gr-qc/yymmnnn
SU–GP–2002/12–1

Black Hole Entropy as Causal Links [1]

Djamel Dou
Institute of Exact Science and Technology, University Center of Eloued, Eloued, Algeria
internet email address: djsdou@yahoo.com

Rafael D. Sorkin
Department of Physics, Syracuse University, Syracuse, NY 13244-1130, U.S.A.
internet email address: sorkin@physics.syr.edu

Abstract

We model a black hole spacetime as a causal set and count, with a certain definition, the number of causal links crossing the horizon in proximity to a spacelike or null hypersurface Σ. We find that this number is proportional to the horizon's area on Σ, thus supporting the interpretation of the links as the "horizon atoms" that account for its entropy. The cases studied include not only equilibrium black holes but ones far from equilibrium.

1 Introduction

Despite all the evidence for an entropy associated with the horizon of a black hole, a full understanding of its statistical origin is still lacking and it remains uncertain what "degrees of freedom" the entropy refers to.[2] Ideally one would appeal for the answer to some more fundamental quantum theory of spacetime structure, but unfortunately no approach to constructing such a "quantum gravity" theory has advanced far enough to offer a definitive account of what the horizon "degrees of freedom" might be. Nevertheless, it is hard to doubt that black hole thermodynamics has opened up a path leading

[1] To appear in a special issue of *Foundations of Physics* in honor of Jacob Bekenstein, "Thirty years of black hole physics", edited by L. Horwitz.

[2] The question is nicely posed in [1].

to a better knowledge of the small scale structure of spacetime. Indeed, the role being played by black hole thermodynamics in this connection looks more and more analogous to the role played historically by the thermodynamics of a box of gas in revealing the underlying atomicity and quantum nature of everyday matter and radiation. We can bring out this analogy more clearly by recalling some facts about thermodynamics in the presence of event horizons.

One often thinks of entropy as measure of missing or "unavailable" information about a physical system, and from this point of view, one would have to expect some amount of entropy to accompany an event horizon, since it is by definition an information hider *par excellence*. In particular, one can associate to each quantum field in the presence of a horizon the "entanglement entropy" that necessarily results from tracing out the interior (and therefore inaccessible) modes of the field, given that these modes are necessarily correlated with the exterior modes. In the continuum, this entanglement entropy turns out to be infinite, at least when calculated for a free field on a fixed, background spacetime. However, if one imposes a short distance cutoff on the field degrees of freedom, one obtains instead a finite entropy; and if the cutoff is chosen around the Planck length then this entropy has the same order of magnitude as that of the horizon. Based on this appealing result, there have been many speculations attributing the black hole entropy to the sum of all the entanglement entropies of the fields in nature.

Whether or not the entanglement of quantum fields furnishes all of the entropy or only a portion of it, contributions of this type must be present, and any consistent theory must provide for them in its thermodynamic accounting. The case appears to be similar to that of an ordinary box of gas, where we know that, fundamentally, the finiteness of the entropy rests on the finiteness of the number of molecules, and to lesser extent on the discreteness of their quantum states. Indeed, at temperatures high enough to avoid quantum degeneracy, the entropy is, up to a logarithmic factor, merely the number of molecules composing the gas. The similarity with the black hole becomes evident when we remember that the picture of the horizon as composed of discrete constituents gives a good account of the entropy if we suppose that each such constituent occupies roughly one unit of Planck area and carries roughly one bit of entropy. A proper statistical derivation along these lines would require a knowledge of the dynamics of these constituents, of course. However, in analogy with the gas, one may still anticipate that the horizon entropy can be estimated by counting suitable discrete structures, analogs of the gas molecules, without referring directly to their dynamics.

Clearly, this type of estimation can succeed only if well defined, discrete entities can be identified which are available to be counted. Within a continuum theory, it is hard to think of such entities. Indeed, if one accepts the estimates carried out below, the entropy would come out infinite were spacetime a true continuum. It would diverge with the cutoff at the same rate as the aforementioned entropy of entanglement of an ambient quantum field. In causal set theory, on the other hand, the elements of the causal set serve as "spacetime atoms", and one can ask whether these elements, or some related structures, are suited to play the role of "horizon molecules". In this paper, we will identify a certain kind of "causal link" as one such structure and we will show that the black hole entropy can be equated to the number of such links crossing the horizon H in proximity to the hypersurface Σ for which the entropy is sought. Moreover, almost all of these links will turn out to be localized very near to H. In consequence, conditions deep inside the black hole will become irrelevant to the counting, as indeed they must do if any interpretation of the entropy in terms of "horizon degrees of freedom" is to succeed.

2 Counting Links

Before proceeding, let us briefly review the terminology we will use. For a fuller introduction to causal sets, see [2] and references therein.

A *causal set* (or "causet") is a locally finite, partially ordered set. We use $\prec$ to represent the order relation and adopt (in this paper) the reflexive convention, according to which every element precedes itself: $x \prec x$. Let C be a causet and let x and y be elements of C. The past of x is the subset $\mathrm{past}(x) = \{y \in C \,|\, y \prec x\}$ and its future is $\mathrm{future}(x) = \{y \in C \,|\, x \prec y\}$. If $x, y \in C$, $x \prec y$, and $\mathrm{future}(x) \cap \mathrm{past}(y) = \{x, y\}$ then we call the relation $x \prec y$ a *link*. Note that (thanks to the local finiteness) if the links of a causet are given, then all the other relations are implied by transitivity; hence the whole structure of the causet is encoded in its irreducible relations or links. An element of a causet (or of a subcauset) is *maximal* (resp. *minimal*) iff it is to the past (resp. future) of no other element in the causet (or subcauset).

