Causal Sets: Discrete Gravity

Notes for the Valdivia Summer School, Jan. 2002

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These are some notes in lieu of the lectures I was scheduled to give, but had to cancel at the last moment. In some places, they are more complete, in others much less so, regrettably. I hope they at least give a feel for the subject and convey some of the excitement felt at the moment by those of us working on it.

An extensive set of references and a glossary of terms can be found at the end of the notes. For a philosophically oriented discussion of some of the background to the causal set idea, see reference [1]. For general background see [2] [3] [4] [5] [6] [7].

Introduction

It seems fair to say that causal set theory has reached a stage in which questions of phenomenology are beginning to be addressed meaningfully. This welcome development is due on one hand to improved astronomical observations which shed light on the magnitude of the cosmological constant (in apparent confirmation of a long-standing prediction of the theory) and on the other hand to theoretical advances which for the first time have placed on the agenda the development of a quantum dynamical law for causal sets (and also for a scalar field residing on a background causal set). What we have so far are: (*i*) an apparently confirmed order of magnitude prediction for the cosmological constant; (*ii*) a method of counting black hole horizon "states" at the kinematical level; (*iii*) the beginnings of a framework in which two-dimensional Hawking radiation can be addressed; (*iv*) a *classical* causal set dynamics which arguably is the most general consistent with the discrete analogs of general covariance and relativistic causality; and in consequence of this, both (*v*) the formulation of a "cosmic renormalization group" which indicates how one might in principle solve some of the large number puzzles of cosmology without recourse

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to a post-quantum era of "inflation"; and (*vi*) a hint of how non-gravitational matter might arise at the fundamental level from causal sets rather than having to be added in by hand or derived at a higher level à la Kaluza-Klein from an effective spacetime topology arising from the fundamental structures via coarse-graining. In addition, a good deal of computer code has been written for use in causal set simulations, including a library of over 5000 lines of Lisp code that can be used by anyone with access to the Emacs editor. At present, the principal need, in addition to fleshing out the developments already outlined, is for a quantum analog of the classical dynamics alluded to above. It looks as if a suitable quantum version of Bell causality (see below) could lead directly to such a dynamics, that is to say, to a theory of quantum spacetime, and in particular to a theory of quantum gravity.

The remainder of these notes rapidly reviews the subject in its current state, progressing broadly from kinematics to dynamics to phenomenology. Although this sequence does not always reflect exactly the chronological development of the theory, it is not far off, and it also fits in well with Taketani's "3-stages" schema of scientific discovery [8].

Origins of the causet idea

The tradition of seeing the causal order of spacetime as its most fundamental structure is almost as old as the idea of spacetime itself (in its Relativistic form). In [9], Robb presented a set of axioms for Minkowski space analogous to Euclid's axioms for plane geometry. In so doing, he effectively demonstrated that, up to an overall conformal factor, the geometry of 4-dimensional flat spacetime (which I'll denote by $\mathbb{M}^4$, taken always with a definite time-orientation) can be recovered from nothing more than the underlying point set and the order relation $\prec$ among points (where $x \prec y \iff$ the vector from x to y is timelike or lightlike and future-pointing). Later, Reichenbach [10] from the side of philosophy and Zeeman [11] from the side of mathematics emphasized the same fact, the latter in particular by proving the theorem, implicit in [9], that any order-isomorphism of $\mathbb{M}^4$ onto itself must — up to an overall scaling — belong to the (isochronous) Poincaré group.

In a certain sense, however, these results appear to say more than they really do. Informally, they seem to tell us that $\mathbb{M}^4$ can be reconstructed from the relation $\prec$, but in actually carrying out the reconstruction (see below), one needs to know that what one is trying to recover is a flat spacetime and not just a conformally flat one. Clearly, there's nothing in the relation $\prec$ *per se* which can tell us that. This difficulty shows itself, in a sense, in the failure of Zeeman's theorem for $\mathbb{M}^2$ and $\mathbb{M}^1$ (i.e. 1+1 and 0+1 dimensional Minkowski space). But it shows up still more clearly with the curved spacetimes of General

Relativity, where the natural generalization of the flat space theorems is that a Lorentzian geometry M can be recovered from its causal order only up to a *local* conformal factor.

Notice that when one says that a Lorentzian manifold M is recovered, one is talking about *all* the mathematical structures that go into the definition of a spacetime geometry: its topology, its differential structure and its metric. Various special results show how to recover, say, the topology (see e.g. [12]) but the most complete theorems are those of [13] and [14], the latter delineating very precisely how close M can come to violating causality without the theorem breaking down.

The upshot of all these reconstruction theorems is that, in the continuum, something is lacking if we possess only the causal order, namely the *conformal factor* or equivalently the "volume element" $\sqrt{-g}\, d^4x$. On the other hand, if we do give the volume element (say in the form of a measure μ on M) then it is clear that metric g_{ab} will be determined in full. The causal order alone, however, is — in the continuum — incapable of furnishing such a measure.

This failing is perhaps one reason to question the reality of the continuum, but of course it is not the only one. In modern times, doubts show up clearly in Riemann's inaugural lecture (Habilitationsschrift) [15], where he contrasts the idea of what he calls a *discrete manifold* with that of a continuous manifold, which latter he takes to be a relatively unfamiliar and unintuitive idea in comparison with the former! The most evocative quotes from this lecture are perhaps the following:

> Grössenbegriffe sind nur da möglich, wo sich ein allgemeiner Begriff vor-findet, der verschiedene Bestimmungsweisen zulässt. Je nachdem unter diesen Bestimmungsweisen von einer zu einer andern ein stetiger Übergang stattfindet oder nicht, bilden sie eine stetige oder discrete Mannigfaltigkeit; die enzelnen Bestimmungsweisen heissen im ersten Falle Punkte, im let-zten Elemente dieser Mannigfaltigkeit. (p.273)

or in translation,*

> Concepts of magnitude are only possible where a general concept is met with that admits of different individual instances [Bestimmungsweisen]. According as, among these individual instances, a continuous passage from one to another takes place or not, they form a continuous or discrete

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* These translations are not guaranteed, but they're a lot better than what "Google" did (try it for fun!).

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manifold; the individual instances are called in the first case points, in the second elements of the manifold;

> Die Frage über die Gültigkeit der Voraussetzungen der Geometrie im Unendlichkleinen hängt zusammen mit der Frage nach dem innern Grunde der Massverhältnisse des Raumes. Bei dieser Frage, welche wohl noch zur Lehre vom Raume gerechnet werden darf, kommt die obige Bemerkung zur Anwendung, dass bei einer discreten Mannigfaltigkeit das Princip der Massverhältnisse schon in dem Begriffe dieser Mannigfaltigkeit enthalten ist, bei einer stetigen aber anders woher hinzukommen muss. Es muss also entweder das dem Raume zu Grunde liegende Wirkliche eine discrete Mannigfaltigkeit bilden, oder der Grund der Massverhältnisse ausserhalb, in darauf wirkenden bindenden Kräften, gesucht werden.

or in translation,

> The question of the validity of the presuppositions of geometry in the infinitely small hangs together with the question of the inner ground of the metric relationships of space. [I almost wrote "spacetime"!] In connection with the latter question, which probably [?] can still be reckoned to be part of the science of space, the above remark applies, that for a discrete manifold, the principle of its metric relationships is already contained in the concept of the manifold itself, whereas for a continuous manifold, it must come from somewhere else. Therefore, either the reality which underlies physical space must form a discrete manifold or else the basis of its metric relationships must be sought for outside it, in binding forces [bindenden Kräfte] that act on it;

and finally,

> Bestimmte, durch ein Merkmal oder eine Grenze unterschiedene Theile einer Mannigfaltigkeit heissen Quanta. Ihre Vergleichung der Quantität nach geschieht bei den discreten Grössen durch Zählung, bei den stetigen durch Messung. (p.274)

or in translation,

> Definite portions of a manifold, distinguished by a criterion [Merkmal] or a boundary, are called quanta. Their quantitative comparison happens for discrete magnitudes through counting, for continuous ones through measurement.

