Indecomposable Ideals in Incidence Algebras[⋆]

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Abstract

The elements of a finite partial order P can be identified with the maximal indecomposable two-sided ideals of its incidence algebra $\mathfrak{A}$, and then for two such ideals, $I \prec J \iff IJ \neq 0$. This offers one way to recover a poset from its incidence algebra. In the course of proving the above, we classify all of the two-sided ideals of $\mathfrak{A}$.

In contemporary physical theory, the concept of a "space" or, more formally, of a set with structure, plays a central role. Most notably, spacetime itself is conceived of in this manner – as a differentiable manifold. However, one can observe a certain tension between two ways of conceptualizing such structures and working with them. Tangent vectors, for example, can be thought of either as infinitesimal displacements or as sets of numbers obeying a linear transformation law. In present day language, the two opposed tendencies of thought can to some extent be characterized by the words "geometrical" and "algebraic", although neither term is really suitable. Perhaps, "intrinsic" vs "coordinate based" comes closer; and sometimes the words "synthetic" and "analytic" have been used to convey the same opposition (as in synthetic versus analytic geometry).[†]

Consider, for example, Minkowski spacetime $\mathbb{M}^4$. From the "intrinsic" side it can be understood, on one hand, as a topological space of dimension four supporting such

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[†] In the philosophical world, these two attitudes manifest themselves to some degree as "materialism" and "instrumentalism" although the correspondence is obviously very imperfect (cf. the oft stated idea that an instrument reading is always a number.)

concepts as straight line (inertial motion), light cone, and parallelogram. Or by focusing on its causal relationships rather than its metric and topological ones, one can understand $\mathbb{M}^4$ as a certain partially ordered set (poset), a second "intrinsic" characterization that is nevertheless very different from the first. In contrast to both these characterizations, $\mathbb{M}^4$ would be described from the "coordinate based" side by four real variables t, x, y, z which geometrically have the meaning of numerical functions on spacetime. Here, the algebraic relationships among the four variables take center stage, while the actual elements of the space (the points) withdraw into the background.

Of course the "intrinsic" and "coordinate based" descriptions of $\mathbb{M}^4$ are mathematically equivalent. The most highly developed and general instance of this sort of equivalence is the Gel'fand isomorphism, which implies in particular that any manifold can be recovered, as a topological space, from the C^*-algebra of scalar functions that it supports (in effect its coordinate functions). However, a manifold *per se* is not yet a spacetime because it lacks metrical information. To recover that as well, one can proceed as in [1] or following the more detailed scheme of [2]. Neither approach captures in any essential manner the *Lorentzian* character of the metric, however. (Indeed, the latter scheme is actually incompatible with Lorentzian signature.) The question thus arises whether there exists a similar "algebraization" of spacetime based not on its topological and metrical attributes but on its causal order.

The finding of such a correspondence could be expected to hold interest for more than one reason. On one hand, some workers, going back to [3], have viewed algebraization as potentially a means by which to introduce a fundamental spacetime discreteness, a view which appears to account for much of the current interest in "non-commutative geometry". On the other hand, algebraization has from the outset been one of the royal roads to the "quantization" of a theory, so that one might hope that any new equivalence between intrinsic and algebraic descriptions of spacetime – or of whatever hypothetical substratum one takes to replace spacetime – would open up new avenues for building a theory of quantum gravity. It is this second prospect that primarily animates the considerations of the present paper, which are inspired by the hypothesis that the deep structure of spacetime is that of a *causal set* [4] [5]. Since this structure is already inherently discrete, there is no need to *introduce* discreteness and therefore no reason to appeal to algebraization on that score. Nonetheless, one may still feel it useful to attempt various algebraic reformulations of causal set kinematics in the hope that one of them might help lead us to the correct quantum theory of causal set dynamics. I will have a bit more to say about this in the

conclusion, but for now let us turn to the mathematical question that will primarily concern us herein: that of finding a suitable "algebraization" of the poset concept.

As I remarked earlier, a relativistic spacetime is inherently a partial order [♭] (at least to the extent that one can count on the impossibility of "time travel"). A causal set is also a partial order, but with the crucial difference that it is *locally finite*.[★] Rather than seeking an algebra to capture the structure of an arbitrary poset, then, let us confine ourselves to the simpler case of locally finite orders; in fact let us simplify still further to the case of orders whose cardinality is strictly finite.