Now the basic hypothesis of causal set theory is that spacetime, ultimately, is discrete, and its deep structure is that of a partial order rather than a differentiable manifold. The macroscopic spacetime continuum of experience must then be recovered as an approximation to the causet. Although a more sophisticated notion of approximation might ultimately be needed [3], the intuitive idea at work here is that of a "faithful embedding". If M is a Lorentzian manifold and C a causal set, then a faithful embedding of C into M is an injection $f : C \to M$ of the causet into the manifold that satisfies the following requirements: (1) The causal relations induced by the embedding agree with those of C itself, i.e $f(x) \in J^-(f(y))$ iff $x \prec y$, where $J^-(p)$ stands for the causal past of p in M; (2) The embedded points are distributed with unit density, and (3) the characteristic length over which the geometry varies appreciably is everywhere much greater than the mean spacing between the embedded points. When these conditions are satisfied, the spacetime M is said to be a continuum approximation to C. From the point of view of an M, the causet resembles a "random lattice" obtained by "sprinkling in points" until the required density is reached. Thus, the probability that there will be n embedded points in a given volume V is given by the Poisson distribution, $(\varrho_c V)^n e^{-\varrho_c V}/n!$, where the fundamental density ϱ_c is unknown but presumed to be of Planckian magnitude.

Let us now consider the entropy associated with a horizon in a spacetime M in which a causet C is faithfully embedded. As discussed in the introduction, we expect that the entropy can be understood as entanglement in a sufficiently generalized sense, and we may hope to estimate its leading behavior by counting suitable discrete structures that measure the potential entanglement in some way. At the same time, we know that the entropy essentially just measures the horizon area, whence, phenomenologically, our discrete structures must turn out to be equal in number to the horizon area, up to small fluctuations.[3] From both points of view, a natural candidate for the structure we seek is a *link* of the causet. Indeed, we may think heuristically of "information flowing along links" and producing entanglement when it flows across the horizon during the course of the causet's growth (or "time development"). Since links are irreducible causal relations (in some sense the building blocks of the causet), it seems natural that by counting links between elements that lie outside the horizon and elements that lie inside, one would measure the degree of entanglement between the two regions. Equally,

[3]In fact, it seems far from obvious that such structures must exist. If they do, then they provide a relatively simple, order theoretic measure of the area of a cross section of a null surface, and, unlike what one's Euclidean intuition might suggest, it is known that such measures are not easy to come by. For example, no one knows such a measure of spacelike distance between two sprinkled points that works in even such a comparatively simple case as a sprinkling of Minkowski spacetime [4].

it seems natural that the number of such causal links might turn out to be proportional to the horizon area, as desired.

2.1 An equilibrium black hole

Let us now consider a spherically symmetric collapsing star which produces a black hole with horizon H, and let Σ be a (null or spacelike) hypersurface on which we wish to evaluate the horizon entropy. For simplicity we shall ignore the influence of the collapse and treat the black hole metric as exactly Schwarzschild in the region of interest. If our picture is consistent, doing so cannot change anything, and we will see evidence for this further into the calculation. Thus, we will work with an eternal black hole spacetime M, as shown in Figure 1. Notice, however, that only the portion of the extended Schwarzschild spacetime that could have arisen from a collapse is to be taken into consideration (i.e. the region exhibited in the diagram).

Now let C be a causet produced by randomly sprinkling points into M with density $\varrho_c = 1$ in causal set units; by definition, then, C is faithfully embedded in M. Let x be a sprinkled point in the region $J^-(H) \cap J^-(\Sigma)$, and let y be a second sprinkled point in $J^+(H) \cap J^+(\Sigma)$. (In other words, x is outside the black hole and to the past of Σ, while y is inside the black hole and to the future of Σ.) To say that $x \prec y$ is a link of C means that the "Alexandrov interval", $J(x, y) := J^+(x) \cap J^-(y)$, is empty of sprinkled points except for x and y: no sprinkled point lies causally between x and y. Such a pair (x, y) might seem to be a good candidate for the sort of "horizon molecule" we wish to count.

In fact the counting reduces to the calculation of an integral, since, as a simple consideration shows [5], the expected number of such pairs is

$$<n> = \int_D e^{-V(x,y)} dV_x dV_y \ . \tag{1}$$

Here $V(x, y)$, whose presence serves to ensure the link condition, is the volume of $J(x, y)$, and D is the domain of integration for x and y. Unfortunately, if we impose no further conditions on x and y, then the integral (1) can be shown to diverge when Σ is spacelike. Therefore, the links we have identified cannot be the ones we want.