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With the subsequent development of physics, more compelling reasons emerged for questioning the continuum, including the singularities and infinities of General Relativity, of Quantum Field Theory (including the standard model), and of black hole thermodynamics. Einstein, for example, voiced doubts of this sort very early [16]:

> But you have correctly grasped the drawback that the continuum brings. If the molecular view of matter is the correct (appropriate) one, i.e., if a part of the universe is to be represented by a finite number of moving points, then the continuum of the present theory contains too great a manifold of possibilities. I also believe that this too great is responsible for the fact that our present means of description miscarry with the quantum theory. The problem seems to me how one can formulate statements about a discontinuum without calling upon a continuum (space-time) as an aid; the latter should be banned from the theory as a supplementary construction not justified by the essence of the problem, which corresponds to nothing "real". But we still lack the mathematical structure unfortunately. How much have I already plagued myself in this way!

and at a later stage stressed the importance of the causal order in this connection, writing [17] that it would be "especially difficult to derive something like a spatio-temporal quasi-order" from a purely algebraic or combinatorial scheme.

The causal set idea is, in essence, nothing more than an attempt to combine the twin ideas of discreteness and order to produce a structure on which a theory of quantum gravity can be based. That such a step was almost inevitable is indicated by the fact that very similar formulations were put forward independently in [4], [5] and [2], after having been adumbrated in [18]. The insight underlying these proposals is that, in passing from the continuous to the discrete, one actually *gains* certain information, because "volume" can now be assessed (as Riemann said) *by counting*; and with both order *and* volume information present, we have enough to recover geometry.

In this way the topology, the differential structure and, the metric of continuum physics all become unified with the causal order (much as mass is unified with energy in Special Relativity). Moreover the Lorentzian signature (namely $(-+++)$ in 4 dimensions) is singled out as the only one compatible with a consistent distinction between past and future, hence the only one that can make contact with the idea of causal order.

To see how these basic ideas work themselves out, we need first a more precise statement of what a causal set is.

What is a causal set?

As a mathematical structure, a causal set (or *causet* for short) is simply a *locally finite ordered set*. In other words, it is a set C endowed with a binary relation $\prec$ possessing the following three properties:

  (*i*) transitivity: $(\forall x, y, z \in C)(x \prec y \prec z \Rightarrow x \prec z)$

  (*ii*) irreflexivity: $(\forall x \in C)(x \nprec x)$

  (*iii*) local finiteness: $(\forall x, z \in C)\,(\mathrm{card}\,\{y \in C \mid x \prec y \prec z\} < \infty)$

where 'card' stands for "cardinality".In the presence of transitivity, irreflexivity automatically implies acyclicity, i.e. the absence of cycles $x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n = x_0$, and this is often taken as an axiom in place of (*ii*). The condition (*iii*) of local finiteness is a formal way of saying that a causet is *discrete*. Thus the real number line, for example does not qualify as a causet, although it is a partial order.[†]

A structure satisfying the above axioms can be thought of as a graph, and in this sense is conveniently represented as a so-called Hasse diagram in which the elements of C appear as vertices and the relations appear as edges. (The sense of the relation is usually shown, just as in the spacetime diagrams of Relativity theory, by making the line between x and y be a rising one when $x \prec y$.) Actually, it is not necessary to draw in all the relations, but only those not implied by transitivity (the "links"), and this convention is almost always adopted to simplify the appearance of the diagram. A causet can also be thought of as a matrix M (the "causal matrix") with the rows and columns labeled by the elements of C and with the matrix element M_{jk} being 1 if $j \prec k$ and 0 otherwise. Perhaps, however, the most suggestive way to think of a causet for the purposes of quantum gravity is as a relation of "descent", effectively a family tree that indicates which of the elements C are "ancestors" of which others.

The multiplicity of imagery associated with causets (or partial orders more generally) is part of the richness of the subject and makes it natural to use a variety of language in discussing the structural relationships induced by the basic order relation $\prec$. Thus, the

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[†] The above definition utilizes the so called "irreflexive convention" that no element precedes itself. Axioms (*i*) and (*ii*) define what is variously called an "order", a "partial order", a "poset", an "ordered set" or an "acyclic transitive digraph". Axiom (*iii*), which expresses the condition of local finiteness, can also be stated in the form "every order-interval has finite cardinality".

relationship $x \prec y$ itself, is variously described by saying that x *precedes* y, that x is an *ancestor* of y, that y is a *descendant* of x, or that x lies to the *past of* y (or y to the *future* of x). Similarly, if x is an *immediate* ancestor of y (meaning that there exists no intervening z such that $x \prec z \prec y$) then one says that x is a *parent* of y, or y a *child* of x, or that y *covers* x, or that $x \prec y$ is a *link*. (See the glossary.)

Still other interpretations of the relation $\prec$ are possible and can also be useful. For example a causet of finite cardinality is equivalent to a T_0 topological space of finite cardinality, allowing one to use the language of topology in talking about causets (which indeed may turn out to have more than just a metaphorical significance). A causet can also be treated as a *function* by identifying C with the function 'past' that associates to each $x \in C$ the set past(x) of all its ancestors, and this is in fact the representation on which the Lisp code of [19] is based.

For the purposes of quantum gravity, a causal set is, of course, meant to be the deep structure of spacetime. Or to say this another way, the basic hypothesis is that spacetime ceases to exist on sufficiently small scales and is superseded by an ordered discrete structure to which the continuum is only a coarse-grained, macroscopic approximation.

Now, at first sight, a structure based purely on the concept of order might seem to be too impoverished to reproduce the geometrical and topological attributes in terms of which general relativistic spacetime is normally conceived. However, if one reflects that light cones can be defined in causal terms and that (in the continuum) the light cones determine the metric up to a conformal rescaling, then it becomes understandable that (given minimal regularity conditions like the absence of closed causal curves) the causal order of a Lorentzian manifold (say J^+ in the usual notation) captures fully the conformal metric, as well as the topology and the differential structure. The volume element $\sqrt{-g}d^n x$ cannot be recovered from J^+, but in the context of a *discrete* order, it can be obtained in another way — by equating the *number* of causet elements to the *volume* of the corresponding region of the spacetime continuum that approximates C. As discussed above, these observations provide the kinematical starting point for a theory of discrete quantum gravity based on causal sets. The dynamics must then be obtained in the form of a "quantum law of motion" for the causet. Let us consider the kinematics further.