So let C be a poset of finite cardinality. Does there exist an algebra $\mathfrak{A}$ naturally associated to C and from which, conversely, C can be recovered? One algebra that people have studied in this connection is the so called *incidence algebra* of C, which one might view as an algebra of retarded (or, dually, advanced) functions on $C \times C$, the product being given by convolution. In the finite case, this is just a matrix algebra, or more accurately, it is the algebra of all matrices with zeros in certain specified locations which reflect directly the defining order relation $\prec$ of C.

It is known that the incidence algebra does indeed capture the structure of C, at least if one interprets $\prec$ reflexively in the sense that the diagonals of the matrices representing the elements of $\mathfrak{A}$ are left free. Remarkably, the nature of the space $\leftrightarrow$ algebra correspondence in this case can be arranged to be closely analogous to that of the Gel'fand isomorphism, even though the algebra which figures in the latter is commutative and semisimple, whereas an incidence algebra is not only non-commutative, but almost nilpotent (which is about as far from semi-simplicity as one can get). Despite these differences, one can in both cases choose to identify the elements of the underlying space with the maximal two-sided ideals of $\mathfrak{A}$, as explained in [9] and [10].

[♭] *Order*, *partial order*, *partially ordered set*, *poset*, and *ordered set* are all synonyms.

[★] A poset is called locally finite if all its *intervals* are finite. If this is strengthened to the requirement that its "upward" and "downward" sets be separately finite, the poset becomes suitable to represent a "finitary topological space". Posets of this type have been the subject of much work by the dedicatee and his co-workers, but in a spatial context rather than a spacetime one [6]. (See also [7] [8]). The results to be proven below will of course apply equally well in this context.

As we will see in a moment, however, there is quite a different way to set up a correspondence between C and $\mathfrak{A}$ in the poset case, and this is perhaps fortunate, in that one might feel uncomfortable with certain features of the Gel'fand-like scheme as adapted to partial orders. A first concern arises from the circumstance that it is not the fundamental order relation $\prec$ as such that one directly recovers between elements regarded as maximal ideals, but rather the "nearest neighbor" or *link* relation [†] [9] [11]. The full precedence relation must then be re-generated via transitive closure. In the purely discrete case, this is always possible, so viewing one relation as more or less "fundamental" than the other seems largely a matter of taste. However, the failure to recover $\prec$ directly could bode ill in a continuum context where the simple precedence relation continues to make sense but links no longer exist (since between any two causally related points, one can always interpolate a third). Moreover, even for finite posets C, the correspondence between elements and maximal ideals falls apart if one adopts the irreflexive convention [♭] for C, and this is disturbing because one would hope that a choice of convention would not influence the underlying relationships so strongly. Again, one might hope that some other scheme would be more robust in this regard.

In view of such doubts, it seems worth exploring other ways to recover a poset from its incidence algebra, a task we begin here by showing that it is equally possible to identify the poset elements, not with the *maximal* ideals, but with certain *indecomposable* ideals; and by doing so one obtains the precedence relation directly as a relation between the corresponding ideals. Of course, it would also make sense to explore alternatives to the incidence algebra itself, but I will not attempt that in this paper.

Some Definitions

A *finite order* is a set C comprising a finite number of elements and carrying a "precedence relation" $\prec$ which is transitive and asymmetric. That is, for arbitrary elements x, y, z of C we always have $x \prec y \prec z \Rightarrow x \prec z$, and we never have $x \prec y \prec x$ when x and y are distinct. In addition we will always assume, unless stated otherwise, that $\prec$ is *reflexive*, i.e. that every element x precedes itself: $x \prec x$. Although, this is in some sense merely

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[†] Also called "covering relation" in the mathematical literature.

[♭] defined below

a convention that one can adopt or reject at will, it turns out to influence profoundly the structure of the incidence algebra. [*]

We can now define the *incidence algebra* of C by introducing for each related pair $x \prec y$ a generator $[xy]$ and taking $\mathfrak{A}$ to be the set of formal linear combinations of these generators. The order structure of C is then further encoded in the rule for multiplication of algebra elements as given by the relation $[xy] \cdot [yz] = [xz]$, for all triples $x \prec y \prec z$. For definiteness, we will take the field of scalars to be the complexes $\mathbb{C}$, although nothing will depend on this.