To help understand the meaning of this divergence, let us remember that, intuitively, we are trying to estimate, not the sum total of all "lost information", but only that corresponding "to a given time", meaning in the vicinity

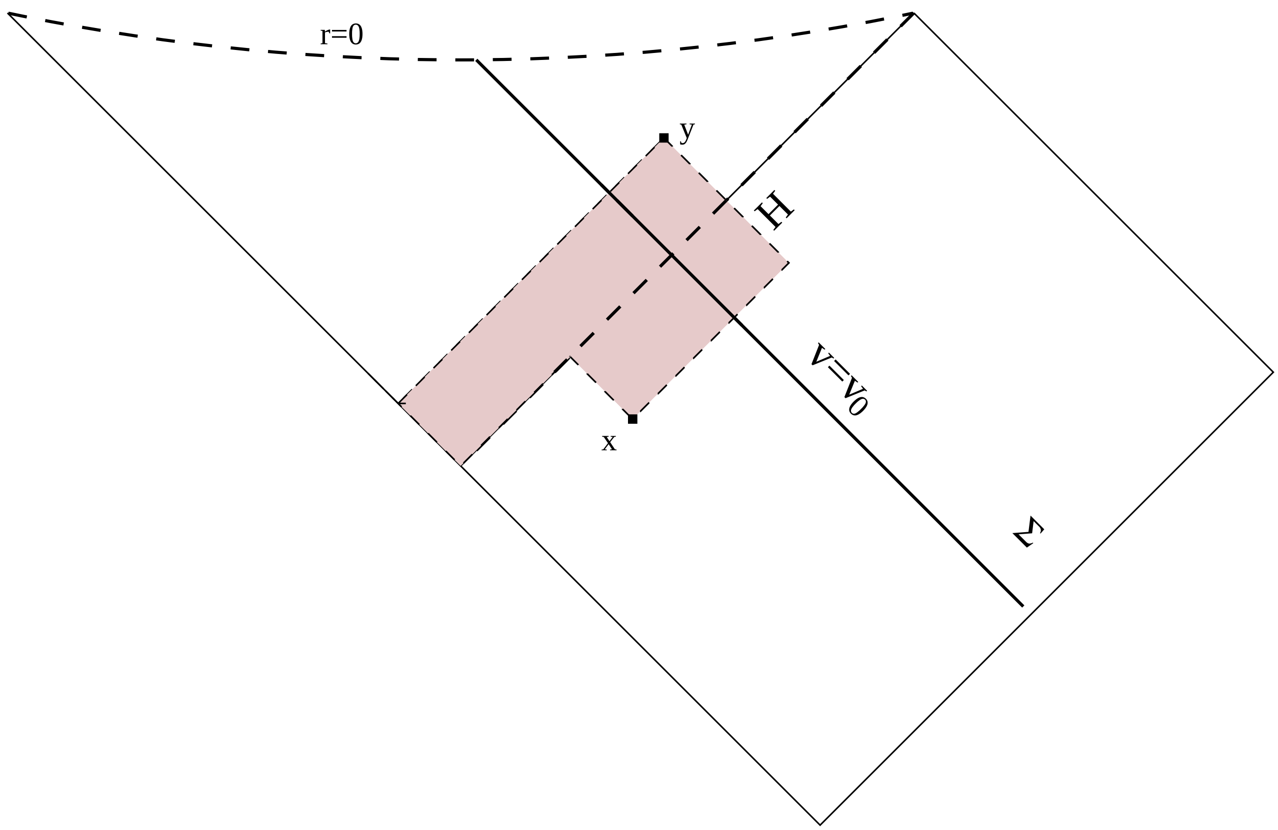

Figure 1: An equilibrium black hole and null hypersurface Σ

of the given hypersurface Σ. Hence, to associate one and the same causal link with more than one hypersurface would be to "overcount" it in forming our estimate, and it is this overcounting that seems to be the source of our divergent answer. Thus, what we need is a further condition or conditions on x and y that would be satisfied only by links that truly belong to Σ rather than to some earlier or later hypersurface. Several possibilities suggest themselves for this purpose, for example the requirement that x be *maximal* in $J^{-}(\Sigma)$, but none seems to be clearly best. Fortunately, the end result seems to be relatively insensitive to which choice one makes. The precise conditions we will use will be specified below, and the general issue will be discussed further in Section 3.

Now, ideally we would have evaluated $<n>$ for a fully four dimensional Schwarzschild black hole, but unfortunately, this is rendered difficult by the need to know all the Alexandrov neighborhoods $J(x,y)$ of the Schwarzschild metric. For this reason, we will simplify the calculation by working with a "dimensionally reduced" two dimensional metric instead of the true, four dimensional one. As the calculation proceeds, it will become very plausible that (for macroscopic black holes) the full four-dimensional answer would differ from the two-dimensional one only by a fixed (albeit still unknown) proportionality coefficient of order unity, together with a factor of the horizon area. This will effectively accomplish our primary aim of demonstrating that the expected number of links is proportional to the area of the horizon in causet units.

A radial section of a four dimensional Schwarzschild spacetime has a line element obtained by omitting the angular coordinates from the four dimensional line element, namely

$$ds^2 = -\frac{4a^3}{r} e^{-r/a} du dv \ ,$$

where a is the radius of the black hole (Schwarzschild radius) and u and v are the usual Kruskal-Szekeres coordinates, with r defined implicitly by the equation[4]

$$uv = \left(1 - \frac{r}{a}\right) e^{r/a} \ . \tag{2}$$

The associated volume element is

$$d^2V = \sqrt{-g}\, du dv = \frac{2a^3}{r} e^{-r/a} du dv \ . \tag{3}$$

Now let Σ be the ingoing null hypersurface defined by the equation $v = v_0$, and let (x,y) be a pair of sprinkled points satisfying the following conditions:

$$\left\{ \begin{array}{c} x \in J^-(\Sigma) \cap J^-(H) \\ y \in J^+(\Sigma) \cap J^+(H) \\ x \prec y \text{ a link} \\ x \text{ maximal in } J^-(\Sigma) \cap J^-(H) \\ y \text{ minimal in } J^+(H) \end{array} \right\} \tag{4}$$

[4]Our signs are such that $u \sim t - r$, $v \sim t + r$, and the horizon H will correspond to $u = 0$.

