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On the "renormalization" transformations induced by cycles of expansion and contraction in causal set cosmology

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Abstract

We study the "renormalization group action" induced by cycles of cosmic expansion and contraction, within the context of a family of stochastic dynamical laws for causal sets derived earlier. We find a line of fixed points corresponding to the dynamics of transitive percolation, and we prove that there exist no other fixed points and no cycles of length ≥ 2. We also identify an extensive "basin of attraction" of the fixed points but find that it does not exhaust the full parameter space. Nevertheless, we conjecture that every trajectory is drawn toward the fixed point set in a suitably weakened sense.

1 Introduction

There is good reason to believe that the appropriate dynamical framework for "quantum gravity" will be — just as the ordinary quantum dynamical framework already is — a generalization of classical probability theory. If in addition, we take the deep structure of spacetime to be that of a causal set [1], then the required dynamics will be expressible as a generalized probability measure on the space of all causal sets, the generalization being in the sense of [2] and [3] or something similar.

In the search for such a dynamics, principles like general covariance and relativistic causality can offer much needed guidance. Indeed, in the limit in which quantal probabilities reduce to classical ones, it has proved possible to derive [4] from versions of these two principles a unique family of dynamical laws (Markov processes) parameterized by a sequence of "coupling constants"

$$t_0,\ t_1,\ t_2,\ \cdots, \tag{1}$$

where $t_0 \equiv 1$ and $t_n \geq 0$ for $n = 1, 2, 3, \cdots$. Although this family of stochastic processes is tremendously special compared to the most general relevant Markov process, it still contains a countable infinity of coupling constants, and the question arises how further to narrow down the field of possibilities. Any answer to this question can be expected to be of use in the attempt to find a quantal generalization of the dynamical scheme in question; it might also, with luck, lead us to a choice of the t_n which would define a causal set version of classical general relativity, arising as the classical limit of quantum causet dynamics. (We will often use the contraction "causet" in place of the longer "causal set".)

In the absence of any further general principle[1] that might be brought to bear in narrowing down the possibilities (1), one can consider an "evolutionary" approach to this problem, that is, one can try to let the causet "choose its own dynamics". Such a possibility was discussed in [5], where it was shown that each cosmic cycle of expansion and contraction for the causet has the effect of "renormalizing" the parameters t_n in a definite manner.[2] The renormalization reduces, in fact, to the repeated action of a single transformation M, given by equation (6) below. By understanding better the action of M, one might hope to identify a distinguished sub-family of the dynamical laws parameterized by (1). For example, if M had a single fixed point (t_n^0) and if every other set of parameters (t_n) were carried toward (t_n^0) by the action of M, then one might designate (t_n^0), or more generally some neighborhood thereof, as distinguished since it would be the dynamics toward which the universe would naturally evolve.[3]

We will see below that the actual situation is more complicated than this, but with many points of similarity. Instead of a single fixed point, there is a one-parameter family of them, and not all sequences (t_n) are attracted to the fixed point set, at least in the most obvious sense. We will also find that the transformation M has no "limit cycles", but we do not know whether it might have "attractors" of a more complicated sort. Thus, we will make a start on determining the nature of the "RG flow" associated with M, but we will not characterize it fully. Nor will we obtain much information about the flow in the neighborhood of the fixed point set itself. Some such results for specific trajectories can be found in [6] and [5], however. Indeed, the very suggestive nature of some of those results for helping to explain the most salient features of the early universe furnished a good part of the motivation for the present investigation.

[1] One principle that suggests itself is that of "locality". Unfortunately, this does not seem to have meaning (at a fundamental level) for causets, though it must, of course, emerge in a suitable continuum approximation if the theory is to reproduce low energy physics correctly.

[2] The question of which dynamical laws actually lead to such cycles (as opposed to monotonic expansion, for example) is very much open. Only in the case of percolation is it known that an infinite number of cycles occur with probability 1.

[3] In much the same manner, one has explained why low energy physics "must" be described by a *renormalizable* quantum field theory. In that case, however, the RG flow in question is conceptual, being an evolution in energy scale, whereas here it is truly a change with time. (Of course, evolution in time gets correlated with evolution in energy/length scale in the context of big bang cosmology.)

2 Formulation of the problem and summary of our main results

Let us now pose more precisely the problem we intend to address. In the course of the growth/evolution of a causal set C, a *post* occurs when an element ω is born such that every other element of C is (or will be) either an ancestor or a descendant of ω. That is, a post is an element ω of C for which

$$C = (\text{past } \omega) \cup \{\omega\} \cup (\text{future } \omega)\,,$$

where $\text{past}(\omega) := \{x \in C | x \prec \omega\}$ and $\text{future}(\omega) := \{x \in C | \omega \prec x\}$ (Our basic definitions and notation are as in [4], in particular $\prec$ is irreflexive.) Cosmologically speaking, this may be interpreted as a kind of collapse of the universe to zero diameter followed by a subsequent re-expansion. It is a simple consequence of the form taken by the dynamical law derived in [4] that the portion $\{\omega\} \cup (\text{future } \omega)$ of C comprising the post and its descendants is governed by an effective dynamics of the same nature as that valid for C as a whole, but with different values of the t_n.