Two other representations of the incidence algebra are useful and (for finite C) strictly equivalent to the definition just given. First one may think of $\mathfrak{A}$ as an algebra of $n \times n$ *matrices*, where $n = |C|$ is the cardinality of C and the generator $[xy]$ corresponds to the matrix with a single 1 in row x, column y and zeros everywhere else. [†] The asymmetry of the relation $\prec$ then translates into the fact that if one chooses a suitable labeling for the elements of C (a so called *natural labeling*) then all the matrices representing members of $\mathfrak{A}$ are upper triangular (but not strictly so, inasmuch as diagonal generators like $[xx]$ are also part of $\mathfrak{A}$ thanks to our standing assumption that $\prec$ is reflexive). A slightly different representation of $\mathfrak{A}$ comes from thinking of the entries of the matrix as the values of a "two-point function" $f : C \times C \to \mathbb{C}$. With this representation a member of the incidence algebra is an arbitrary *advanced* function and the algebra product is convolution: $f = g \cdot h \iff f(x,z) = \sum_y g(x,y)h(y,z)$. Of course this is just the formula for matrix multiplication in a slightly different notation.

By an *ideal I* of $\mathfrak{A}$, I will always mean, in this paper, a two-sided ideal, in other words a nonempty subset of $\mathfrak{A}$ closed under addition, scalar multiplication, and left or right multiplication by an arbitrary element of $\mathfrak{A}$. Note in this connection that, by virtue of our choice of reflexive convention for C, $\mathfrak{A}$ is automatically "unital": it has an identity element given by $1 = [xx] + [yy] + [zz] + \cdots$, where the sum extends over all elements of C.

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[*] For example, with the reflexive convention, $\mathfrak{A}$ has in general more idempotents than C has elements, whereas with the opposite, *irreflexive* convention, $\mathfrak{A}$ has no idempotents at all.

[†] In the Dirac notation, this correspondence reads $[xy] = |x\rangle\langle y|$, a notation that was used in [9].

Consequently there is no need to distinguish, for example, "regular ideals" from irregular ones [12].

The *sum* $I_1 + I_2$ of two ideals I_1 and I_2 is the collection of all sums $a_1 + a_2$ where $a_1 \in I_1$ and $a_2 \in I_2$. Equivalently $I_1 + I_2$ is the least ideal containing both I_1 and I_2: it is their "join" in the lattice of ideals of $\mathfrak{A}$. An *indecomposable ideal* in $\mathfrak{A}$ is then a non-zero ideal that cannot be expressed as the sum of two ideals distinct from itself. (In the language of lattice theory, such an ideal is said to be "join irreducible". A closely related notion was employed in the definitions of TIP and TIF in [13].) By a *maximal indecomposable ideal* I will mean an indecomposable ideal that is contained properly in no other indecomposable ideal. One also defines simply a *maximal ideal* $I \neq \mathfrak{A}$ as one which cannot be enlarged without coinciding with $\mathfrak{A}$. (Thus, a maximal ideal is a maximal element in the family of all ideals not equal to $\mathfrak{A}$, while a maximal indecomposable ideal is a maximal element in the family of all indecomposable ideals.)

The *product* IJ of two ideals I and J is the collection of sums of products of members of I with members of J; in other words IJ is the linear span of $\{xy \,|\, x \in I, y \in J\}$. From the definitions it is immediate that IJ is also an ideal and is contained in both I and J. It is, however, not in general equal to their intersection $I \cap J$, which would be their "meet" in the lattice of ideals.

Finally, we define within an arbitrary order P a *downward set* as a subset $D \subseteq P$ that is closed under "taking of pasts": $x \prec y \in D \Rightarrow x \in D$. An *upward set* U is defined dually. (In spacetime language, these could be called, respectively, "past sets" and "future sets".)

Recovery of the poset from its incidence algebra

Our main result can be stated as a theorem:

THEOREM Let C be a finite order (with reflexive convention) and let $\mathfrak{A}$ be its incidence algebra. Then the elements x of C correspond bijectively with the maximal indecomposable ideals I of $\mathfrak{A}$, and under this correspondence the relation $x_1 \prec x_2$ goes over to $I_1 I_2 \neq 0$.