(For a null Σ in two dimensions, the fourth condition is actually redundant, but it would be needed with a spacelike Σ.) In order that these conditions be fulfilled, no sprinkled point (other than x or y) must fall into the shaded region depicted in Figure 1. The volume of this excluded region is readily evaluated and is given by

$$V = a^2 + r_{xy}^2 - r_{xx}^2 - r_{yy}^2 \ ,$$

where we have adopted the notation,

$$u_i v_j = \left(1 - \frac{r_{ij}}{a}\right) e^{r_{ij}/a} \ . \tag{5}$$

In analogy with equation (1), the expected number of links satisfying our conditions is therefore

$$<n> = \left(2a^3\right)^2 \int_0^{v_0} dv_x \int_{-\infty}^0 du_x \int_{v_0}^{\infty} dv_y \int_0^{1/v_y} du_y \frac{e^{-r_{xx}/a - r_{yy}/a}}{r_{xx} r_{yy}} \, e^{-V}$$

A change of integration variables from (u_x, v_x, u_y, v_y) to $(r_{xx}, r_{x0}, r_{xy}, r_{yy})$, followed by the notational substitutions $x = r_{xy}$, $y = r_{x0}$, $z = r_{xx}$, now reduces $<n>$ to the form,[5]

$$<n> = 4\, I(a)\, J(a) \ ,$$

where

$$I(a) = \int_a^{\infty} dx \frac{x}{x-a} e^{-x^2} \int_a^x dy \frac{y}{y-a} \int_a^y e^{z^2} dz \tag{6}$$

and

$$J(a) = e^{-a^2} \int_0^a e^{r_{yy}^2} dr_{yy} \ . \tag{7}$$

Notice that $<n>$ does not depend on v_0, reflecting the stationarity of the black hole.

Now, inasmuch as comparison with the Bekenstein-Hawking entropy is meaningful only for macroscopic black holes, we might as well assume that $a \gg 1$, and in that regime, $I(a)$ can be shown [5] to have the following asymptotic behavior:

$$I(a) = \frac{\pi^2}{12}\, a + O\left(\frac{1}{a}\right) \ .$$

[5]Here r_{x0} is of course the radial variable corresponding via (5) to the product $u_x v_0$. To avoid confusion, notice that the dummy integration variables x, y and z are real numbers entirely distinct from the sprinkled points x and y.

On the other hand it is not difficult to see that

$$J(a) = \frac{1}{2a} + O\left(\frac{1}{a^3}\right) \ .$$

Putting everything together, we end up with

$$<n> = \frac{\pi^2}{6} + O\left(1/a\right) \ . \tag{8}$$

Although our calculation has been carried out in two dimensions, a study of the integrals $I(a)$ and $J(a)$ indicates that, were we to redo it in four dimensions, the expected number of links would reduce to essentially the same expression. Indeed, the dominant contribution to the integral $J(a)$ plainly comes from $r_{yy} \approx a$, but since r_{yy} is the radial coordinate r of sprinkled point y, and since $r = a$ is the horizon, this implies that y resides near the horizon. Similarly, the dominant contribution to the integral $I(a)$ comes from $z \approx a$, which, since $z = r_{xx}$, implies in turn that sprinkled point x resides near the horizon as well. Consequently our counting can be said to be controlled by the near horizon geometry. But in four dimensions, this geometry is locally just the two dimensional one times the Euclidean plane. Thus, one would expect $<n>$ to be simply proportional to the area of the horizon. Moreover, from (8), one would expect the coefficient of proportionality to be of order unity, although there is of course no reason for it to be exactly $\pi^2/6$.

It is interesting that part of what makes the near horizon pairs special is the vanishing of the denominators in $I(a)$ when the dummy integration variables x and y tend to a. To the extent that it is this divergence which makes the horizon such a strong source for the links, we may be reminded of the analogous fact that the strong redshift in the vicinity of the horizon allows modes of arbitrarily high (local) frequency to contribute to the entanglement entropy without influencing the energy as seen from infinity. Notice also that the clustering of x and y near the horizon is not simply a consequence of the maximality and minimality conditions we imposed on them. For instance, pairs (x, y) sitting arbitrarily close to the hypersurface Σ, with y arbitrarily close to the horizon, still do not contribute to the leading term in $I(a)$ if x is far from the horizon, namely with coordinate $|u_x| \gg 1$.