Causal set kinematics in general

Both the study of the mathematics of causets for its own sake and its study for the sake of clarifying how the geometrical and topological properties of a continuous spacetime

translate into order properties of the underlying causet can be regarded as aspects of causal set kinematics: the study of causets without reference to any particular dynamical law.

A large amount is known about causet kinematics as a result of extensive work by both physicists and mathematicians. (See for example [20] [21] [22] [23] [24] [25] [26] [27].) (Some of the mathematicians were directly influenced by the causal set idea, others were studying ordered sets for their own reasons.) We know for example, that the length of the longest chain[b] provides a good measure of the proper time (geodesic length) between any two causally related elements of a causet that can be approximated by a region of Minkowski space [28]. And for such a causet, we also possess at least two or three good dimension estimators, one of which is well understood analytically [29].

The next few sections are devoted to some of these topics.

"How big" is a causet element?

Of course the question is badly worded, because a causet element has no size as such. What it's really asking for is the conversion factor v_0 for which $N = V/v_0$. Only if we measure length in units such that $v_0 \equiv 1$ can we express the hypothesis that number=volume in the form $N = V$. On dimensional grounds, one naturally expects $v_0 \sim (G\hbar)^2$ [where I've taken $c \equiv 1$]. But we can do better than just relying on dimensional analysis *per se*. Consider first the entropy of a black hole horizon, which is given by $S = A/4G\hbar = 2\pi A/\kappa$, with $\kappa = 8\pi G$, the rationalized gravitational constant. This formula suggests forcefully that about one bit of entropy belongs to each horizon "plaquette" of size $\kappa\hbar$, and that the effectively finite size of these "plaquettes" reflects directly an underlying spacetime discreteness. Consideration of the so called entanglement entropy leads to the same conclusion, namely that there exists an effective "ultraviolet cutoff" at around $l = \sqrt{\kappa\hbar}$. [30]

A related but less direct train of thought starts by considering the gravitational action-integral $\frac{1}{2\kappa} \int R dV$. Here the "coupling constant" $1/2\kappa$ is an inverse length^2 and conventional Renormalization Group wisdom suggests that, barring any "fine tuning", the order of magnitude of such a coupling constant will be set by the underlying "lattice spacing", or in this case, the fundamental discreteness scale, leading to the same conclusion as before that $l \equiv (v_0)^{1/4}$ is around $l = \sqrt{\kappa\hbar} \sim 10^{-32} cm$.

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[b] The term 'chain' is defined in the glossary.

A noteworthy implication of the formula $l \sim \sqrt{\kappa \hbar}$ is that $l \to 0$ if $\hbar \to 0$ (κ being fixed). That is, the classical limit is necessarily a continuum limit: spacetime discreteness is inherently quantal.

The reconstruction of $\mathbb{M}^4$

In order to get a better feel for how it is that "geometry = order + number", it is useful to work through the reconstruction — in the continuum — of $\mathbb{M}^4$ from its causal order and volume-element. The proof can be given in a quite constructive form which I'll only sketch here.

We start with a copy M of $\mathbb{M}^4$ and let $\prec$ be its causal order. We construct in turn: light rays l, null 3-planes, spacelike 2-planes, spacelike lines, arbitrary 2-planes, arbitrary lines, parallel lines, parallelograms, vectors. Once we have vector addition (affine structure) it is easy to get quadratic forms and in particular the flat metric η_{ab}. (The normalization of η_{ab} uses the volume information.)

You may enjoy working these constructions out for yourself, so I won't give them all here. At the risk of spoiling your fun however, let me give just the first two, which perhaps are less straightforward than the rest. We define a light ray l to be a maximal chain such that $\forall x, y \in l$, $\mathrm{interval}(x, y)$ is *also* a chain; and from any such l we get then the null hyperplane $N(l) = l \cup l^{\natural}$, where $l^{\natural}$ is the set of all points of M spacelike to l.

Sprinkling, coarse-graining, and the "Hauptvermutung"

A basic tenet of causet theory is that spacetime does not exist at the most fundamental level, that it is an "emergent" concept which is relevant only to the extent that some manifold-with-Lorentzian-metric M furnishes a good approximation to the physical causet C. Under what circumstances would we want to say that this occurred? So far the most promising answer to this question is based on the concepts of *sprinkling* and *coarse-graining*.

Given a manifold M with Lorentzian metric g_{ab} (which is, say, globally hyperbolic) we can obtain a causal set $C(M)$ by selecting points of M and endowing them with the order induced from that of M (where in M, $x \prec y$ iff there is a future causal curve from x to y). In order to realize the equality $N = V$, the selected points must be distributed *with unit density* in M. One way to accomplish this (and conjecturally the *only* way!) is to generate the points of $C(M)$ by a *Poisson process*. (To realize the latter, imagine dividing M up into small regions of volume ε and independently putting a point into each region with probability ε. In the limit $\varepsilon \to 0$ this is the Poisson process of unit intensity in M.)

Let us write $M \approx C$ for the assertion that M is a good approximation to C. The idea then is that $M \approx C$ if C "might have been produced by a sprinkling of M" (in a sense to be specified more fully below).

It's important here that the elements of $C(M)$ are selected at random from M. In particular, this fact is an ingredient in the heuristic reasoning leading to the prediction of a fluctuating cosmological constant (see below). But such a *kinematic* randomness might seem gratuitous. Wouldn't a suitable *regular* embedding of points into M yield a subset that was equally uniformly distributed, if not more so? To see what goes wrong, consider the "diamond lattice" in $\mathbb{M}^2$ consisting of all points with integer values of the null coordinates $u = t - x$ and $v = t + x$. This would *seem* to be a uniform lattice, but under a boost $u \rightarrow \lambda u$, $v \rightarrow v/\lambda$ it goes into a distribution that looks entirely different, with a very high density of points along the u=constant lines (say) and large empty spaces in between. In particular, our diamond lattice is far from Lorentz invariant, which a truly uniform distribution *should be* — and which $C(M)$ produced by a Poisson process *actually is*. Examples like this suggest strongly that, in contrast to the situation for Euclidean signature, only a random sprinkling can be uniform for Lorentzian signature.

I have just argued that the idea of random sprinkling must play a role in making the correspondence between the causet and the continuum, and for purely kinematic reasons. A second concept which might be needed as well, depending on how the dynamics works out, is that of *coarse-graining*. Indeed, one might expect that, on very small scales, the causet representing our universe will no more look like a manifold than the trajectory of a point particle looks microscopically like a smooth curve in nonrelativistic quantum mechanics. Rather, we might recover a manifold only after some degree of "averaging" or "coarse-graining" (assuming also that we keep away from the big bang and from black hole interiors, etc., where we don't expect a manifold at all). That is, we might expect not $C \approx M$ but $C' \approx M'$, where C' is some coarse-graining of C and M' is M with a correspondingly rescaled metric. The relevant notion of coarse-graining here seems to be an analog of sprinkling applied to C itself: let C' be obtained from C by selecting a subset at random, keeping each element $x \in C$ with some fixed probability, say $1/2$ if we want a 2:1 coarse-graining.