Before proving the theorem, let us notice a sense in which this (or any other) equivalence between a poset and its incidence algebra is somewhat less satisfactory than the

Gel'fand isomorphism mentioned earlier, the flaw being that $\mathfrak{A}$ has in general more auto-morphisms than C has. [♭] Although the isomorphism equivalence class of $\mathfrak{A}$ can be deduced from that of C and conversely, there is thus some "looseness" in the correspondence that is not present in the Gel'fand case. Possibly related is the failure, pointed out in [14], of the correspondence between a poset and its incidence algebra to be functorial between the category of finite orders with isotone mappings and the category of algebras with algebra homomorphisms. [⋆]

The ideals of $\mathfrak{A}$

We will prove the above theorem by classifying the ideals of $\mathfrak{A}$. To this end, let us notice that every member of $\mathfrak{A}$ has a unique expression as a sum of multiples of the generators $[xy]$ that we defined earlier. To convey the fact that one particular such generator $[xy]$ occurs with a nonzero coefficient in some member of the ideal I, let me say, for lack of a better word, that $[xy]$ "figures in I". In contrast, the statement that $[xy]$ "is an element of I" (in symbols, $[xy] \in I$) means that some $A \in I$ literally coincides with $[xy]$. In the matrix representation of $\mathfrak{A}$, a generator $[xy]$ corresponds to a particular location in the matrix. It then "figures in" I if some matrix of I has a nonzero coefficient in that location, whereas it is an element of I if some matrix of I has a 1 in that location and zeros everywhere else. In these terms we can now state a key lemma.

LEMMA If some $[xy]$ "figures in" the ideal I then it is actually an element of I

[♭] Because the algebra $\mathfrak{A}$ is not commutative, it will in general possess continuous fam-ilies of inner automorphisms, whereas C, being a finite set, can have at most finitely many automorphisms. On the other hand, it looks as if the superfluity due to the inner automorphisms might be the only one. That is, it looks as if we might have $\mathrm{Aut}(C) \simeq \mathrm{Outer}(\mathfrak{A}) := (\mathrm{Aut}\,\mathfrak{A})/(\mathrm{Inner}\,\mathfrak{A})$, where $(\mathrm{Inner}\,\mathfrak{A})$ is the group of inner auto-morphisms of $\mathfrak{A}$.

[⋆] This is not necessarily a "failing" in itself. For example, the association to a manifold M of its *tensor* algebra is not functorial, nor (as remarked to me by Chris Isham) is the association to M of its diffeomorphism group. Nevertheless, for certain purposes the lack of functoriality can be a problem, e.g. if one wished to reproduce the limiting process of [7] in terms of incidence algebras.

PROOF By assumption there is $A \in I$ such that $A = \alpha[xy] + B$ where α is a nonzero scalar and B is a sum of multiples of pairs $[uv]$ such that either u differs from x or v from y. But this means that $[xx]B[yy] = 0$ because the same holds for every one of its constituent pairs $[uv]$. Hence $[xx]A[yy] = \alpha[xx][xy][yy] = \alpha[xy]$ is a multiple of $[xy]$, and this implies immediately that $[xy]$ itself belongs to I by the definition of an ideal. (Namely $[xy] = (1/\alpha)[xx]A[yy]$ must be an element of I if A is one.)

COROLLARY Every ideal I of $\mathfrak{A}$ is the set of all linear combinations of some unique set $\mathcal{U}(I)$ of generators $[xy]$.

That is, we have for every ideal of the incidence algebra, $I = \operatorname{span}\mathcal{U}(I)$, where $\mathcal{U}(I) = \{[x_1y_1], [x_2y_2], [x_3y_3], \cdots [x_ny_n]\}$, with the $[x_jy_j]$ being uniquely determined by I and conversely. It is moreover, easy to figure out which sets of pairs can belong to an ideal in this way. Let $S = \mathcal{U}(I)$ be such a set. If $[xy] \in S$ and $u \prec x \prec y \prec v$ then $[uv] = [ux][xy][yv] \in I$ and therefore $[uv] \in S$. Thus S is necessarily closed under the process of "passing to nested pairs". To express this succinctly let us formally introduce this "nesting" as an order relation among pairs.

