Discreteness without symmetry breaking: a theorem

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Abstract

This paper concerns random sprinklings of points into Minkowski spacetime (Poisson processes). It proves that there exists no equivariant measurable map from sprinklings to spacetime directions (even locally). Therefore, if a discrete structure is associated to a sprinkling in an intrinsic manner, then the structure will not pick out a preferred frame, locally or globally. This implies that the discreteness of a sprinkled causal set will not give rise to "Lorentz breaking" effects like modified dispersion relations. Another consequence is that there is no way to associate a finite-valency graph to a sprinkling consistently with Lorentz invariance.

1 Introduction

Most approaches to quantum gravity aspire to replace the continuous spacetime of general relativity by something more fundamental, often something discrete. The continuum can survive in such an approach only as an "emergent" description of the more fundamental structure, and so the question arises of how to set up the correspondence between the discrete structure and the approximating spacetime (or space, in the case of canonical quantum gravity). Appealing to the ideas of statistical geometry is one possibility for doing this. Indeed, this method has for some time been central to the causal set program [1, 2, 3] and has more recently been applied to loop quantum gravity as well [4]. In the case of causal sets, a statistical correspondence is the only kind that has ever been found to be consistent with the basic postulate that spacetime volume reflects number of causet elements. It is then an inevitable (and presumably welcome) consequence that Lorentz invariance is recovered in the effective continuum description. (In this sense, causal set theory *must* respect Lorentz invariance.) The case of loop quantum gravity is similar even though one is dealing with space rather than spacetime. Because of the positive signature of the metric, a "regular lattice" (*e.g.*, cubic) can now adequately reproduce volumes of regions with slowly varying boundary surfaces. However, if one wishes to recover surface area too, then — once again — a statistical correspondence is the only known possibility. (In this sense, loop quantum gravity must respect rotation invariance.)

Let us assume that our discrete structure is a set of elements (or "vertices") endowed with some extra information (an order relation, a labelling by spins, *etc.*). Let there also be a method of inducing one of these discrete structures on each locally finite subset of spacetime (or space), *i.e.*, a map C from the space of such sets of points of the continuum to the space of discrete structures. If the fundamental structure were a causal set (a locally finite partial order), this map would associate to each set of points drawn from a

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Lorentzian manifold the order induced on them by the causal order of the manifold.[1] If the structure were a graph playing the role of a spatial configuration, the map would associate to each set of points drawn from a Riemannian manifold a set of vertices, identified with the points themselves or an appropriately defined dual set, and a set of edges, obtained for example as a result of applying the Delaunay or Voronoi construction, respectively.[2]

A discrete/continuum correspondence is needed to let us connect properties of the new fundamental structure with known physics. Given a discrete structure $\mathcal{D}$ and a candidate approximating manifold $\mathcal{M}$, the methods of statistical geometry establish criteria for answering the question "Is $\mathcal{M}$ a good approximation to $\mathcal{D}$?". (We will write $\mathcal{M} \approx \mathcal{D}$ for this.) In any manifold possessing a volume measure, we can choose a locally finite, uniformly random set of points by the *Poisson process* in which the probability of finding n points in a region of volume V is

$$P(n) = \frac{(\rho V)^n \, \mathrm{e}^{-\rho V}}{n!} \; , \tag{1}$$

where ρ is some fundamental density; we call such a set of points a "sprinkling". Let us now induce our discrete structure on the sprinkled points, and notice that the map from sets of points in $\mathcal{M}$ to discrete structures is in general many-to-one. We then want to say that $\mathcal{M} \approx \mathcal{D}$ iff $\mathcal{D}$ arises "with high probability" from a sprinkling of $\mathcal{M}$. Conversely, if we begin with $\mathcal{M}$ and derive a $\mathcal{D}$ from it by sprinkling, then we will almost always obtain a $\mathcal{D}$ to which $\mathcal{M}$ is a good approximation.[3] As we will see, this way of setting the discrete/continuum correspondence helps to preserve the well-observed symmetries of the continuum. (And, as remarked above, it is the only known way, consistent with the indefinite signature of the spacetime metric, to implement the assumption that number equals volume.)