To see what these new parameters are, recall from [4] that any given transition probability leading from an n-element causet to one of $n+1$ elements takes the form

$$\frac{\lambda(\varpi, m)}{\lambda(n, 0)}\,, \tag{2}$$

where

$$\lambda(\varpi, m) := \sum_{k=m}^{\varpi} \binom{\varpi - m}{k - m} t_k\,. \tag{3}$$

Here the transition is occasioned by the birth of some new element x and $\varpi = |\,\text{past}(x)|$ is the number of all its ancestors, while $m = |\,\text{maximal}\,(\text{past}(x))|$ is the number of its *immediate* ancestors or "parents". Now, by definition, no element x born to the future of a post can have vanishing m (or ϖ). In view of (3), it follows simply from this that the parameter t_0 becomes irrelevant subsequent to a post.[4] By the same token, we see from (2) that only the ratios of the remaining t's, $(t_1, t_2, t_3, \cdots)$, have meaning in such a region of C, not their overall normalization. Moreover, if ϖ and m are the *effective* sizes of $\text{past}(x)$ and of $\text{maximal}\,(\text{past}(x))$ for some element x born after the post ω, then their true values are plainly $\varpi + p$ and m, respectively, where $p = |\,\text{past}\,\omega|$ is the number of ancestors of the post. The (un-normalized) probability of this birth is therefore given by the effective quantity $\widetilde{\lambda}(\varpi, m)$, where

$$\widetilde{\lambda}(\varpi, m) = \lambda(\varpi + p, m)\,. \tag{4}$$

[4]In more detail, the reasoning is that the probabilities of the possible transitions (those that respect ω being a post) are, according to (2), given by the $\lambda(\varpi, m)$, up to an overall normalization, and since $m = 0$ is excluded, t_0 never occurs in these, according to (3). Since the normalization then follows from the requirement that all allowed probabilities sum to 1, it also cannot depend on t_0. This is made explicit in (7) below (see also the Appendix).

Given these facts, it is an easy matter to derive the effective coupling constants for the after-post dynamics, and we claim that $\widetilde{t}_n^{(p)}$, the effective or "renormalized" value of t_n in the after post region, is given by

$$\widetilde{t}_n^{(p)} = \sum_{k=0}^{p} \binom{p}{k} t_{n+k}\,, \tag{5}$$

or equivalently by p applications of the transformation M to the fundamental coupling constants, where $p = |\,\mathrm{past}\,\omega|$ and M takes $(t_1, t_2, t_3, \cdots)$ to $(\widetilde{t}_1, \widetilde{t}_2, \widetilde{t}_3, \cdots)$ via

$$\widetilde{t}_n = t_n + t_{n+1} \qquad (n = 1, 2, 3, \cdots)\,. \tag{6}$$

Perhaps the simplest way to demonstrate these claims, if we accept that a relationship of the form (3) holds good for the effective dynamics[5], is just to observe that, as a trivial special case of that equation,

$$t_n = \lambda(n, n)$$

from which we find immediately, for the special case $p = 1$,

$$\widetilde{t}_n = \widetilde{\lambda}(n, n) = \lambda(n+1, n) = t_n + t_{n+1}\,.$$

Our claim that the $\widetilde{t}_n^{(p)}$ are given by p applications of M is then a consequence of the manifest fact that adding $p > 1$ elements to past(ω) all at once is no different from adding them one by one. A logically impeccable proof is almost as easy, but requires a short calculation, given in the Appendix, to establish (40). In the light of the latter, one sees immediately from (4) that the effective coupling constants are indeed those of (5), and these in turn are readily seen to be given by $M^p(t_n)$, for example by noting that M can be expressed as $1 + a$, where a is a shift operator acting on the t_n and comparing (4) with the binomial expansion of $(1 + a)^p$.

The "renormalization (semi)group" we will study in the sequel is that generated by the transformation M. For consistency with ω being a post, at least one of the parameters $t_1, t_2, t_3 \cdots t_{p+1}$ must be nonzero. Henceforth we will assume for simplicity that $t_1 > 0$. If this is not true initially, it will become so after a sufficient number of applications of the operator M.

For completeness, we record here the correctly normalized transition probability T for the birth of an element x to the future of our post ω. For this purpose, let $\varpi = |\,\mathrm{past}\,x| - p$ be the *effective* number of ancestors of x (excluding the p elements preceding ω), let m be its number of parents, and let n be the effective size of the pre-existing causet (the size of $C\backslash(\mathrm{past}\,\omega)$ before x is born). Then T can be written as

$$\frac{\widetilde{\lambda}(\varpi, m)}{\widetilde{\lambda}(n, 0) - \widetilde{\lambda}(0, 0)} \tag{7}$$

where $\widetilde{\lambda}(\varpi, m)$ is the same function of the $\widetilde{t}_k$ as $\lambda(\varpi, m)$ is of the t_k:

$$\widetilde{\lambda}(\varpi, m) = \sum_{k=m}^{\varpi} \binom{\varpi - m}{k - m} \widetilde{t}_k = \lambda(\varpi + p, m)\,,$$

[5]Essentially, this is proven in [4], however in strict logic, one would have to redo the whole derivation for the present "originary" case, where no new element is born with an empty past, this being part of the meaning of a post.

and where, consequently, the denominator can be written more properly as

$$\sum_{k=1}^{n} \binom{n}{k} \widetilde{t}_k$$

(eliminating the apparent reference to $\widetilde{t}_0$).

