classical Correspondence: Proceedings of the 4th Drexel Symposium on Quantum Nonintegrability, held Philadelphia, September 8-11, 1994, pages 229-251 (International Press, Cambridge Mass. 1997) D.H. Feng and B-L Hu, editors, arXiv:gr-qc/9507057

[9] R.D. Sorkin, *An exercise in "anhomomorphic logic"*, *Journal of Physics: Conference Series (JPCS)* **67:** 012018 (2007), a special volume edited by L. Diosi, H-T Elze, and G. Vitiello, and devoted to the Proceedings of the DICE-2006 meeting, held September 2006, in Piombino, Italia. arXiv:quant-ph/0703276 http://www.perimeterinstitute.ca/personal/rsorkin/some.papers/

[10] X. Martin, D. O'Connor and R.D. Sorkin *Random walk in generalized quantum theory* Phys. Rev. D 71, 024029 (2005), arXiv:gr-qc/0403085

[11] P.R. Halmos, *Measure Theory*, Van Nostrand, 1964.

[12] V.P. Belavkin, *Reconstruction theorem for quantum stochastic processes*, Translated from: Teoreticheskaya i Matematicheskaya Fizika, 62 (3) 409–431 (1985), arXiv:math/0512410

[13] D. E. Dutkay, *Positive definite maps, representations and frames*, Reviews in Mathematical Physics, vol. 16, No. 4 (2004), arXiv:math/0511137

[14] D.E. Evans and J.T. Lewis, *Dilations of Irreversible Evolutions in Algebraic Quantum Theory*, Dublin Institute for Advanced Studies 1977.

[15] R. Geroch, *Special Topics in Particle Physics*, Unpublished

[16] K.G. Binmore, *The Foundations of Analysis: Book 2, Topological Ideas*, Cambridge University Press, 1981.

[17] C.J. Isham, *Quantum Logic and the Histories Approach to Quantum Theory*, J.Math.Phys. 35 (1994) 2157-2185, arXiv:gr-qc/9308006.

[18] M. Chaichian, A. Demichev, *Path integrals in physics. Vol.1, Stochastic processes and quantum mechanics* Institute of Physics, Bristol. 2001.

[19] R.L. Wheeden, A. Zygmund, *Measure and Integral: An Introduction to Real Analysis*, Marcel Dekker, Inc. 1977.

[20] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, 1978.

[21] C. Grosche, F. Steiner, *Handbook of Feynman Path Integrals*, Springer, 1998

[22] L.S. Schulman, *Techniques and Applications of Path Integration*, Wiley-Interscience publication, 1981.

[23] David Craig, Fay Dowker, Joe Henson, Seth Major, David Rideout and Rafael D. Sorkin, "A Bell Inequality Analog in Quantum Measure Theory", *J. Phys. A: Math. Theor.* **40:** 501-523 (2007), `quant-ph/0605008`, `http://www.perimeterinstitute.ca/personal/rsorkin/some.papers/`

[24] Rafael D. Sorkin, "Quantum dynamics without the wave function" *J. Phys. A: Math. Theor.* **40:** 3207-3221 (2007) (http://stacks.iop.org/1751-8121/40/3207) `quant-ph/0610204` `http://www.perimeterinstitute.ca/personal/rsorkin/some.papers/`

[25] Sumati Surya and Petros Wallden, "Quantum Covers in Quantum Measure Theory" `arXiv:0809.1951`

[26] Steven Johnston, "Feynman Propagator for a Free Scalar Field on a Causal Set" *Phys. Rev. Lett* **103:** 180401 (2009) `arXiv:0909.0944 [hep-th]`

[27] R. Geroch, *Path Integrals*, Unpublished notes available at http://physics.syr.edu/ sorkin/lecture.notes/

[28] Marie Ericsson and David Poulin (unpublished)