2.2 A black hole far from equilibrium

Turning now to a case which, though still spherically symmetric, is very far from equilibrium, let us consider a shell of null matter which collapses to

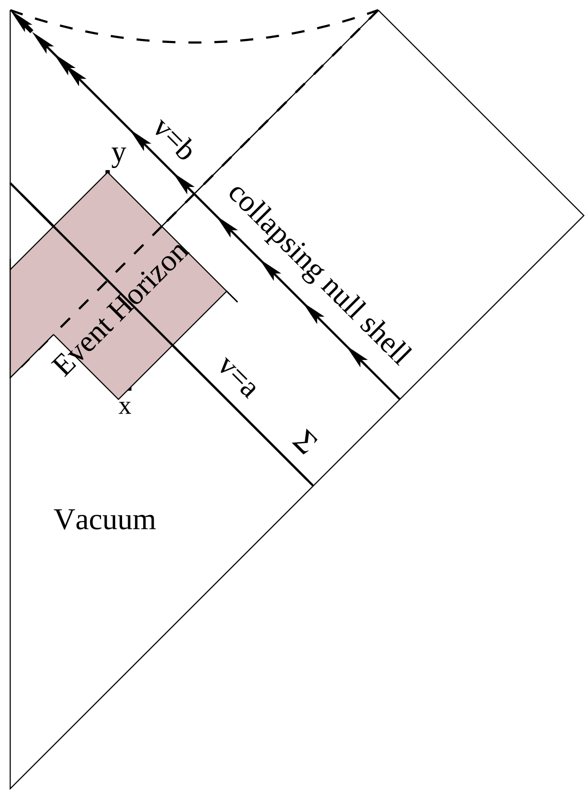

Figure 2: A non-stationary horizon and null hypersurface Σ

form a Schwarzschild black hole. The Penrose diagram for this spacetime[6] is shown in Figure 2. Let the shell sweep out the world sheet $v = b$ and let us choose for our hypersurface Σ a second ingoing null surface defined by $v = a$, with $a < b$ so that Σ lies wholly in the flat region. Here u and v are null coordinates, chosen so that the horizon first forms at $u = v = 0$ and normalized for convenience such that

$$ds^2 = -2dudv + r^2 d\Omega^2 \ .$$

Since our interest is again in macroscopic black holes, we will assume as before that the horizon radius is large in units such that $\varrho_c = 1$; and to simplify matters further, we will also restrict ourselves to a time well before

[6]A fuller description of this spacetime may be found, for example, in [6].

the infalling matter arrives (as judged in the center of mass frame). We thus have the double inequality, $b \gg a \gg 1$. Once again, we will perform the calculation for the two dimensional radial section rather than the full four dimensional spacetime.

Now since we are assuming that the infalling matter is far to the future of the hypersurface Σ, points y sprinkled into that region should not contribute significantly when our minimality and link conditions are taken into account. For this reason, we shall, for convenience, restrict our counting to pairs (x, y) with $v_y < b$. Imposing, then, the same conditions (4) introduced above, we obtain for the expected number of causal links

$$< n > = \int_a^b dv_y \int_0^{v_y} du_y \int_{-\infty}^0 du_x \int_0^a dv_x \, e^{-V}$$

where $V = u_y v_y - u_x(v_y - v_x) - u_y^2/2$.

It is not difficult to derive the leading behavior of this integral for large a, and here we quote only the final result:

$$< n > = \frac{\pi^2}{6} - l\left(\frac{a}{b}\right) + O(1/a) \ ,$$

where $l(x) \equiv \sum_{k=1}^{\infty} x^k/k^2$, a convergent series that vanishes in the limit $x \to 0$. Since we have assumed that $a \ll b$, we can write this more simply as

$$< n > = \frac{\pi^2}{6} + O(a/b) + O(1/a) \ . \tag{9}$$

Notice that the presence of a negative contribution like $-l(a/b)$ was to be expected, since we have omitted to count links that extend past the shell into the Schwarzschild region. For Σ near to the shell, one obviously should not neglect such links, and our counting is incomplete.

Two features of the result (9) are especially noteworthy. The first is its independence of the value of a. As with the equilibrium black hole above, this indicates that the analogous four dimensional computation would produce (at leading order) an answer proportional to the horizon area. What is then even more striking is the occurrence of the same numerical coefficient $\pi^2/6$ in both (8) and (9). This agreement furnishes a nontrivial consistency check of the suggestion that one can attribute the horizon entropy to the "causal links" crossing it.

Now what can one say about the case where the hypersurface Σ is spacelike? In two dimensions it can be shown that $< n >$ is again finite and independent of a to leading order. Although we have not carried the calculation far enough to verify explicitly that one obtains for it the same numerical answer $\pi^2/6$, one can make it plausible on general grounds that this would have to happen. The point is that (in the flat case) the definition of $<n>$ is manifestly Lorentz invariant, whence any spacelike plane (or in this case line) Σ must give the same answer as any other related to it by a boost. But in the limit of tilting, a spacelike line becomes null, and by continuity the corresponding $<n>$ should go over to $\pi^2/6$ in this limit. Now observe that a suitable boost transformation will convert any nearly null line Σ into one which is "purely spacelike" (and with a larger value of a). This gives a good reason to expect that both null and spacelike Σ must yield the same result. Observe also, that a similar argument can be made for the Schwarzschild case, using the time-translation Killing vector instead of the boost Killing vector.