In the next section, we will prove those properties of the "renormalization map" $M : t_n \to \widetilde{t}_n$ that we have been able to establish. We assume throughout that $(t_n) = (t_1, t_2, t_3, \cdots)$ is a sequence of nonnegative real numbers, with $t_1 > 0$, and we let M act by (6), it being understood that the space $\mathcal{T}$ on which it acts is actually the set of equivalence classes of sequences (t_n), where (t_n) and (t'_n) are equivalent iff $t_n = \lambda t'_n$ for all n and for some fixed $\lambda > 0$. Our principal results are then as follows.

(i) The fixed points of M are given by the sequences (t_n) such that

$$t_n = t^n \tag{8}$$

for some $t \geq 0$ [6]. They thus form a 1-parameter set, whose parameter t is related to the parameter p of originary percolation [4] by $p = t/(t+1)$. In (8) the $t = 0$ case is to be interpreted by taking the limit $t \searrow 0$, which is equivalent to putting $t_n = \delta_{n1}$, a dynamics which produces originary causets C that are always trees [5, 4]. Note that the limit $t \to \infty$ also makes sense, and corresponds to originary percolation with $p = 1$ (cf. (3) and (7)), a dynamics which always produces the same causet: the "purely one dimensional" poset, or chain.

(ii) Aside from its fixed points, M possess no other cycles.

(iii) Suppose that (t_n), though not necessarily of the form (8), is such that $t_n^{1/n}$ has a limit in $[0, \infty)$ as $n \to \infty$. Then, under repeated action of M, the sequence (t_n) converges pointwise to (8) with $t = \lim_{n\to\infty} t_n^{1/n}$. (Of course, this convergence can only be modulo the overall scale ambiguity in (t_n). One way to lift this ambiguity is to deal with the *ratios* t_{n+1}/t_n, and what we prove below is that

$$\lim_{p\to\infty} \widetilde{t}^{(p)}_{n+1}/\widetilde{t}^{(p)}_n = t \,,$$

where $(\widetilde{t}^{(p)}_n)$ is the result of acting p times with M on (t_n). Notice that this asserts more than simply pointwise convergence to $t_n = \delta_{n1}$ in the case $t = 0$. Notice also that we have not included the case $t = \infty$ in the result just stated.)

For the case where $t_n^{1/n}$ has no $n \to \infty$ limit, we have no general result, although one can show using the generating functions defined below that if $(\widetilde{t}^{(p)}_n)$ *does* converge pointwise to some fixed point (8), then the latter must be the one with parameter $t = \limsup t_n^{1/n}$. In general, however, one can not expect pointwise convergence to any sequence (see the counterexample in Section 7 below.)

As a matrix, our transformation M is just

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Given the simplicity of this matrix (it is already in "Jordan normal form" and in fact is just the identity plus a shift operator), one might think it an easy matter to understand the "flow" its powers define. However one meets with two complicating circumstances: the infinite dimensionality and the fact that M only *appears* to be linear, because the space $\mathcal{T}$ on which it acts is really a projective space (rather than a vector space), since its points are given by the *ratios*, $t_1 : t_2 : t_3 \cdots$. For these reasons, we have chosen to study the transformation (6) directly, without attempting to utilize any of the general results (such as the spectral theorem, for example) which one might have tried to bring to bear on the problem.

3 Functional representation

We are working on sequences t_n $(n = 1, 2, 3, \cdots)$ defined up to an overall multiplicative constant, and such that

$$t_n \geq 0, \;\; t_1 \neq 0 \,. \tag{9}$$

One way to remove the overall multiplicative constant freedom is to set $t_1 = 1$, however it will be more convenient to keep the freedom in the following. The renormalization scheme on these sequences is given by

$$\widetilde{t}_n = t_n + t_{n+1}. \tag{10}$$

It is rather clear that after renormalization, the new sequence satisfies the same conditions (9) as the initial sequence. Thus it is possible to iterate the renormalization, and the issue is then to decide what kind of limiting behavior arises after a large number of iterations.

It is sometimes useful to represent these sequences as formal power series defined up to a multiplicative constant:

$$G(z) = \sum_{n=1}^{\infty} t_n z^n.$$

The sequence can be recovered from the power series by differentiation:

$$t_n = \frac{G^{(n)}(0)}{n!}\,.$$

In this new representation, the renormalization (10) is mapped to a functional relation between power series which can be derived easily as

$$\begin{aligned} \widetilde{G}(z) &= \sum_{n=1}^{\infty} \widetilde{t}_n z^n = \sum_{n=1}^{\infty} t_n z^n + \sum_{n=2}^{\infty} t_n z^{n-1} \\ &= (1 + \frac{1}{z})G(z) - G'(0). \end{aligned} \tag{11}$$

Iterating this functional relation is still rather difficult, however a change of variable

$$\begin{aligned} \frac{1}{y} &= 1 + \frac{1}{z}, \\ g(y) &= G(z) \end{aligned}$$

induces a new renormalization map on $g(y)$ which is simply given by

$$\tilde{g}(y) = \frac{g(y)}{y} - g'(0)\,. \tag{12}$$

Note that since the change of variable is analytic, $g(y)$ is still a formal power series

$$g(y) = \sum_{n=1}^{\infty} g_n y^n$$

with new coefficients which depend on the original ones as

$$g_n = \sum_{p=1}^{n} \binom{n-1}{p-1} t_p\,.$$

There is no obvious sufficient condition on the g_p which will ensure the positivity of the t_n, so that this condition can not be checked easily in the y representation. However, it is easy to see that it implies that g_p is bounded below by some strictly positive number (because $g_p \geq t_1 > 0$). The renormalization equation (12) written in terms of the coefficients g_n takes a particularly simple form