If the set of all discrete structures $\mathcal{D}$ is to be a "history-space", rather than a space of possible spatial configurations, it is reasonable to require that at least some of the $\mathcal{D}$ should admit Minkowski space as a good approximation (at some sufficiently large scale, in the same way as some paths of the non-relativistic path-integral track classical trajectories at large scales, even though they are fractal on short scales — such paths dominate the integral in semi-classical situations). It is therefore interesting to ask how well the symmetries of Minkowski space can be preserved in the emergent continuum.

In what sense can we expect our discrete structure to be Lorentz invariant — what sense of the term is physically relevant here? An answer was proposed in reference [5]: that the discreteness must not, in and of itself, serve to pick out a preferred frame (or frames) of reference. This requirement is more general than asking that any characteristic scale of a quantum theory in which states are based on discrete structures be invariant under Lorentz tranformations [6, 7], and it is plausibly the criterion most relevant for phenomenology.[4] In reference [5] we presented strong evidence that causets produced by sprinkling into Minkowski spacetime meet this criterion, but a skeptic could still have found grounds for doubt. In this paper, we prove a theorem that we believe removes most of the remaining doubt.

[1] Although this is the most obvious possibility for C, one has also considered generalizations in which the map C itself contains a random element. We would not expect such a generalization to alter the main conclusions of this paper.

[2] Such constructions cover most of the approaches to quantum gravity that are popular today, but they do not represent the most general case. For example, a covering of a manifold by open sets would not give rise to a discrete structure whose vertices corresponded to specific points in the manifold.

[3] The underlying idea is that $\mathcal{M} \approx \mathcal{D}$ iff $\mathcal{D}$ is a "typical result" of sprinkling $\mathcal{M}$. When $\mathcal{M}$ is compact (with boundary) one can try to make the idea of typicality precise by identifying it with the quoted requirement of "high probability". For $\mathcal{M}$ of infinite volume (*e.g.*, $\mathcal{M} = \mathbb{M}^n$) this idea evidently won't work as such; for the probability to obtain any given $\mathcal{D}$ is clearly zero. Nevertheless, any attribute of $\mathcal{D}$ that is present with probability 1 (relative to the Poisson process in $\mathcal{M}$) can be taken to obtain whenever $\mathcal{M} \approx \mathcal{D}$.

[4] We do not mean to imply that it is the only criterion of interest. For example, translation invariance in this sense does not, in and of itself, imply that some notion of momentum will be defined and obey a conservation law.

The fact that the process of "causet sprinkling" in Minkowski space is Lorentz invariant is an important first step in the argument. (In this process we include both the Poisson sprinkling as such and the subsequent induction of the causal order. Both steps are manifestly Lorentz invariant since they depend only on the volume element and the causal structure of the spacetime, respectively). But Lorentz invariance of the resulting causal set in the above sense does not immediately follow. Consider by analogy a game of fortune in which a circular wheel is spun to a random orientation. While the *distribution* of final directions is indeed rotationally invariant, a *particular outcome* of the process is certainly not. (A form of "spontaneous symmetry breaking", perhaps). Likewise, a particular outcome of the Poisson process might be able to prefer a frame, even though the process itself does not.

So, the question becomes: Is it possible to use a sprinkling of Minkowski space to select a preferred frame? We will prove a theorem that answers "no" to this question. In fact, it answers the slightly more general question whether a sprinkling can pick out a preferred time-direction (which is certainly possible if an entire frame can be derived.) Below, we formalise the notion of deriving a direction from a sprinkling, and we prove a theorem showing that this cannot be done. In this sense, the situation with sprinklings of Minkowski space is even more comfortable than that with sprinklings of Euclidean space. It is possible to associate a direction from the rotation group to a point in such a sprinkling, as discussed later (although this will not stop anyone from maintaining that a gas behaves isotropically in the continuum approximation; these locally defined directions have little significance at that level), but the non-compactness of the Lorentz group makes the Lorentzian case different.