DEFINITION $[xy] \ll [uv] \iff u \prec x$ and $y \prec v$

This definition makes the set of all pairs $[xy]$ into a poset Γ, and with reference to this auxiliary poset, we see that the sets $\mathcal{U}(I)$ are precisely the *upward sets* of Γ. Thus we have proved:

THEOREM The ideals of $\mathfrak{A}$ correspond bijectively with the upward sets of the poset Γ of pairs $[x, y]$

Moreover, the relation of inclusion between ideals obviously mirrors exactly the relation of inclusion between the corresponding upward sets. In particular, we have

LEMMA $\mathcal{U}(I + J) = \mathcal{U}(I) \cup \mathcal{U}(J)$

PROOF Recall that $I + J$ is precisely the smallest ideal including both I and J; and notice that the union of two upward sets is also an upward set.

It follows immediately that an ideal I is indecomposable iff its corresponding upward set, $\mathcal{U}(I)$, is not the union of two upward sets distinct from itself. But in any (finite) poset an upward set has this property iff it is (empty or) what might be called a *principal upward set*, that is iff it has the form $\{\eta \mid \xi \ll \eta\}$ for some ξ. We have thus shown:

THEOREM   The *indecomposable ideals* of $\mathfrak{A}$ correspond bijectively with the pairs $[xy]$, $x, y \in C$, $x \prec y$

From this we can easily conclude that the *maximal* indecomposable ideals correspond precisely with the principal upward sets of *minimal* elements of Γ, which in turn are clearly the diagonal pairs $[xx]$, for $x \in C$. In other words:

LEMMA   An ideal $I \subseteq \mathfrak{A}$ is *maximal indecomposable* $\iff I = \mathfrak{A}[xx]\mathfrak{A}$ for some $x \in C$

This substantiates the first assertion of our main theorem. [†] In order to complete the proof, we have only to verify that $I_x I_y \neq 0$ iff $x \prec y$, where I've written I_x for $\mathfrak{A}[xx]\mathfrak{A}$. But this is completely straightforward. In fact, we have:

LEMMA   Let $\mathfrak{I}(x,y) = \mathfrak{A}[xy]\mathfrak{A}$ denote the principal ideal generated by $[xy]$. Then the product $\mathfrak{I}(x,y)\mathfrak{I}(u,v)$ equals $\mathfrak{I}(x,v)$ when $y \prec u$, and zero otherwise.

PROOF   Write $\mathfrak{B}$ for the product in question. Because $\mathfrak{A}$ is unital, $\mathfrak{A}\mathfrak{A} = \mathfrak{A}$, whence $\mathfrak{B} = \mathfrak{A}[xy]\mathfrak{A}\mathfrak{A}[uv]\mathfrak{A} = \mathfrak{A}[xy]\mathfrak{A}[uv]\mathfrak{A}$. Now, $[xy]\mathfrak{A}[uv] \neq 0 \iff [yu] \in \mathfrak{A} \iff y \prec u$, and in that case, clearly, $\mathfrak{B} = \mathfrak{A}[xy][yu][uv]\mathfrak{A} = \mathfrak{A}[xv]\mathfrak{A}$.

COROLLARY $\mathfrak{I}(x,x)\mathfrak{I}(y,y)$ equals $\mathfrak{I}(x,y)$ when $x \prec y$, and zero otherwise.

Our main theorem is thus demonstrated. Along the way, we have seen that every indecomposable ideal is a principal ideal (of some pair $[xy]$). We have also found (in view of the most recent lemma) an equivalent way to characterize the maximal indecomposable ideals: An indecomposable ideal I is maximal iff it is *idempotent* in the sense that $II = I$ (which happens iff $II \neq 0$).