$$\tilde{g}(y) = \sum_{n=1}^{\infty} g_{n+1} y^n,$$

which can evidently be iterated p times to get

$$\tilde{g}^{(p)} = \sum_{n=1}^{\infty} g_{n+p} y^n\,.$$

4 Fixed points

Fixed points are sequences (t_n) which do not change under renormalization. Since our sequences are only defined up to a multiplicative constant, this means that

$$\tilde{t}_n = c t_n$$

or equivalently

$$t_n + t_{n+1} = c t_n$$

the solution of which is

$$t_{n+1} = (c-1) t_n$$

where we must have $c \geq 1$ in consequence of the positivity of the t_j. For $c > 1$, this implies $t_n = t^{n-1} t_1$, where we have put $c - 1 = t$; this can be written most compactly if we use the scale freedom to set t_1 itself to be t, in which case we get simply

$$t_n = t^n \,. \tag{13}$$

For $c = 1$, we just obtain the $c \to 1$ limit of these relationships, namely

$$t_1 = 1,\ t_2 = t_3 = t_4 = \cdots = 0 \,.$$

The equivalent relationships in terms of the generating functions introduced in the last section are as follows. Fixed points are power series which do not change under renormalization. Since these power series are only defined up to a multiplicative constant, this means that

$$\widetilde{G}(z) = tG(z), \tag{14}$$

t a real (positive) constant. Putting Eq. (11) in (14) then gives

$$G(z) \propto \frac{z}{1 - tz}.$$

This corresponds for the sequence (t_n) to a geometric series of ratio t, and gives a power series

$$g(y) \propto \frac{y}{1 - cy}$$

of the same type after change of variable, where we have put $t + 1 = c$.

5 Cycles

Cycles are such that, after a finite number of renormalizations p, one gets back to the initial sequence. Taking into account the multiplicative constant freedom, this gives the equation

$$\widetilde{t}_n^{(p)} = c^p t_n, \tag{15}$$

for all $n \geq 1$, where the constant, which is necessarily positive since both t_n and $\widetilde{t}_n^{(p)}$ are, was written as a power c^p. In light of (5), the equations (15) can be rewritten as a recursion relation for the sequence t_n as

$$t_{n+p} = (c^p - 1)t_n - \sum_{q=1}^{p-1} \binom{p}{q} t_{n+q}. \tag{16}$$

For the sequence to be positive, this implies in particular $c \geq 1$. A sequence given by such a linear recursion relation with constant coefficients independent of the index n can always be rewritten as a linear combination of p geometric progressions satisfying (16). The corresponding polynomial characteristic equation for the ratios can be solved easily as

$$q_j = -1 + c\mathrm{e}^{2i\pi j/p},$$

and a cycle must therefore have the general form

$$t_n = \alpha_0(c-1)^n + 2\sum_{1\le j<p/2} \mathrm{Re}(\alpha_j q_j^n) \quad (+\,\alpha_{p/2}(-c-1)^n)$$

where the last term (in parentheses) is only there if p is even α_0 and $\alpha_{p/2}$ are real constants, the other α_j are complex, and the combination was chosen such that t_n be real.

Note that $\alpha_{p/2}(-c-1)^n = (-)^n\alpha_{p/2}(c+1)^n$ will never be purely positive for n large. Similarly, $\mathrm{Re}(\alpha_j q_j^n)$ can never be purely positive for large n. Indeed, if the phase of q_j is called θ_j, it is well known that for $\theta_j/2\pi$ irrational, $\cos(n\theta_j)$ is dense in $[-1,1]$, whereas if it is rational, it is periodic with values of the form $\cos(2k\pi/r)$, $0 \le k < r$, which is positive for $k=0$ and negative for $k=[r/2]$, the integer part of $r/2$.

As a linear combination of geometric progressions, the behavior, and the sign, of t_n for large n is dominated by the geometric progression with non zero coefficient α_j and ratio q_j with largest modulus. For t_n to be positive for large n, the geometric progression $(c-1)^n$ must therefore dominate there. As it happens, the modulus of q_j grows with j from $j=0$ to $j=[p/2]$, which implies that for $0<j\le p/2$, all the coefficients α_j must be 0. This gives for the sequence t_n the necessary form

$$t_n = \alpha_0(c-1)^n,$$

which is percolation with $t=c-1\ge 0$, and therefore a fixed point.

Thus, we have proved that the only cycles are the fixed points.

6 Flows

The issue is now to determine the behavior of a given initial sequence after a large number of renormalizations. Among other possibilities, one may expect either convergence to a fixed point, or some kind of oscillatory behavior. The convergence studied here will simply be pointwise convergence of sequences. In the absence of cycles, and since fixed points are geometric progressions (percolation), a reasonable hypothesis is that an initial sequence will converge to a fixed point described by $t=\lim(t_n^{1/n})$, assuming this limit exists. In the following, this hypothesis will be validated in two qualitatively different cases: $t>0$ finite, and its limiting case, $t=0$. If $t_n^{1/n}$ does not have a limit (or converges to ∞), the issue becomes more complex, as will be discussed in the last part of this section and subsequently.

t finite and non-zero

In this subsection, the assumption will be that

$$\lim_{n\to\infty}\left(t_n^{1/n}\right) = t \neq 0. \tag{17}$$

As a consequence, one can write

$$t_n = t^n(1+\varepsilon_n)^n, \tag{18}$$

where ε_n is an auxiliary sequence which goes to 0 as $n \to \infty$. Another useful auxiliary sequence is

$$E_n = \sup_{i \geq n}(|\varepsilon_i|)$$

which also goes to 0 as $n \to \infty$.