Based on the theorem, we can assert the following. Not only is the Poisson process in Minkowski space Lorentz invariant, but the *individual realizations* of the process are also Lorentz invariant in a definite and physically important sense.

Another question one might ask is: how easy is it to come up with maps taking sprinklings to discrete structures in general? Is this feature of causal sets unusual, or is it generic? Our theorem is relevant to this question as well. As a corollary, it implies that no finite valency graph can be associated to a sprinkling of Minkowski space consistently with Lorentz invariance. This rules out the use of the sprinkling technique to find Lorentz invariant spin-foams or relativistic spin-lattices. Precisely this problem was encountered by T.D. Lee's "random lattices" in Minkowski space (see, for example, reference [8]).

2 A theorem

Let $\mathbb{M}^n$ be n-dimensional Minkowski space, and let $\mathcal{L}_0$ be the connected component of the identity in $\mathrm{O}(n-1,1)$, the full Lorentz group of $\mathbb{M}^n$. We will call $\mathcal{L}_0$ simply "the Lorentz group". Fix a point O ("the origin") in $\mathbb{M}^n$ and let $\mathcal{L}_0$ act on $\mathbb{M}^n$ with O as its fixed point.

An individual realization of a Poisson process in $\mathbb{M}^n$ is almost surely a locally finite subset of $\mathbb{M}^n$ (*i.e.*, a collection of point-events of $\mathbb{M}^n$ with no accumulation point anywhere). The space of all such subsets or "possible sprinklings", we will denote by Ω. A Poisson process is captured mathematically by a probability measure μ on Ω. Formally, it is the stochastic process defined by the triple (Ω, Σ, μ), where $\mu : \Sigma \to \mathbb{R}$ and Σ is the σ-algebra of all measurable subsets of Ω, as defined, for example, in reference [9].

Clearly the action of $\mathcal{L}_0$ on spacetime points induces an action on collections of spacetime points. In particular it induces an action on Ω under which Σ is left invariant. It is known [9] that a Poisson process in $\mathbb{M}^n$ is invariant against any volume-preserving, linear transformation of $\mathbb{M}^n$, the ultimate reason being that equation (1) only refers to volumes. (So in fact the hypothesis of linearity can be dropped.) Consequently it is invariant under the action of $\mathcal{L}_0$; that is, the probability of a (measurable) set of possible sprinklings is the same as that of the set obtained applying a Lorentz transformation to it:

$$\mu = \mu \circ \Lambda \, , \quad \forall \Lambda \in \mathcal{L}_0 \; . \tag{2}$$

Since we want to prove a local theorem and not just a global one, let us consider the existence of a preferred direction relative to a selected point of $\mathbb{M}^n$, which we can take to be the origin O. (In cases of interest, O will actually be a point of the sprinkling, but we don't need to assume that for purposes of the proof.) The assertion that every sprinkling determines a preferred direction at O states at a formal level that there exists a map D from Ω to $\mathcal{H}$, the hyperboloid of unit future-timelike vectors. This map D is the "rule" by which each individual set ω of sprinkled points gives rise to the direction $D(\omega)$ at O. Not every function $D : \Omega \to \mathcal{H}$ is a valid candidate, however, because we want the direction chosen by D "to have come from the sprinkling and nothing else". As an example of what we *don't* want, consider the map D_X that takes every sprinkling ω to the same vector X. The "distinguished direction" defined by this map (namely X) was clearly put in by hand; it has nothing to do with the sprinkling from which it supposedly came. In order to eliminate such specious rules, we should require *equivariance* with respect to $\mathcal{L}_0$, *i.e.*, we should require that a Lorentz transformed sprinkling $\Lambda\,\omega$ give rise to the correspondingly transformed direction $\Lambda\,X$. In other words, we should require that the process D of deducing a direction from a sprinkling commute with Lorentz transformations (which is a special case of the more general requirement, which we could impose with equal justice, that D commute with all of $\mathrm{Aut}\,\mathbb{M}^n$). The equivariance of D can be expressed by a commutative diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{\Lambda} & \Omega \\ \big\downarrow{\scriptstyle D} & & \big\downarrow{\scriptstyle D} \\ \mathcal{H} & \xrightarrow{\Lambda} & \mathcal{H} \end{array}$$