In four dimensions, the calculation of $<n>$ needs a much more elaborated technique, both for null and spacelike hypersurfaces. The calculation of the volumes needed to insure the conditions (4) is lengthy, and it turns out that one has to distinguish many cases depending on the relative positions of the linked points x and y, each case making its own contribution to $<n>$. Fortunately, only a few of these contributions survive for macroscopic black holes ($a \gg 1$), and it should be possible to evaluate them all with sufficient effort. Here we give only the final result for one such contribution, referring the reader to reference [5] for further detail:

$$< n >_1 = \frac{\pi^3 a^2}{16} \left(c + O\left(1/a \right) \right)$$

where

$$c = \int_0^\infty dx \int_0^x dy \, (x-y)^4 \, e^{-\frac{\pi}{3}\left(x^4+y^4\right)} \approx 0.0419$$

As indicated above, it is not difficult to convince oneself on the basis of our two dimensional experiences that the number of links in four dimensions must turn out to be proportional to the area of the horizon, or more precisely, to the area of the two-surface $S = H \cap \Sigma$. To recall the reasoning: The surfaces H and Σ will look locally like their two dimensional analogs extended trivially by a portion of $\mathbb{R}^2$, but since, as we saw, the main contribution to $<n>$ in two dimensions came from pairs just straddling $H \cap \Sigma$, and since locally Σ

will also look flat (like our two dimensional Σ was), and since (as we argued) all flat Σ (null or spacelike) give the same (finite) answer in two dimensions, so in four dimensions the density of links per unit surface area of S will be constant, that is to say, their total number will be proportional to the area of S, modulo subleading corrections. Moreover, the same should hold for arbitrary hypersurfaces Σ and arbitrarily curved horizons H, as long as neither is so badly distorted as to exhibit significant curvature in the vicinity of a horizon point. Accepting all this, we can anticipate the general formula for four dimensions:

$$<n> = \gamma \frac{A\left(H \cap \Sigma\right)}{l_c^2} \left(1 + O\left(\frac{l_c}{\sqrt{A\left(H \cap \Sigma\right)}}\right)\right) \ ,$$

where $A(H \cap \Sigma)$ is the area of the 2-surface in which the horizon meets Σ, $l_c = \varrho_c^{-1/4}$ is the fundamental causal set length, and γ is a number of order unity. For macroscopic black holes we can safely neglect the second term and conclude that the number of links will just be proportional to the area of the horizon in causal set units, with a coefficient of order unity. From this we can infer that, if the entropy really does measure the number of causal links, then l_c must be of Planckian order, as was anticipated a long time ago.

3 On the minimality and maximality conditions

Now we return to the "max/min" conditions we introduced in Section 2, in order to prevent the double counting of causal links to which we attributed the initially divergent character of our integral for $< n >$. In (4), these conditions are the last two in the list. Other possibilities exist, however, and we know of nothing particularly sacred about the conditions used in (4), which we selected partly with an eye to the simplicity (for evaluation) of the resulting integral. One must be careful not to use something like "y minimal in $J^+(\Sigma)$", which would drive $<n>$ to zero in the limit of null Σ, but this does not rule out, for example, a condition like "x maximal in $J^-(\Sigma)$".

Fortunately, the finiteness of the answer — and even its exact numerical value — seems to be insensitive to variations in the max/min conditions. Consider, for example, repeating the calculation of Section 2.2 with the different set of conditions illustrated in Figure 3. (We have weakened the fifth

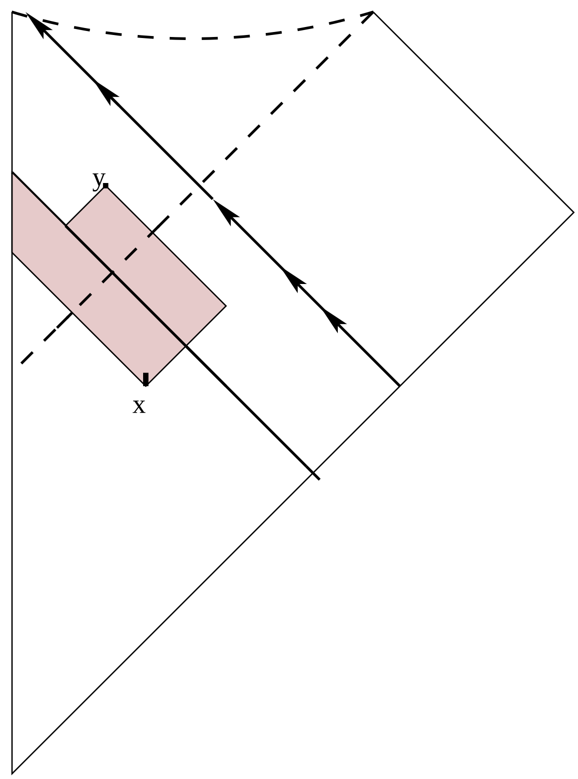

Figure 3: A second variation on the "max/min" theme

condition to "y minimal in $J^+(H) \cap J^+(\Sigma)$" and strengthened the fourth to "x maximal in $J^-(\Sigma)$".) With this alternative set of conditions, the integral for $<\!n\!>$ is modified (because V is modified), but it can be shown [5] to have the same asymptotic behavior as before, namely

$$<\!n\!> = \frac{\pi^2}{6} + O(1/a) \ .$$

Thus, in this case at least, we obtain exactly the same numerical answer as in Section 2.