Taking into account the scaling freedom of the sequences, the pointwise convergence of the sequence to a fixed point in the limit of a large number of renormalizations is expressed as

$$\lim_{p\to+\infty} \frac{\widetilde{t}_n^{(p)}}{\widetilde{t}_1^{(p)}} = t^{n-1}\,,$$

for n a fixed integer number. This is equivalent to showing that for any given $n \geq 1$,

$$\lim_{p\to+\infty} \frac{\widetilde{t}_{n+1}^{(p)}}{\widetilde{t}_n^{(p)}} = t\,. \tag{19}$$

First, it will be useful to prove the following Lemma.

Lemma If $t_n \geq 0$ is a sequence satisfying (17) for some particular $t > 0$ and if $i_1, i_2, i_3, \cdots$ is a sequence of positive integers such that $i_p = o(p/\ln p)$, then

$$\sum_{i=0}^{i_p} \binom{p}{i} t_i = o\left(\sum_{i=0}^{p} \binom{p}{i} t_i\right)\,. \tag{20}$$

We will need the lemma only in the special case $i_p = \sqrt{p}$.

The proof will be obtained by showing first that the sequence t_n can be replaced by another sequence u_n with no zeros so that the sequence $u_n^{1/n}$ be bounded from below by a strictly positive number. From (17), it is clear that an integer i_0 can be fixed such that $t_i > 0$ for all $i \geq i_0$. Then introducing the auxiliary sequence u_n such that

$$\begin{aligned} u_n &= t_n + 1 \text{ if } n \leq i_0 \\ u_n &= t_n \text{ otherwise} \end{aligned}$$

one gets (when $i_p > i_0$)

$$\frac{\sum_{i=0}^{i_p} \binom{p}{i} t_i}{\sum_{i=0}^{p} \binom{p}{i} t_i} = \frac{\sum_{i=0}^{i_p} \binom{p}{i} u_i - \sum_{i=0}^{i_0} \binom{p}{i}}{\sum_{i=0}^{p} \binom{p}{i} u_i - \sum_{i=0}^{i_0} \binom{p}{i}}\,.$$

Moreover,

$$\sum_{i=0}^{i_0} \begin{pmatrix} p \\ i \end{pmatrix} = o(\begin{pmatrix} p \\ i_0+1 \end{pmatrix} u_{i_0+1})$$

because the left hand side is a polynomial in p of degree i_0 while the right hand side is a polynomial of degree i_0+1. Thus (with '$A \sim B$' meaning as usual that $A/B \to 1$),

$$\frac{\sum_{i=0}^{i_p} \begin{pmatrix} p \\ i \end{pmatrix} t_i}{\sum_{i=0}^{p} \begin{pmatrix} p \\ i \end{pmatrix} t_i} \sim \frac{\sum_{i=0}^{i_p} \begin{pmatrix} p \\ i \end{pmatrix} u_i}{\sum_{i=0}^{p} \begin{pmatrix} p \\ i \end{pmatrix} u_i}$$

and the sequence t_n has been replaced by a strictly positive sequence that fulfills (20) iff the original sequence does. So in the following, it will be simply assumed that t_n was strictly positive to start with.

Let $0 < T_- \leq T_+$ be the lower and upper bounds of the sequence $t_n^{1/n}$. Then,

$$\frac{\sum_{i=0}^{i_p} \begin{pmatrix} p \\ i \end{pmatrix} t_i}{\sum_{i=0}^{p} \begin{pmatrix} p \\ i \end{pmatrix} t_i} \leq \frac{\sum_{i=0}^{i_p} \begin{pmatrix} p \\ i \end{pmatrix} T_+^i}{(1+T_-)^p}.$$

The numerator can be bounded from above by comparing it with a geometric series. Thus, call

$$v_i = \begin{pmatrix} p \\ i \end{pmatrix} T_+^i \ ,$$

then

$$\frac{v_i}{v_{i+1}} = \frac{i+1}{p-i} \, \frac{1}{T_+} \leq \frac{i_p+1}{p-i_p} \, \frac{1}{T_+} \to 0$$

since $i_p = o(p)$. So, by taking p large enough, it can be ensured that

$$\frac{v_i}{v_{i+1}} \leq \frac{1}{2}.$$

Then,

$$\frac{\sum_{i=0}^{i_p} \begin{pmatrix} p \\ i \end{pmatrix} t_i}{\sum_{i=0}^{p} \begin{pmatrix} p \\ i \end{pmatrix} t_i} \leq \frac{2 \begin{pmatrix} p \\ i_p \end{pmatrix} T_+^{i_p}}{(1+T_-)^p} \ . \tag{21}$$

The numerator on the right hand side can be bounded using that

$$\begin{pmatrix} p \\ k \end{pmatrix} = \frac{(p-0)(p-1)(p-2)\cdots(p-[k-1])}{k!} \leq \frac{p^k}{k!} \ ,$$

where we have put $i_p = k$ for short. From this it follows that

$$\frac{2\binom{p}{k}{T_+}^k}{(1+T_-)^p} \leq \frac{2(pT_+)^k}{(1+T_-)^p}$$

or taking logarithms,

$$\ln \frac{2\binom{p}{k}{T_+}^k}{(1+T_-)^p} \leq \ln 2 + k\ln(pT_+) - p\ln(1+T_-) \sim -p\ln(1+T_-) \to -\infty$$

which implies in turn that the right hand side of (21) converges to zero for $p \to \infty$ and concludes the proof of the Lemma.