Theorem 1 *In dimensions $n > 1$ there exists no measurable equivariant map $D : \Omega \to \mathcal{H}$, i.e., there exists no measurable D such that*

$$D \circ \Lambda = \Lambda \circ D \, , \quad \forall \Lambda \in \mathcal{L}_0 \; . \tag{3}$$

Proof: Suppose that such a map D exists. Its inverse D^{-1} yields a well-defined map from subsets of $\mathcal{H}$ to subsets of Ω, and this in turn lets us define a probability distribution μ_D on $\mathcal{H}$, as follows. Since D is measurable, it follows by definition that the inverse image of each measurable subset of $\mathcal{H}$ is measurable in Ω, and we set

$$\mu_D := \mu \circ D^{-1}. \tag{4}$$

Equation (3) then implies that $\Lambda \circ D^{-1} = D^{-1} \circ \Lambda$, and using equations (4), (2), and (3), respectively, we can see that

$$\begin{aligned} \mu_D &= \mu \circ D^{-1} = \mu \circ \Lambda \circ D^{-1} \\ &= \mu \circ D^{-1} \circ \Lambda = \mu_D \circ \Lambda \, , \qquad \forall \Lambda \in \mathcal{L}_0 \; . \end{aligned} \tag{5}$$

This means that μ_D is a probability measure on the unit hyperboloid $\mathcal{H}$ that is invariant under the action of $\mathcal{L}_0$. But the non-compactness of the hyperboloid, related to that of the Lorentz group, forbids this. To see why, consider an open set $U \subset \mathcal{H}$ of compact closure; such a set is measurable and has a finite measure $\mu_D(U)$. We can assume without loss of generality that $\mu_D(U) > 0$. Now apply a boost Λ such that U and its images $U_n := \Lambda^n\,U$ are all disjoint. By Lorentz invariance, $\mu_D(U_n) = \mu_D(U)$, and therefore by the additivity of the measure, $\mu_D(\cup_{i=1}^n U_i) = n\,\mu_D(U)$ for any n. But this is impossible if μ is a probability measure, because for n sufficiently large, $n\,\mu_D(U)$ would exceed any pre-assigned value, which is absurd since probabilities cannot exceed unity. Therefore no measurable D obeying equation (3) can exist.[5] $\square$

[5] Notice that the measurability requirement on D is extremely weak; we know of no use of non-measurable maps in physics, and in fact no such map can be explicitly specified.

The theorem extends easily to other, similar statements about the impossibility of sprinklings in Minkowski space breaking Lorentz invariance:

(1) No *partially defined* equivariant measurable map from sprinklings to directions can exist if its domain has nonzero measure. (This rules out the possibility that a nontrivial subset of the sprinklings could break Lorentz invariance, even if not all of them could.)

(2) A sprinkling cannot determine a timelike direction globally, in Minkowski space as a whole (or in any Lorentz-invariant subset of Minkowski space). Otherwise that direction could be defined as the preferred one at each point, contradicting the result just proved.

(3) A sprinkling cannot determine a preferred *spacelike* direction at any point of $\mathbb{M}^n$. The proof proceeds like the one for the timelike case, with $\mathcal{H}$ replaced by the hyperboloid of unit spacelike vectors at the origin.

(4) No finite set of timelike and/or spacelike directions at a point (this includes a reference frame) can be associated to a sprinkling consistently with Lorentz invariance. In this case, the map D would go from Ω to a product of various copies of the two hyperboloids, possibly quotiented by the action of a permutation group.

(5) A sprinkling cannot determine a preferred *location* in Minkowski spacetime. In other words, a Poisson-sprinkled set of points is as homogeneous as it is isotropic. (Thus, our results hold for the Poincaré group as well as the Lorentz group.)