Another feature that our counting must have if it is to yield the horizon area is that, within reason, the expected number of links should depend only on the intersection $H \cap \Sigma$, and not on how the surface Σ is prolonged outside or (especially) inside the horizon H. For example one should get the same

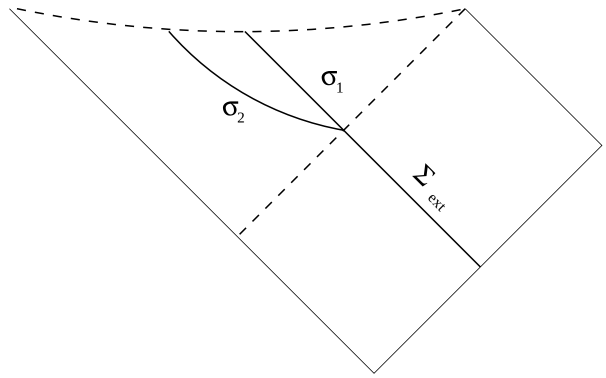

Figure 4: Two continuations of a hypersurface to the interior region.

answer for both of the continuations shown in Figure 4. (The case where the difference is confined to the interior black hole region is of particular significance for the entanglement interpretation of horizon entropy, since such a difference cannot, by definition, influence the effective density operator for the external portion of Σ (at least to the extent that unitary quantum field theory is a good guide).) From this point of view, those max/min conditions are most satisfactory that depend least on conditions outside the black hole. In this sense, the condition used in Section 2 that y be minimal in $J^+(H)$ has an advantage over the alternative, "dual" condition that x be maximal in $J^-(\Sigma)$; for the former, at least in the case of null Σ, refers only to the interior region.

4 Summary and Discussion

In Sections 2 and 3 we have reported some calculations, in the context of causal set theory, of the expected number of irreducible causal relations (links) that cross the horizon H of a black hole in proximity to a specified spacelike or null hypersurface Σ, as determined by the satisfaction of certain "max/min" conditions. Limiting ourselves to the case of spherical symmetry, we considered both equilibrium and nonequilibrium examples of macroscopic black holes in both 3+1 and 1+1 dimensions, together with both null and spacelike hypersurfaces. We also considered variants of the max/min conditions. In all these cases one obtains finite answers, but we computed exact numbers only for null Σ, and only for the two dimensional reductions of the corresponding four dimensional black holes. The expected number of links was always $\pi^2/6$. Moreover, we saw that the bulk of the links always resided in close proximity to the horizon, meaning that the result was being controlled by the near horizon geometry. From this we inferred the likelihood of a universal relationship in four dimensions, with the number of links being proportional to the horizon area, modulo corrections down by a factor of l_c/R where l_c is the fundamental discreteness length and R the black hole size.[7]

What seems significant about these results is not so much the proportionality to horizon area *per se*. One might have expected as much. However the coefficient of proportionality might have turned out to be either infinite or zero in the limit of large black hole radius (as in fact it does if one omits the max/min conditions introduced to prevent "double counting" of links). Moreover, for the nonequilibrium horizon, the coefficient might have varied with time or it might have differed from its equilibrium (Schwarzschild) value. In the event, none of these things occurred, at least in the cases checked. Rather we found a universal answer which took the same value in all cases where we succeeded in evaluating it exactly. The agreement between the equilibrium and nonequilibrium cases seems especially noteworthy, inasmuch as this is the first time, to our knowledge, that such an entropy has been computed for a non-stationary horizon.

The weakness of our result, of course, is that it remains at a purely kinematic level: we believe to have found something like the number of "horizon atoms", whose multiplicity is the ultimate source of black hole entropy, but this belief cannot be substantiated or refuted before we possess a fully quantum dynamics for causal sets. Short of this, many interesting extensions and cross-checks of our conclusions can still be pursued, however.

Of greatest immediacy is the need to carry out a full calculation in four-dimensions. Not only would this provide an important test of our reduction to two dimensions, but it would furnish the correction to the two-dimensional value of $\pi^2/6$, thereby laying the basis for a future determination of the fundamental length l_c. (Comparison of the known entropy with a calculation from first principles is probably the most reliable way to get a handle on

[7]Interestingly, these corrections are – in 4 dimensions – comparable in order of magnitude to the inherent $\sqrt{n}$ fluctuations that one would expect in n itself purely for statistical reasons.

the basic parameters of any quantum gravity theory, as entropy, being an absolute number, should not be subject to "renormalization".)

Completing the evaluation of the two dimensional integrals for spacelike Σ is also desirable in order to decide whether they indeed give the same result as for null Σ.[8] It would also be good to explore further the extent to which the results we have obtained depend on the details of the max/min conditions chosen (or indeed, whether other conditions, not of the max/min type might possibly offer a better solution to the "double counting" problem). The generalization to rotating and deformed black holes is another obvious direction for further work.

With respect to the causal set, there can of course be no horizon as such, only a division of the elements into those which can and cannot communicate with distant regions. The closest one can come to the horizon as a null surface is probably the collection of linked pairs we have counted in this paper.[9] But these correspond to a "thickened hypersurface" in the continuum. It would be interesting to compute the amount of this "kinematical thickening", especially as there are hints from a very different direction of a pronounced "dynamical thickening" of the horizon (possibly of order $a^{1/3}$) resulting from the influence of quantum fluctuations in fields propagating near H [8].

A further direction for generalization would be the substitution of some different structure for the links we have considered in this paper. One such

[8]In this connection, we note that the case of null Σ is of particular interest for the "Generalized Second Law" of entropy increase. It is difficult to imagine proving this law — or even formulating it — without being able to specify in a well defined manner the hypersurface Σ to which the entropy is being referred. Within a semiclassical spacetime with its fixed metric, this is not a problem, but the semiclassical framework is overly restrictive, since it cannot accommodate, for example, such a mundane entropy as that due to the spread in position of the individual members of "gas" of black holes. Fortunately, recourse to a semiclassical spacetime is unnecessary in the case of a null hypersurface, since then one can specify Σ by "anchoring it to the environment" (say to the walls of the proverbial thermodynamic box), and with this accomplished, one can envisage proving the second law as sketched in [7]. But no similarly robust technique seems available in the case of a spacelike Σ.