Main proof The desired limiting behavior, equation (19), can be rearranged using Eqs. (5) and (18) as

$$\frac{\tilde{t}_{n+1}^{(p)}}{\tilde{t}_n^{(p)}} - t = \frac{\sum_{i=1}^p \begin{pmatrix} p \\ i \end{pmatrix} (\frac{i}{p-i+1} - t)t_{n+i} + t_{n+p+1} - t\, t_n}{\sum_{i=0}^p \begin{pmatrix} p \\ i \end{pmatrix} t_{n+i}} \,. \tag{22}$$

The goal is to prove that the latter expression goes to 0 when $p \to \infty$. To do so, we will split the sum in the numerator into four pieces corresponding to the ranges

$$1 \leq i < \sqrt{p}\,,\ \sqrt{p} \leq i < r_p - n_p\,,\ r_p - n_p \leq i < r_p + m_p\,,\ r_p + m_p \leq i \leq p\,,$$

where

$$r_p = \frac{t}{1+t}(p+1) \tag{23}$$

and where we leave the auxiliary sequences n_p and m_p free for the moment, subject only to the conditions,

$$n_p, m_p = o(p) = o(r_p)\,, \qquad n_p, m_p \to \infty \,\text{as}\, p \to \infty\,. \tag{24}$$

We will also assume, without loss of generality, that p is large enough so that $0 < \sqrt{p} < r_p - n_p$, whence we will have

$$0 < \sqrt{p} < r_p - n_p < r_p + m_p < p\,.$$

Observe here that r_p has been chosen to make the expression

$$\frac{i}{p-i+1} - t \tag{25}$$

vanish when $i = r_p$. Now, on performing our split in (22), we obtain

$$\frac{\tilde{t}_{n+1}^{(p)}}{\tilde{t}_n^{(p)}} - t = S_1 + S_2 + S_3 + S_4\,,$$

where (with '$\sum_{i=a}^b$' interpreted to imply $a \leq i < b$ in the first three cases) we can write

$$0 \leq -S_1 \quad = \quad \frac{\sum_{i=1}^{\sqrt{p}} \begin{pmatrix} p \\ i \end{pmatrix} (t - \frac{i}{p-i+1})t_{n+i} + tt_n}{\sum_{i=0}^p \begin{pmatrix} p \\ i \end{pmatrix} t_{n+i}} \leq \frac{t\sum_{i=0}^{\sqrt{p}} \begin{pmatrix} p \\ i \end{pmatrix} t_{n+i}}{\sum_{i=0}^p \begin{pmatrix} p \\ i \end{pmatrix} t_{n+i}} = o(1) \tag{26}$$

$$
\begin{aligned}
0 \le -S_2 \;&=\; \frac{\sum_{i=\sqrt{p}}^{r_p-n_p} \binom{p}{i} (t - \frac{i}{p-i+1}) t_{n+i}}{\sum_{i=0}^{p} \binom{p}{i} t_{n+i}} \\
&\le\; \frac{\sum_{i=\sqrt{p}}^{r_p-n_p} \binom{p}{i} t^{i+1}(1+E_{\sqrt{p}})^{n+i}}{\sum_{i=\sqrt{p}}^{p} \binom{p}{i} t^{i}(1-E_{\sqrt{p}})^{n+i}} \\
&\sim\; \frac{\sum_{i=\sqrt{p}}^{r_p-n_p} \binom{p}{i} t^{i+1}(1+E_{\sqrt{p}})^{i}}{(1+t-tE_{\sqrt{p}})^p} \le \frac{(1+E_{\sqrt{p}})^p}{(1+t-tE_{\sqrt{p}})^p} \left[\sum_{i=\sqrt{p}}^{r_p-n_p} \binom{p}{i} t^{i+1}\right] = A_p \qquad (27) \\
|S_3| \;&\le\; \frac{\sum_{i=r_p-n_p}^{r_p+m_p} \binom{p}{i} |\frac{i}{p-i+1} - t| t_{n+i}}{\sum_{i=0}^{p} \binom{p}{i} t_{n+i}} \le \sup_{i=r_p-n_p, r_p+m_p} \left(\left|\frac{i}{p-i+1} - t\right|\right) = o(1) \qquad (28) \\
0 \le S_4 \;&=\; \frac{\sum_{i=r_p+m_p}^{p} \binom{p}{i} (\frac{i}{p-i+1} - t) t_{n+i} + t_{n+p+1}}{\sum_{i=0}^{p} \binom{p}{i} t_{n+i}} \\
&\le\; \frac{\sum_{i=r_p+m_p}^{p+1} \binom{p}{i-1} t_{n+i}}{\sum_{i=0}^{p} \binom{p}{i} t_{n+i}} \\
&\le\; \frac{\sum_{i=r_p+m_p}^{p+1} \binom{p}{i-1} t^{i}(1+E_{\sqrt{p}})^{n+i}}{\sum_{i=\sqrt{p}}^{p} \binom{p}{i} t^{i}(1-E_{\sqrt{p}})^{n+i}} \\
&\sim\; \sum_{i=r_p+m_p-1}^{p} \frac{\binom{p}{i} t^{i+1}(1+E_{\sqrt{p}})^{i}}{(1+t-tE_{\sqrt{p}})^p} \le \frac{(1+E_{\sqrt{p}})^p}{(1+t-tE_{\sqrt{p}})^p} \left[\sum_{i=r_p+m_p-1}^{p} \binom{p}{i} t^{i+1}\right] = B_p \,. \qquad (29)
\end{aligned}
$$