Implicit in the idea of a manifold approximating a causet is that the former is relatively unique; for if two very different manifolds could approximate the same C, we'd have no objective way to understand why we observe one particular spacetime and not some very different one. (On the other hand, considering things like AdS/CFT duality, who knows...!) The conjecture that such ambiguities don't occur has been called the "Hauptvermutung". In the $G\hbar \rightarrow 0$ limit, it has already been proven in [31]. Moreover, the fact that we know

how to obtain dimensional and proper time information in many situations (see below) is strong circumstantial evidence for its truth at finite $G\hbar$. Nevertheless it would be good to prove it in full, in something like the following form.

CONJECTURE If $M_1 \approx C$ and $M_2 \approx C$ then $M_1 \approx M_2$

Here $M_1 \approx M_2$ means that the manifolds M_1 and M_2 are "approximately isometric". As the quotation marks indicate, it is surprisingly difficult to give this conjecture rigorous meaning, due ultimately to the Lorentzian signature of g_{ab} (cf. [32]). Here's a sketch of how it might be done: interpret $M \approx C$ to mean that $\mathrm{Prob}(C(M) = C)$ is relatively large (in comparison with $\mathrm{Prob}(C(M) = C')$) for most of the other C'; and interpret $M_1 \approx M_2$ to mean that the random variables $C(M_1)$ and $C(M_2)$ share similar probability distributions (on the space of causets). These definitions almost make the conjecture look tautological, but it isn't! (cf. [33])

Dimension and length

Assuming that — as seems very likely — causal sets do possess a structure rich enough to give us back a macroscopically smooth Lorentzian geometry, it is important to figure out how in practice one can extract geometrical information from an order relation. But before we can speak of a geometry we must have a manifold, and the most basic aspect of a manifold's topology is its dimension. So an obvious first question is whether there is a good way to recognize the effective continuum dimension of a causal set (or more precisely of a causal set that is sufficiently "manifold like" for the notion of its dimension to be meaningful). In fact several workable approaches exist. Here are three of them. All three estimators will assign a dimension to an interval I in a causet C and are designed for the case where $I \approx A$ for some interval ("double light cone") A in Minkowski space $\mathbb{M}^d$.

Myrheim-Meyer dimension [5] [29]. Let $N = |I|$ be the number of elements in I and let R be the number of relations in I (i.e. pairs x, y such that $x \prec y$). Let $f(d) = \frac{3}{2}\left(\frac{3d/2}{d}\right)^{-1}$. Then $f^{-1}(R/\binom{N}{2})$ is a good estimate of d when $N \gg (27/16)^d$.

REMARK The Myrheim-Meyer estimator is coarse-graining invariant on average (as is the next)

Midpoint scaling dimension. Let $I = \mathrm{interval}(a,b)$ and let $m \in I$ be the (or a) "midpoint" defined to maximize $N' = \min\{|\mathrm{interval}(a,m)|, |\mathrm{interval}(m,b)|\}$. Then $\log_2(N/N')$ estimates d.

A third dimension estimator. Let K be the total number of chains in I. Then $\ln N/\ln\ln K$ estimates d. However, the logarithms mean that good accuracy sets in only for exponentially large N.

A length estimator

Again this is for $C \approx \mathbb{M}^d$ (or some convex subspace of $\mathbb{M}^d$). Let $x \prec y$. The most obvious way to define a distance (or better a time-lapse) from x to y is just to count the number of elements L in the longest chain joining them, where a joining chain is by definition a succession of elements z_i such that $x \prec z_1 \prec z_2 \prec z_3 \cdots \prec y$. Clearly a maximal path in this sense is analogous to a timelike geodesic, which maximizes the proper-time between its endpoints. It is known that this estimator L converges rapidly to a multiple of the true proper time T as the latter becomes large. However the coefficient of proportionality depends on the dimension d and is known exactly only for $d = 1$ and $d = 2$. For $d > 2$ only bounds are known, but they are rather tight.

Thus, it seems that we have workable tools for recovering information on both dimensionality and length (in the sense of timelike geodesic distance). However, these tools have been proven so far primarily in a flat context, and it remains to be shown that they continue to work well in the presence of generic curvature.

Dynamics

A priori, one can imagine at least two routes to a "quantum causet dynamics". On one hand, one could try to mimic the formulation of other theories by seeking a causet-invariant analogous to the scalar curvature action, and then attempting to build from it some discrete version of a gravitational "sum-over-histories". On the other hand, one could try to identify certain general principles or rules powerful enough to lead, more or less uniquely, to a family of dynamical laws sufficiently constrained that one could then pick out those members of the family that reproduced the Einstein equations in an appropriate limit or approximation. (By way of analogy, one could imagine arriving at general relativity either by seeking a spin-2 analog of Poisson's equation or by seeking the most general field equations compatible with general covariance and locality.)

The recent progress in dynamics has come from the second type of approach, and with the causet's "time-evolution" conceived as a process of what may be termed *sequential growth*. That is, the causet is conceived of as "developing in time",[*] rather than as

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[*] It might be more accurate to say that the growth of the causet *is time*.

"existing timelessly" in the manner of a film strip. At the same time the growth process is taken to be random rather than deterministic — classically random to start with, but ultimately random in the quantum sense familiar from atomic physics and quantum field theory. (Thus the quantum dynamical law is being viewed as more analogous to a classical stochastic process like Brownian motion than to a classical deterministic dynamics like that of the harmonic oscillator. [34] [35]) Expressed more technically, the idea is to seek a quantum causet dynamics by first formulating the causet's growth as a classical stochastic process and then generalizing the formulation to the case of a "quantum measure" [36] or "decoherence functional" [37].

The growth process in question can be viewed as a sequence of "births" of new causet elements and each such birth is a transition from one partial causet to another. A dynamics or "law of growth" is then simply an assignment of probabilities to each possible sequence of transitions. Without further restriction, however, there would be a virtually limitless set of possibilities for these probabilities. The two principles that have allowed us to narrow this field of possibilities down to a (hopefully) manageable number are *discrete general covariance* and *Bell causality*. To understand the first of these, notice that taking the births to be sequential implicitly introduces a labeling of the causet elements (the first-born element being labeled 0, the second-born labeled 1, etc). Discrete general covariance is simply the requirement that this labeling be "pure gauge", that it drop out of the final probabilities in the same way that the choice of coordinates drops out of the equations of general relativity. This requirement has the important side effect of rendering the growth process Markovian, so that it is fully definable in terms of *transition probabilities* obeying the Markov sum rule. The requirement of Bell causality is slightly harder to explain, but it is meant to capture the intuition that a birth taking place in one region of the causet cannot be influenced by other births that occur in regions spacelike to the first region.

Taken together, these assumptions lead to a set of equations and inequalities that — remarkably — can be solved explicitly and in general [38] [39]. The resulting probability for a transition $C \to C'$ in which the new element is born with ϖ ancestors and m parents (immediate ancestors) is given by the ratio

$$\frac{\lambda(\varpi, m)}{\lambda(n, 0)} \ , \tag{1}$$

where $n = \mathrm{card}(C)$ is the number of elements before the birth in question and where the function λ is defined by the formula

$$\lambda(\varpi, m) = \sum_{k=m}^{\varpi} \binom{\varpi - m}{k - m} t_k \tag{2}$$

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with $t_n \geq 0$ and $t_0 > 0$. A particular dynamical law, then, is determined by the sequence of "coupling constants" t_n (or more precisely, by their ratios).