In each of these cases, there is no probability-measure on the corresponding space that is invariant under the Lorentz (or Poincaré) group. However, this is not true for infinite subsets of $\mathcal{H}$, and a countably infinite set of directions *can* be equivariantly associated to a sprinkling. An example is the set of directions from the origin to all the sprinkled points in its future. This is why the theorem does not exclude the possibility of consistently associating a causal set to a sprinkling.

If a finite set of directions cannot be associated to a point in a sprinkling of Minkowski space, consistently with Lorentz invariance, it is clear that any method used to associate a finite-valency graph to such a sprinkling would have to violate Lorentz invariance, regardless of any other properties the discrete structure might have (such as labels on the edges or vertices, or the presence of higher-dimensional cells). Some preferred direction would have to be introduced before one could fill in edges between the points of the sprinkling.

2.1 An example

The theorem is rather abstract and might conflict with the intuition of people who are accustomed to working with Lorentz-violating discretisations. As an aid to intuition, we give an example in which the construction of a direction-map D fails.

In sprinklings of flat Euclidean space $\mathbb{E}^n$, it *is* possible to associate a direction to a point in an equivariant (rotationally covariant) way. An example is the map from a point in $\mathbb{E}^n$ (possibly one belonging to the sprinkling ω) to the direction toward the nearest sprinkled point. This map *is* equivariant, since rotating ω around our chosen point and then finding the direction, gives the same result as finding the direction and then rotating it.[6]

So, why does the analogous construction fail in the Lorentzian case? The answer is that there is no nearest neighbour to a given point in a sprinkling of $\mathbb{M}^4$. To put it another way, there is no lower bound on the distance from the origin to a sprinkled point. Consider a point sprinkled at Lorentzian distance d from the origin. The region within distance d of the origin is of infinite volume, extending all the way up the light-cone, and we see from equation (1) that there are one or more points sprinkled into this volume with

[6]Whether a direction can be associated to the sprinkling as a whole is not clear to us, although one might expect that the answer is no.

probability 1. Therefore, no matter how close a sprinkled point is to the origin, there is always another point sprinkled closer.

3 Conclusion

As an example of how the theorem proved above can carry phenomenological implications, consider a model of scalar field propagation in which we discretise Minkowski space by replacing it with a sprinkled causal set, as in reference [10]. The theorem shows that no special frame or direction u^a can be picked out with respect to which one could introduce Lorentz violating effects, even a u^a that varies stochastically with position.

The theorem also shows that a finite-valency graph like a "spin-foam" cannot be equivariantly associated to a sprinkling of Minkowski space. If we wanted our discrete structure to involve such a graph, we could attempt to use some other continuum approximation scheme than sprinkling. The problem is that no other way is known that preserves Lorentz symmetry, when Minkowski space is to be the effective continuum description. Rather, the two requirements of Lorentz invariance and discreteness seem to lead to an unavoidable randomness already at the kinematic level: a randomness in the discrete/continuum correspondence. The causal set is a simple example of a structure amenable to this sort of correspondence, and in fact the only example so far proposed in the literature. Others could be imagined (*e.g.*, adding distance information to the relations in the causal set), but we would conjecture that the causal set is, in some sense, the minimal Lorentz invariant discrete structure from which the continuum can be reconstructed at macroscopic scales. Other such structures might then be expected to over-specify the information needed to reconstruct the continuum.

Strictly speaking, our results apply only to full $\mathbb{M}^4$, whereas we know that in reality the Minkowskian portions of spacetime are of limited extent. What of other Lorentzian manifolds? For example, what about a large but finite region of $\mathbb{M}^4$? In this case the boundary of the region would contain directional information that in principle could be used to introduce graph-like structures, even in the limit of infinite sprinkling density. Indeed, we would guess that the boundary would always enter essentially into any such scheme, rendering any resulting phenomenological theory radically nonlocal. Even without the influence of a boundary, spacetime curvature will generically also contain directional information, which in addition would now be local. Nevertheless, one would still have an approximate Lorentz invariance in regions which were large compared to the Planck length but small compared to the radius of curvature. Thus we would not expect the underlying discreteness to disrupt the effective local Lorentz invariance, except in extreme situations where the curvature was approaching unity in natural units.

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