In these deductions, we used the Lemma in the final step of (26), and we used it also (in the rather trivial special case, $t_n \to [t(1-E_{\sqrt{p}})]^n$) to extend the sums in the denominators for S_2 and S_4 to the full range, $0 \le i \le p$. In (28), by narrowing the sum in its denominator, we converted the second expression into a weighted mean of $|\frac{i}{p-i+1} - t|$ with the positive weights, $\binom{p}{i} t_{n+i}$, thereby obtaining the third expression, which is $o(1)$ thanks to (24). Obviously, the only thing left to prove is that S_2 and S_4 go to zero as $p \to \infty$. For that, it will be sufficient to show that their bounds A_p and B_p go to zero.

For future reference, we quote here the Stirling formula,

$$
n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n
$$

and its direct consequence (valid for $m, n \gg 1$)

$$\log \frac{(m+n)!}{m!\,n!} = m \log \frac{m+n}{m} + n \log \frac{m+n}{n} + \frac{1}{2} \log \frac{m+n}{2\pi mn} + o(1) \tag{30}$$

In order to bound A_p and B_p, one can observe that the general term of the sums appearing in them grows for $0 \leq i \leq r_p$ and decreases for $r_p \leq i \leq p$. This is because r_p is effectively defined such that the terms with $i = r_p$ and $i = r_p - 1$ be equal. Then the sums can be bounded by their largest term times the number of terms in the sum (which is itself smaller than p). From there, the following bounds for A_p and B_p are easily deduced:

$$A_p, B_p \leq \frac{p \begin{pmatrix} p \\ r_p + m \end{pmatrix} t^{r_p+m}(1+E)^p}{(1+t-tE)^p} \ ,$$

where the simplified notations

$$m = -n_p + 1, m_p \qquad E = E_{\sqrt{p}}$$

were introduced. This expression can be estimated with the aid of the asymptotic formula (30). Substituting and expanding in powers of small quantities like m/p produces, after some tedium,

$$\ln(A_p, B_p) \leq -\frac{1+t}{2}\frac{m^2}{r_p} + \frac{1}{2}\ln p + 2r_pE + O(1) + O(m^3/p^2) + O(r_pE^2) \ ,$$

which we will need only in the simplified form,

$$\ln(A_p, B_p) \leq \frac{-(1+t)^2}{2t}\frac{m^2}{p} + O(\ln p) + O(pE) + O(m^3/p^2) \ . \tag{31}$$

The only question now is whether (consistently with (24)) we can choose the auxiliary sequence m (representing either m_p or $-n_p + 1$) so that the right hand side of (31) diverges to $-\infty$, i.e. whether we can choose $m = m(p)$ so that the first term in (31) dominates the others for large p. But this is not difficult. For example,

$$m^2 = p^2(E_{\sqrt{p}})^{1/2} + p^{3/2}$$

meets all our requirements. With this choice, both A_p and B_p converge to zero as $p \to \infty$, and thus S_2 and S_4, which concludes the proof that the sequence $\widetilde{t}^{(p)}_{n+1}/\widetilde{t}^{(p)}_n$ converges to t.

t=0

This case can be treated in a manner very similar to the previous case. So, suppose that

$$\lim_{n\to\infty}(t_n^{1/n}) = 0,$$

and let us introduce the auxiliary sequence

$$E_n = \sup_{i \geq n} (t_n^{1/n}).$$

To prove that the sequence (t_n) flows to the fixed point $t_n = \delta_{1n}$ corresponding to $t = 0$, it will be sufficient to prove (the somewhat stronger assertion) that

$$\lim_{p \to +\infty} \frac{\widetilde{t}_{n+1}^{(p)}}{\widetilde{t}_n^{(p)}} = 0 \tag{32}$$

for all $n \geq 1$.

In writing (32) this way, we have implicitly assumed that none of the $\widetilde{t}_n^{(p)}$ vanish, and this will hold automatically in the generic case where the original t_n are themselves all nonzero. If, on the other hand, some of the t_n do vanish, then there are two possibilities. Either the set of nonzero t_n is infinite or finite. If it is infinite, then, for any fixed n, only a finite number of the $\widetilde{t}_n^{(p)}$ can vanish, so that the formulation (32) remains valid as it stands, if we agree to omit a finite number of initial values of the index p. However, if all of the t_n vanish after some point, then our assertion must be reworded as follows. Let t_{n_0} be the last nonzero t_n (recall that, by definition, not *all* of the t_n can vanish). Then, for $p > n_0$, $\widetilde{t}_1^{(p)}, \widetilde{t}_2^{(p)}, \cdots, \widetilde{t}_{n_0}^{(p)}$ will all be > 0, while $\widetilde{t}_n^{(p)}$ will $= 0$ for $n > n_0$. Thus, our renormalized sequence $(\widetilde{t}_n^{(p)})$ will *already* have converged to δ_{n1} for $n > n_0$, and we can limit the assertion (32) to $n < n_0$. Having thus dealt with these special cases, we will assume henceforth that all of the $\widetilde{t}_n^{(p)}$ occurring in our discussion are strictly positive.