(It turns out that the probabilities resulting from these rules can be re-expressed in terms of an "Ising model" whose spins reside on the relations $x \prec y$ of the causet, and whose "vertex weights" are governed directly by the parameters t_n [38]. In this way, a certain form of "Ising matter" emerges indirectly from the dynamical law, albeit its dynamics is rather trivial if one confines oneself to a fixed background causet. This illustrates how one might hope to recover in an appropriate limit, not only spacetime and gravity, but also certain forms of non-gravitational matter (here unified with gravity in a way reminiscent of earlier proposals for "induced gravity" [40]).)

With some progress in hand concerning both the kinematics and dynamics of causets, it is possible to start to think about applications, or if you will "phenomenology". Several projects of this nature are under way, with some interesting results already obtained. In the remaining sections I will mention some of these results and projects.

Fluctuations in the cosmological constant

From the most basic notions of causal set theory, there follows already an order of magnitude prediction for the value of the cosmological constant Λ. More precisely, one can argue that Λ should *fluctuate* about its "target value", with the magnitude of the fluctuations decreasing with time in proportion to $N^{-1/2}$, where N is the relevant number of ancestors (causet elements) at a given cosmological epoch. If one assumes that (for reasons yet to be understood) the target value for Λ is zero, and if one takes for the N of today the spacetime volume of the currently visible universe from the big bang until the present, then one deduces for Λ a magnitude consistent with the most recent observations, as predicted already in [41] (and with refined arguments in [35] and [42]).

Four basic features of causet theory enter as ingredients into the refined version of the argument: the fundamental discreteness, the relation $n = V$, the Poisson fluctuations in n associated with sprinkling, and the fact that n serves as a parameter time in the dynamics of sequential growth.

From the first of these we derive a finite value of n (at any given cosmic time). From the fourth we deduce that, since time is not summed over in the path-integral of non-relativistic quantum mechanics, neither should one expect to sum over n in the gravitational path integral that one expects to result as an approximation to the still to be formulated quantum dynamics of causets. But holding n fixed means holding spacetime

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volume V fixed, a procedure that leads in the continuum to what is called "unimodular gravity".

In the classical limit, this unimodular procedure leads to the action principle

$$\delta \left(\int (\frac{1}{2\kappa}R - \Lambda_0)dV - \lambda V \right) = 0$$

where Λ_0 is the "bare" cosmological constant, $V = \int dV$, $\kappa = 8\pi G$, and λ is a Lagrange multiplier implementing the fixation of V. Plainly, the last two terms combine into $-\int \Lambda dV$ where $\Lambda = \Lambda_0 + \lambda$, turning the effective cosmological constant Λ into a free constant of integration rather than a fixed microscopic parameter of the theory. Moreover, the fact that Λ and V enter into the action-integral in the combination $-\Lambda V$ means that they are conjugate in the quantum mechanical sense, leading to the indeterminacy relation

$$\delta\Lambda \, \delta V \sim \hbar \ .$$

Finally the Poisson fluctuations in n of size $\delta n \sim \sqrt{n}$ at fixed V imply that, at fixed n, there will be fluctuations in V of the same magnitude: $\delta V \sim \sqrt{n} \sim \sqrt{V}$, which correspond to fluctuations in Λ of magnitude $\delta\Lambda \sim \hbar/\delta V \sim 1/\sqrt{V}$ (taking $\hbar = 1$). The observed Λ would thus be a sort of residual quantum gravity effect, even though one normally associates the quantum with the very small, rather than the very big!

Of course, this prediction of a fluctuating Λ remains at a heuristic level until it can be grounded in a complete "quantum causet dynamics". Nevertheless, given its initial success, it seems worthwhile to try to extend it by constructing a model in which not only the instantaneous magnitude of the fluctuations could be predicted, but also their correlations between one time and another. In this way, one could assess whether the original prediction was consistent with important cosmological constraints such as the extent of structure formation and the abundances of the light nuclei. In this respect, it is worth noting that current fits of nucleosynthesis models to the observed abundances favor a non-integer number of light neutrinos falling between two and three. If this indication holds up, it will require some form of effective negative energy density at nucleosynthesis time, and a negative fluctuation in the contemporaneous Λ is perhaps the simplest way to realize such an effective density.

Added note: A concrete model of the sort suggested above has been developed in [43].

Links across the horizon

An important question on which one can hope to shed light while still remaining at the level of kinematics is that of identifying the "horizon states" that underpin the entropy of

a black hole. Indeed, just as the entropy of a box of gas is, to a first approximation, merely counting the molecules in the box, one might anticipate that the entropy of a black hole is effectively counting suitably defined "molecules" of its horizon. With this possibility in mind, one can ask whether any simply definable sub-structures of the causets associated with a given geometry could serve as candidates for such "horizon molecules" in the sense that counting them would approximately measure the "information content" of the black hole.

Perhaps the most obvious candidates of this sort are the causal links crossing the horizon in the neighborhood of the hypersurface Σ for which the entropy is sought. (Recall that a link is an irreducible relation of the causet.) Of course, the counting of any small scale substructures of the causet is prone to produce a result proportional to the area of the horizon, but there is no reason *a priori* why the coefficient of proportionality could not be divergent or vanishing, or why, if it is finite, it could not depend on the details of the horizon geometry.

Djamel Dou [44] has investigated this question for two very different black hole geometries, one in equilibrium (the 4 dimensional Schwarzschild metric) and one very far from equilibrium (the conical horizon that represents the earliest portion of a black hole formed from the collapse of a spherical shell of matter). For the Schwarzschild case, he made an ad hoc approximation that reduced the problem to 2 dimensional Schwarzschild and found, for a certain definition of "near horizon link", that the number N of such links has an expectation value which reduces in the $\hbar \to 0$ limit to $c(\pi^2/6)A$, where A is the horizon area and c is a constant arising in the dimensional reduction. (By $\hbar \to 0$ I mean equivalently $l^2/R^2 \to 0$ where l is the fundamental causet scale and R the horizon radius.) For the expanding horizon case (again dimensionally reduced from 4 to 2) he obtained exactly the same answer, $c(\pi^2/6)A$, despite the very different geometries. Not only is this a nontrivial coincidence, it represents the first time, to my knowledge, that something like a number of horizon states has been evaluated for any black hole far from equilibrium. The first step in solidifying and extending these results would be to control the dimensional reduction from 4 to 2, evaluating in particular the presently unknown coefficient c. Second, one should check that both null and spacelike hypersurfaces Σ yield the same results. (The null case is the best studied to date. Conceptually, it is important for possible proofs of the generalized second law [45].) Also one should assess the sensitivity of the answer to changes in the definition of "near horizon link", since there exist examples showing that the wrong definition can lead to an answer of either zero or infinity. And of course one should extend the results to other black hole geometries beyond the two studied so far.