[9]Another possibility might be the minimal layer L of the subcauset corresponding to the interior region of the black hole. However, L is by definition an antichain and therefore more akin to a spacelike surface than a horizon, which, though not everywhere null, is ruled by null geodesics. More importantly, as one moves along H toward the future, the elements of L probably become sparse too rapidly to mark out H correctly. This difficulty is even clearer for the dually defined set L' of *maximal* elements of the *exterior* region, which probably is empty!

possibility might be a "triad" of elements, say x, y and z with x and y to the past of Σ and z to its future, and with x inside the black hole and y outside it. The requirement that the triad be "small" in a suitable sense might then be able to replace our max/min conditions. In the same spirit one could consider inverted triads or even "diamonds" containing both types of triad simultaneously. There is however some suggestion that triads of the first type are naturally related to the kind of correlation responsible for entanglement entropy in a quantum field theory framework [9].

A final remark concerns the finiteness of our integrals in two spacetime dimensions. Although this was necessary in order that the four dimensional result scale correctly with area, it could nonetheless seem surprising that the counting of two dimensional links remains finite even in the continuum limit where the fundamental length is sent to zero. In this sense, the replacement of continuous spacetime by a causal set could appear in two dimensions as more of a regularization device than something fundamental. We do not know whether this has any deeper meaning, or whether it might be related to some of the other special properties that both quantum field theory and quantum gravity possess in two dimensions (cf. [10]).

In concluding, we would like to dedicate this article to our friend and colleague, Jacob Bekenstein. Not only does Jacob's work lie at the origin of our understanding of black hole entropy and the "generalized second law", but it also raised explicitly the theme of missing information which forms the backdrop to, and inspiration for, the work reported herein.

The work of RDS was partly supported by NSF grant PHY-0098488, by a grant from the Office of Research and Computing of Syracuse University, and by an EPSRC Senior Fellowship at Queen Mary, University of London. The work of DjD was partly supported by grant number ERBFMRXCT 960090. RDS would like to thank Goodenough College, London for its hospitality during the writing of this paper. DjD would like to thank the General Relativity group of Syracuse University and the High Energy group of ICTP for their kind hospitality during different stages of the work.

References

[1] J.D. Bekenstein, "Do we understand black hole entropy?", in *The Seventh Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field The-*

ories, proceedings of the MG7 meeting, held Stanford, July 24–30, 1994, edited by R.T. Jantzen, G. Mac Keiser and R. Ruffini (World Scientific 1996) ⟨gr-qc/9409015⟩

[2] L. Bombelli, J. Lee, D. Meyer and R.D. Sorkin, "Spacetime as a Causal Set", *Phys. Rev. Lett.* **59:** 521-524 (1987);
David Porter Rideout and Rafael Dolnick Sorkin, "A Classical Sequential Growth Dynamics for Causal Sets", *Phys. Rev. D* **61:** 024002 (2000) ⟨gr-qc/9904062⟩

[3] L. Bombelli, "Statistical Lorentzian geometry and the closeness of Lorentzian manifolds", *J. Math. Phys.* **41:** 6944-6958 (2000) ⟨gr-qc/0002053⟩

[4] G. Brightwell and R. Gregory, "The Structure of Random Discrete Spacetime", *Phys. Rev. Lett.* **66:** 260-263 (1991)

[5] Djamel Dou, *Causal Sets, a Possible Interpretation for the Black Hole Entropy, and Related Topics*, Ph. D. thesis (SISSA, Trieste, 1999) ⟨gr-qc/0106024⟩

[6] Rafael D. Sorkin and Daniel Sudarsky, "Large Fluctuations in the Horizon Area and What They Can Tell Us About Entropy and Quantum Gravity", *Class. Quant. Grav.* **16:** 3835-3857 (1999) ⟨gr-qc/9902051⟩

[7] R.D. Sorkin, "Toward an Explanation of Entropy Increase in the Presence of Quantum Black Holes", *Phys. Rev. Lett.* **56:** 1885-1888 (1986);
R.D. Sorkin, "The Statistical Mechanics of Black Hole Thermodynamics", in R.M. Wald (ed.) *Black Holes and Relativistic Stars*, (U. of Chicago Press, 1998), pp. 177-194 ⟨gr-qc/9705006⟩

[8] R.D. Sorkin, "How Wrinkled is the Surface of a Black Hole?", in David Wiltshire (ed.), *Proceedings of the First Australasian Conference on General Relativity and Gravitation*, held February 1996, Adelaide, Australia, pp. 163-174 (University of Adelaide, 1996) ⟨gr-qc/9701056⟩

[9] R.D. Sorkin, "Quantum Measure Theory and its Interpretation", in *Quantum Classical Correspondence: Proceedings of the* 4^{th} *Drexel Symposium on Quantum Nonintegrability*, held Philadelphia, September 8-11, 1994, edited by D.H. Feng and B-L Hu, pages 229–251 (International Press, Cambridge Mass. 1997) ⟨gr-qc/9507057⟩

[10] Thomas M. Fiola, John Preskill, Andrew Strominger, and Sandip P. Trivedi, "Black hole thermodynamics and information loss in two dimensions", ⟨hep-th/9403137⟩ *Phys.Rev. D* **50:** 3987-4014 (1994)