Finally, it bears remarking that the last lemma actually furnishes an order on the full set of indecomposable ideals. Since every such ideal has the form $\mathfrak{A}[xy]\mathfrak{A}$ for some $[xy]$,

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[†] Very similar reasoning yields the "Gel'fand correspondence" between maximal ideals and elements of C, since a maximal upward set in Γ is precisely the complement of a single minimal element of Γ, i.e. (as we have just observed) of a single pair $[xx]$. However, if our only purpose were to classify the maximal ideals of $\mathfrak{A}$, it would be simpler just to adopt the representation of an element of $\mathfrak{A}$ as a matrix and pay attention to the set of zeros on its diagonal.

the substrate of this order can be taken to be the set of pairs $^\flat$ $x \prec y$ of elements of C and its precedence relation is then $[xy] \prec [uv] \iff y \prec u$. (Such an order is often called an "interval order".) It is actually this order that we recover most directly from the algebra $\mathfrak{A}$. Interestingly enough, it is neither reflexive nor irreflexive.* Rather, it is precisely its reflexive elements that correspond to elements of the underlying poset C.

Remarks

With the proof of our main theorem, we are in the curious position of being able to discern two very different algebraic images of our original poset C within its incidence algebra $\mathfrak{A}$. On one hand, we can identify the element $x \in C$ with the ideal *generated by* $[xx]$, on the other hand with the ideal of all algebra elements *omitting* $[xx]$. Depending on which possibility one selects, one recovers from the ideals either the relation $\prec$ or its associated link relation, respectively. Either way, one can conclude that the full structure of C is captured by $\mathfrak{A}$.

However, with the second scheme, this conclusion rests heavily on our assumption that C is finite, which (trivially) guarantees enough discreteness so that any two related elements of C can be joined by a chain of links. In a continuum such as a Lorentzian manifold (or for a countable dense set thereof), this is assuredly not the case because no links are present at all. Hence the construction which recovers $\prec$ directly (that based on maximal indecomposable ideals) seems more robust from this point of view. It might be interesting to test whether this is indeed true by generalizing $\mathfrak{A}$ to, say, a globally hyperbolic spacetime M and asking whether some suitable analog of the construction of this paper would give back the metric of M. (This would require one to recover the volume density $\sqrt{-g}$ in addition to the causal order of M, but that is not obviously impossible, since $\sqrt{-g}$ does enter into $\mathfrak{A}$, via the definition of the convolution product.)

Of course, it would also be interesting to investigate possible inter-relationships between the two schemes for recovering the elements of C. Perhaps they could already be

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$^\flat$ Thus, the order in question shares its substrate with the order Γ. The two precedence relations are obviously very different, however.

* providing a counter-argument to the opinion that the choice between the two possibilities is always purely conventional!

seen to be equivalent at the algebraic level, or perhaps they complement each other in some way.

Concerning physical applications of the duality between a poset and its incidence algebra, there seems not much to say at present. If one is thinking in terms of quantum gravity and causal sets, then trading a causal set C for its incidence algebra $\mathfrak{A}$ is not necessarily a step in the right direction, because a quantal "sum over causal sets" seems easier to imagine (as in [5]) than a corresponding "sum over algebras $\mathfrak{A}$". On the other hand, the physical causal set C is not the only poset to figure in the theory. The "poset of stems" played an important role in the considerations of [5], where it served as a fixed arena allowing one to define conveniently the Markov process of "classical sequential growth". (This poset is illustrated in Figure 1 of [5]. Its elements are the finite orders, and its precedence relation is "inclusion as a downward set".) Perhaps the incidence algebra of *this* poset could play a role in the search for a physically appropriate dynamics of "quantal sequential growth". Such a dynamics would provide the quantal analog for causal sets of the classical Einstein equations for continuum Lorentzian manifolds. That is, it would provide a theory of quantum gravity.

Returning to the realm of pure mathematics for a moment, one can ask whether algebras other than the incidence algebra might have a role to play in the "algebraization" of the poset concept. If so, the new algebra might be constructed either from the incidence algebra or directly from the elements of the poset. Here, just to illustrate the type of thing one could consider, is an algebra of the second sort. It might even be trivial as far as I know, but at least it is not as obviously trivial as some similar possibilities I played with first! One takes for generators the elements of C itself, and one imposes three sets of relations: (i) $a \prec b \prec c \Rightarrow abc = ac$; (ii) $a \neq b \Rightarrow aba = 0$; (iii) $a \neq b$, $a \prec b \Rightarrow ba = 0$. If this construction is worthy of further consideration, one might begin by asking whether it is functorial and how the resulting algebra's automorphism group relates to that of C.

11

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