Here is one of the definitions of "near horizon link" investigated by Djamel: Let H be the horizon of the black hole and Σ, as above, the hypersurface for which we seek the entropy S. The counting is meant to yield the black hole contribution to S, corresponding to the section $H \cap \Sigma$ of the horizon. We count pairs of sprinkled points (x, y) such that

(i) $x \prec \Sigma, H$ and $y \succ \Sigma, H$.

(ii) $x \prec y$ is a link[‡]

(iii) x is maximal[‡] in $(\text{past}\, \Sigma)$ and y is minimal[‡] in $(\text{future}\, \Sigma) \cap (\text{future}\, H)$.

What are the "observables" of quantum gravity?

Just as in the continuum the demand of diffeomorphism-invariance makes it harder to formulate meaningful statements,[†] so also for causets the demand of discrete general covariance has the same consequence, bringing with it the risk that, even if we succeeded in characterizing the covariant questions in abstract formal terms, we might never know what they meant in a physically useful way. I believe that a similar issue will arise in every approach to quantum gravity, discrete or continuous (unless of course general covariance is renounced).[♭]

Within the context of the classical growth models described above, this problem has been largely solved [47], the "observables" being generated by "stem-predicates". ('stem' is defined in the glossary).

Added note: The conjecture in [47] has been settled in the affirmative by [48].

[‡] see glossary

[†] Think, for example, of the statement that light slows down when passing near the sun.

[♭] In the context of canonical quantum gravity, this issue is called "the problem of time". There, covariance means commuting with the constraints, and the problem is how to interpret quantities which do so in any recognizable spacetime language. For an attempt in string theory to grapple with similar issues see [46].

How the large numbers of cosmology might be understood: a "Tolman-Boltzmann" cosmology

Typical large number is ratio r of diameter of universe to wavelength of CMB radiation. Idea is cycling of universe renormalizes [49] coupling constants such that r automatically gets big after many bounces (no fine tuning). Large numbers thus reflect large age of universe. See [50] and [51].

Fields on a background causet

See [52], [53], [54].

Topology change

See [55].

Acknowledgments

References

[1] R.D. Sorkin, "A Specimen of Theory Construction from Quantum Gravity", in J. Leplin (ed.), *The Creation of Ideas in Physics: Studies for a Methodology of Theory Construction* (Proceedings of the Thirteenth Annual Symposium in Philosophy, held Greensboro, North Carolina, March, 1989) pp. 167-179 (Number 55 in the University of Western Ontario Series in Philosophy of Science) (Kluwer Academic Publishers, Dordrecht, 1995) `gr-qc/9511063`

[2] L. Bombelli, J. Lee, D. Meyer and R.D. Sorkin, "Spacetime as a causal set", *Phys. Rev. Lett.* **59**: 521-524 (1987);
C. Moore, "Comment on 'Space-time as a causal set'," *Phys. Rev. Lett.* **60**: 655 (1988);
L. Bombelli, J. Lee, D. Meyer and R.D. Sorkin, "Bombelli et al. Reply", *Phys. Rev. Lett.* **60**: 656 (1988).

[3] L. Bombelli, *Space-time as a Causal Set*, Ph.D. thesis, Syracuse University (1987)

[4] G. 't Hooft, "Quantum gravity: a fundamental problem and some radical ideas", in *Recent Developments in Gravitation* (Proceedings of the 1978 Cargese Summer Institute) edited by M. Levy and S. Deser (Plenum, 1979)

[5] J. Myrheim, "Statistical geometry," CERN preprint TH-2538 (1978)

[6] G. Brightwell and R. Gregory, "The Structure of Random Discrete Spacetime", *Phys. Rev. Lett.* **66** : 260-263 (1991)

[7] David D. Reid, "Discrete Quantum Gravity and Causal Sets", *Canadian Journal of Physics* **79** : 1-16 (2001) `gr-qc/9909075`

[8] Mituo Taketani, "On formation of the Newton Mechanics", *Suppl. Prog. Theor. Phys.* **50** : 53-64 (1971)

[9] Alfred Arthur Robb, *Geometry of Time and Space* (Cambridge University Press, 1936) (a revised version of *A theory of Time and Space* (C.U.P. 1914))

[10] Hans Reichenbach:
Physikal. Zeitschr. **22** : 683 (1921); and
Axiomatik der relativistische Raum-Zeit-Lehre, translated into English as *Axiomatization of the theory of relativity* (Berkeley, University of California Press, 1969).

[11] E.C. Zeeman, "Causality Implies the Lorentz Group", *J. Math. Phys.* **5** : 490-493 (1964)

[12] Roger Penrose, *Techniques of Differential Topology in Relativity*, AMS Colloquium Publications (SIAM, Philadelphia, 1972)

[13] S.W. Hawking, A.R. King and P.J. McCarthy, "A New Topology for Curved Space-Time which Incorporates the Causal, Differential and Conformal Structures", *J. Math. Phys.* **17** : 174 (1976)

[14] David Malament, "The class of continuous timelike curves determines the toplogy of space-time", *J. Math. Phys* **18** : 1399 (1977)

[15] Bernhard Riemann, "Über die Hypothesen, welche der Geometrie zu Grunde liegen", in *The Collected Works of B. Riemann* (Dover NY 1953)

[16] A. Einstein, Letter to Walter Dällenbach, Nov. 1916, Item 9-072 translated and cited by Stachel in the following article, page 379.

[17] A. Einstein, Letter to H.S. Joachim, August 14, 1954, Item 13-453 cited in J. Stachel, "Einstein and the Quantum: Fifty Years of Struggle", in *From Quarks to Quasars, Philosophical Problems of Modern Physics*, edited by R.G. Colodny (U. Pittsburgh Press, 1986), pages 380-381.

[18] David Finkelstein, "The space-time code", *Phys. Rev.* **184** : 1261 (1969)

[19] R.D. Sorkin, "Lisp Functions for Posets", http://www.physics.syr.edu/~sorkin (version 1.0 1997, version 2.0 2002).
Another poset package (using Maple) is [56].

[20] Alan Daughton, "An investigation of the symmetric case of when causal sets can embed into manifolds" *Class. Quant. Grav.* **15** : 3427-3434 (1998)

[21] G. Brightwell and P. Winkler, "Sphere Orders", *Order* **6** : 235-240 (1989)

[22] Graham Brightwell, "Models of Random Partial Orders", in *Surveys in Combinatorics, 1993*, London Math. Soc. Lecture Notes Series **187**: 53-83, ed. Keith Walker (Cambridge Univ. Press 1993)

[23] B. Pittel and R. Tungol, "A Phase Transition Phenomenon in a Random Directed Acyclic Graph", (Ohio State preprint, 1998) (submitted to *Combinatorics, Probability and Computing*)

[24] Klaus Simon, Davide Crippa and Fabian Collenberg, "On the Distribution of the Transitive Closure in a Random Acyclic Digraph", *Lecture Notes in Computer Science* **726:** 345-356 (1993)

[25] Charles M. Newman, "Chain Lengths in Certain Random Directed Graphs", *Random Structures and Algorithms* **3:** 243-253 (1992)

[26] Jeong Han Kim and Boris Pittel, "On tail distribution of interpost distance" (preprint, Ohio State University and Microsoft Corp., 1999)

[27] Ioannis Raptis, "Algebraic Quantization of Causal Sets" *Int. J. Theor. Phys.* **39** : 1233-1240 (2000) gr-qc/9906103

[28] B. Bollobas and G. Brightwell, "The Height of a Random Partial Order: Concentration of Measure", *Annals of Applied Probability* **2:** 1009-1018 (1992)

[29] D.A. Meyer:
"Spherical containment and the Minkowski dimension of partial orders", *Order* **10** : 227-237 (1993); and
The Dimension of Causal Sets, Ph.D. thesis, M.I.T. (1988).

[30] R.D. Sorkin, "On the Entropy of the Vacuum Outside a Horizon", in B. Bertotti, F. de Felice and A. Pascolini (eds.), *Tenth International Conference on General Relativity and Gravitation (held Padova, 4-9 July, 1983), Contributed Papers*, vol. II, pp. 734-736 (Roma, Consiglio Nazionale Delle Ricerche, 1983);
G. 't Hooft, "On the quantum structure of a black hole", *Nuclear Phys. B* **256:** 727-745 (1985)

[31] L. Bombelli and D.A. Meyer, "The origin of Lorentzian geometry," *Physics Lett. A* **141:** 226-228 (1989)

[32] Johan Noldus, "A new topology on the space of Lorentzian metrics on a fixed manifold" *Class. Quant. Grav* **19:** 6075-6107 (2002)

[33] Luca Bombelli, "Statistical Lorentzian geometry and the closeness of Lorentzian manifolds", *J. Math. Phys.* **41** : 6944-6958 (2000) `gr-qc/0002053`

[34] R.D. Sorkin:
"On the Role of Time in the Sum-over-histories Framework for Gravity", paper presented to the conference on The History of Modern Gauge Theories, held Logan, Utah, July 1987, published in *Int. J. Theor. Phys.* **33** : 523-534 (1994); and
"A Modified Sum-Over-Histories for Gravity", reported in *Proceedings of the International Conference on Gravitation and Cosmology, Goa, India, 14-19 December, 1987*, edited by B. R. Iyer, Ajit Kembhavi, Jayant V. Narlikar, and C. V. Vishveshwara, see pages 184-186 in the article by D. Brill and L. Smolin: "Workshop on quantum gravity and new directions", pp 183-191 (Cambridge University Press, Cambridge, 1988)

[35] R.D. Sorkin, "Forks in the Road, on the Way to Quantum Gravity", talk given at the conference entitled "Directions in General Relativity", held at College Park, Maryland, May, 1993, published in *Int. J. Th. Phys.* **36** : 2759–2781 (1997)
`gr-qc/9706002`

[36] R.D. Sorkin, "Quantum Mechanics as Quantum Measure Theory", *Mod. Phys. Lett. A* **9** (No. 33) : 3119-3127 (1994) `gr-qc/9401003`

[37] J.B. Hartle, "Spacetime Quantum Mechanics and the Quantum Mechanics of Spacetime", in B. Julia and J. Zinn-Justin (eds.), *Les Houches, session LVII, 1992, Gravitation and Quantizations* (Elsevier Science B.V. 1995)

[38] David P. Rideout and Rafael D. Sorkin, "A Classical Sequential Growth Dynamics for Causal Sets", *Phys. Rev. D* **61** : 024002 (2000) `gr-qc/9904062`

[39] David P. Rideout, *Dynamics of Causal Sets*, Ph.D. thesis (Syracuse University 2001)

[40] T. Jacobson, "Black Hole Entropy and Induced Gravity", `gr-qc/9404039`;
S.L. Adler, "Einstein gravity as a symmetry-breaking effect in quantum field theory", *Reviews of Modern Physics* **54** : 729-766 (1982);
V.A. Kazakov, "The appearance of matter fields from quantum fluctuations of 2D-gravity", *Mod. Phys. Lett. A* **4** : 2125-2139 (1989) .

[41] R.D. Sorkin:
"First Steps with Causal Sets", in R. Cianci, R. de Ritis, M. Francaviglia, G. Marmo, C. Rubano, P. Scudellaro (eds.), *General Relativity and Gravitational Physics* (Proceedings of the Ninth Italian Conference of the same name, held Capri, Italy, September, 1990), pp. 68-90 (World Scientific, Singapore, 1991); and
"Spacetime and Causal Sets", in J.C. D'Olivo, E. Nahmad-Achar, M. Rosenbaum, M.P. Ryan, L.F. Urrutia and F. Zertuche (eds.), *Relativity and Gravitation: Classical and Quantum* (Proceedings of the *SILARG VII Conference*, held Cocoyoc, Mexico, December, 1990), pages 150-173 (World Scientific, Singapore, 1991).

[42] R.D. Sorkin, Two Talks given at the 1997 Santa Fe workshop: "A Review of the Causal Set Approach to Quantum Gravity" and "A Growth Dynamics for Causal Sets", presented at: "New Directions in Simplicial Quantum Gravity" July 28 - August 8, 1997.

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The scanned in transparencies may be viewed at
http://t8web.lanl.gov/people/emil/Slides/sf97talks.html ;
Y. Jack Ng and H. van Dam, "A small but nonzero cosmological constant", (Int. J. Mod. Phys D., to appear) `hep-th/9911102`

[43] Maqbool Ahmed, Scott Dodelson, Patrick Greene and Rafael D. Sorkin, "Everpresent Λ", `astro-ph/0209274`

[44] Djamel Dou, "Causal Sets, a Possible Interpretation for the Black Hole Entropy, and Related Topics", Ph. D. thesis (SISSA, Trieste, 1999) `gr-qc/0106024`.
See also Dj. Dou and R.D. Sorkin, "Black Hole Entropy as Causal Links", *Foundations of Physics* (2003, to appear).

[45] R.D. Sorkin, "Toward an Explanation of Entropy Increase in the Presence of Quantum Black Holes", *Phys. Rev. Lett.* **56**: 1885-1888 (1986)

[46] Edward Witten, "Quantum Gravity in De Sitter Space" `hep-th/0106109`

[47] Graham Brightwell, H. Fay Dowker, Raquel S. García, Joe Henson and Rafael D. Sorkin, "General Covariance and the 'Problem of Time' in a Discrete Cosmology", in K.G. Bowden, Ed., *Correlations*, Proceedings of the ANPA 23 conference, held August 16-21, 2001, Cambridge, England (Alternative Natural Philosophy Association, London, 2002), pp 1-17 `gr-qc/0202097`

[48] Graham Brightwell, Fay Dowker, Raquel Garcia, Joe Henson and Rafael D. Sorkin, ""Observables" in Causal Set Cosmology", `gr-qc/0210061`

[49] Xavier Martin, Denjoe O'Connor, David Rideout and Rafael D. Sorkin, "On the "renormalization" transformations induced by cycles of expansion and contraction in causal set cosmology", *Phys. Rev. D* **63**: 084026 (2001) `gr-qc/0009063`

[50] Rafael D. Sorkin, "Indications of causal set cosmology", *Int. J. Theor. Ph.* **39 (7)**: 1731-1736 (2000) (an issue devoted to the proceedings of the Peyresq IV conference, held June-July 1999, Peyresq France) `gr-qc/0003043`

[51] Avner Ash and Patrick McDonald, "Moment problems and the causal set approach to quantum gravity", `gr-qc/0209020`

[52] A.R. Daughton, *The Recovery of Locality for Causal Sets and Related Topics*, Ph.D. thesis, Syracuse University (1993)

[53] Roberto Salgado, Ph.D. thesis, Syracuse University (in preparation)

[54] Richard F. Blute, Ivan T. Ivanov, Prakash Panangaden, "Discrete Quantum Causal Dynamics", `gr-qc/0109053`

[55] Fay Dowker, "Topology Change in Quantum Gravity", `gr-qc/0206020`

[56] John R. Stembridge (jrs@math.lsa.umich.edu), *A Maple Package for Posets: Version 2.1*, (Ann Arbor 1998-may-10) available at www.math.lsa.umich.edu/~jrs

[57] David P. Rideout and Rafael D. Sorkin, "Evidence for a continuum limit in causal set dynamics", *Phys. Rev. D* **63**: 104011 (2001) `gr-qc/0003117`

[58] F. Markopoulou and L. Smolin, "Causal evolution of spin networks", *Nuc. Phys.* **B508**: 409-430 (1997) `gr-qc/9702025`

[59] R.D. Sorkin, "Percolation, Causal Sets and Renormalizability", unpublished talk delivered at the RG2000 conference, held in Taxco, Mexico, January 1999

[60] David P. Rideout and Rafael D. Sorkin, "Evidence for a Scaling Behavior in Causal Set Dynamics", (in preparation)

[61] L. Smolin, "The fate of black hole singularities and the parameters of the standard models of particle physics and cosmology", `gr-qc/9404011`

[62] Roberto B. Salgado, "Some Identities for the Quantum Measure and its Generalizations", *Mod. Phys. Lett.* **A17**: 711-728 (2002) `gr-qc/9903015`

[63] Fay Dowker and Rafael D. Sorkin, "Toward an Intrinsic Definition of Relativistic Causality" (in preparation)

[64] A. Criscuolo and H. Waelbroeck, "Causal Set Dynamics: A Toy Model", *Class. Quant. Grav.* **16**: 1817-1832 (1999) `gr-qc/9811088`

[65] Joohan Lee and Rafael D. Sorkin, "Limits of Sequential Growth Dynamics" (in preparation)

[66] T. Jacobson, "Black-hole evaporation and ultrashort distances", *Phys. Rev. D* **44**: 1731-1739 (1991))

[67] Curtis Greene et al., *A Mathematica Package for Studying Posets* (Haverford College, 1990, 1994) available at www.haverford.edu/math/cgreene.html

[68] C.D. Broad, *Scientific Thought* (Harcourt Brace and Company, 1923)

GLOSSARY

Major deviations from these definitions are rare in the literature but minor ones are common. (In this glossary, we use the symbol $<$ rather than $\prec$.)

ancestor/descendant

> If $x < y$ then x is an ancestor of y and y is a descendant of x.

antichain

> a trivial order in which no element is related to any other (cf. 'chain')

causet = causal set = locally finite order

chain = linear order

> an order, any two of whose elements are related. In particular, any linearly ordered subset of an order is a chain. n-chain = chain of n elements

child/parent

> If $x < y$ is a link we can say that x is a parent of y and y a child of x. One also says that "y covers x".

comparable

> see 'related'

covering relation, covers

> see 'child', 'link'

descendant

> see 'ancestor'

down-set = downward-set = past-set = order-ideal = ancestral set

> a subset of an order that contains all the ancestors of its members

full stem

> A partial stem whose complement is the (exclusive) future of its top layer

future

> see 'past'

inf = greatest lower bound (cf. 'sup')

interval

> see 'order-interval'

level

> In a past-finite causet the level of an element x is the number of links in the longest chain $a < b < ... < c < x$. Thus, level 0 comprises the minimal elements, level 1 is level 0 of the remainder, etc.

linear extension

> Let S be a set and $<$ an order-relation on S. A linear extension of $<$ is a second order-relation $\prec$ which extends $<$ and makes S into a chain.

link = covering relation

> An irreducible relation of an order, that is, one not implied by the other relations via transitivity. Of course we exclude pairs (xx) from being links, in the case where such pairs are admitted into the order relation at all.

> (There's no inconsistency here with the notion that a "chain" ought to be made up of "links": the links in a chain are indeed links relative to the chain itself, even if they aren't links relative to the enveloping order.)

locally finite

> An order is locally finite iff all its order-intervals are finite. (cf. 'past-finite')

maximal/minimal

> A maximal/minimal element of an order is one without descendants/ancestors.

natural labeling

> A natural labeling of a past-finite order is an assignment to its elements of labels 0 1 2 ... such that $x < y \Rightarrow label(x) < label(y)$. Thus it is essentially the same thing as a locally finite linear extension.

order-interval (or just plain interval)

> The interval determined by two elements a and b is the set, $interval(a, b) = \{x | a < x < b\}$.

order = ordered set = poset = partially ordered set = partial order

> An order is a set of elements carrying a notion of "ancestry" or "precedence". Perhaps the simplest way to express this concept axiomatically is to define an order as a transitive, irreflexive relation $<$. Many other, equivalent definitions are possible. In particular, many authors use the reflexive convention, in effect taking $\leq$ as the defining relation.

> It is convenient to admit the empty set as a poset.

order-isomorphic

> isomorphic as posets

origin = minimum element

    a single element which is the ancestor of all others

originary

    a poset possessing an origin is originary

parent

    see 'child'

partial stem (or just plain stem)

    a past set of finite cardinality

partial post

    An element x of which no descendant has an ancestor spacelike to x. The idea is x is the progenitor of a "child universe"

partially ordered set

    see 'order'

past/future

    $\text{past}(x) = \{y|y < x\}$, $\text{future}(x) = \{y|x < y\}$

past-finite

    An order is past-finite iff all its down-sets are finite. (cf. 'locally finite')

path = saturated chain

    a chain all of whose links are also links of the enveloping poset (saturated means it might be "extended" but it can't be "filled in")

poset

    see 'order'

post

    An element such that every other element is either its ancestor or its descendant: a one-element slice.

preorder = preposet = acyclic relation = acyclic digraph

    A preorder is a relation whose transitive closure is an order.

pseudo-order = transitive relation (possibly with cycles)

related = comparable

    Two elements x and y are 'related' (or 'comparable') if $x < y$ or $y < x$.

slice = maximal antichain

(where maximal means it can't be enlarged and remain an antichain)

equivalently, every x in the causet is either in the slice or comparable to one of its elements.

equivalently, its inclusive past is a full stem

spacelike = incomparable

Two elements x and y are spacelike to each other ($x \natural y$) iff they are unrelated (ie neither $x < y$ nor $y < x$).

stem

see 'partial stem'

sup = least upper bound (cf. 'inf')

transitively reduced

The "transitive reduction" of an order is its Hasse digraph, an acyclic relation containing only links.