Again, it is convenient to introduce an auxiliary sequence m_p, to be chosen later subject to the conditions

$$\sqrt{p} \leq m_p = o(p)\,. \tag{33}$$

Then, splitting the sum (5) in a manner similar to before, and applying similar techniques to bound the two resulting terms in $\widetilde{t}_{n+1}^{(p)}/\widetilde{t}_n^{(p)}$, we get

$$\begin{aligned} 0 \leq \frac{\widetilde{t}_{n+1}^{(p)}}{\widetilde{t}_n^{(p)}} &= \frac{\sum_{j=0}^{m_p} \binom{p}{j} t_{n+j} \frac{j}{p-j+1} + \sum_{i=m_p}^{p} \binom{p}{i} t_{n+i+1}}{\sum_{i=0}^{p} \binom{p}{i} t_{n+i}} \\ &\leq \frac{m_p}{p - m_p + 1} + \frac{1}{t_n} \sum_{i=m_p}^{p} \binom{p}{i} t_{n+i+1} \leq o(1) + \frac{E_{\sqrt{p}}^{n+1}}{t_n} \sum_{i=m_p}^{p} \binom{p}{i} E_{\sqrt{p}}^{i}\,. \end{aligned}$$

Calling this last sum A_p, it will be sufficient to show that it goes to zero as $p \to \infty$. As before, we can accomplish this, bounding A_p by a geometric series. Set

$$u_i = \binom{p}{i} E_{\sqrt{p}}^{i}\,,$$

then

$$\frac{u_{i+1}}{u_i} = \frac{p-i}{i+1} E_{\sqrt{p}} \leq \frac{p}{m_p} E_{\sqrt{p}}\,.$$

In light of this result, a useful choice of m_p is evidently

$$m_p = \sup(p(E_{\sqrt{p}})^{1/2}, \sqrt{p})\,; \tag{34}$$

for with this choice $u_{i+1}/u_i \leq \sqrt{E} \to 0$ for $p \to \infty$, and the conditions (33) are also satisfied. Than, by taking p large enough, we can insure

$$\frac{u_{i+1}}{u_i} \leq \frac{1}{2}\,,$$

so that (omitting the subscripts on m_p and $E_{\sqrt{p}}$ for brevity)

$$\sum_{i=m}^{p} \binom{p}{i} E^i = \sum_{i=m}^{p} u_i \leq u_m \sum_{i=0}^{p-m} 2^{-i} \leq 2u_m = 2\binom{p}{m} E^m$$

whence

$$A_p \leq \frac{2}{t_n} \binom{p}{m} E^{m+n+1} \leq \frac{2}{t_n} \binom{p}{m} E^m$$

or

$$\ln A_p \leq O(1) + \ln \binom{p}{m} + m \ln E$$

Invoking the Stirling formula (30) once again then yields, after some simplification,

$$\ln A_p \;\leq\; O(1) + m\ln(pe/m) + m\ln E \;\leq\; m\ln(me/p) + O(1)$$

where the final step used that $\ln(E) \leq 2\ln(m/p)$ because of (34). This proves that $\ln(A_p) \to -\infty$ since $m \geq \sqrt{p} \to +\infty$ and $m/p = o(1) \to 0$ as $p \to \infty$. In consequence, $A_p \to 0$, which entails the desired convergence

$$\lim_{p\to\infty} \left(\frac{\tilde{t}^{(p)}_{n+1}}{\tilde{t}^{(p)}_n} \right) = 0\,.$$

t=∞

In this case, the coefficients t_n grow faster than any geometric progression, and one might think, consistent with the $t \to \infty$ limit of (19), that $\tilde{t}^{(p)}_{n+1}/\tilde{t}^{(p)}_n$ would tend to $+\infty$ after a large number of renormalizations. Unfortunately this is not necessarily so, but it seems clear, at least, that it would follow under the stronger hypothesis that not only the $(t_n)^{1/n}$, but the ratios $\rho_n = t_{n+1}/t_n$ converged to $+\infty$ with n. In such a case, the renormalized dynamics would be trivial in the sense that it would produce only chains, as one can see from (2) and (3) above (because the unique $\lambda(\varpi, m)$ with $\varpi = n$ would swamp all others in the limit $p \to \infty$).

Multiple limits

When the sequence $t_n^{1/n}$ does not converge, it is difficult to conclude anything in general. However, if, for a given integer q, the q subsequences $t_{k(q+1)+i}$ all converge as $k \to \infty$, for $0 \leq i \leq q$:

$$\lim_{k\to\infty} t_{k(q+1)+i}^{\frac{1}{k(q+1)+i}} = T_i, \tag{35}$$

then the p-times renormalized sequence $t_n^{(p)}$ tends as $p \to \infty$ toward the fixed point corresponding to $t = \sup(T_i)$. This is because, as we are going to show, after q renormalizations one has

$$\lim_{n\to\infty}((\tilde{t}_n^{(q)})^{1/n}) = \sup_{0\leq i\leq q}(T_i) = t\,, \tag{36}$$

so that the previous result applies. We can restrict ourselves to the case when $t \neq 0$, because otherwise, since $T_i \geq 0$ automatically from the positivity of t_n, we would have $T_i = 0$ for all $0 \leq i \leq q$ and therefore $t_n^{1/n}$ itself would converge to 0, a case treated in the previous subsection.

To show (36), we define the auxiliary sequence

$$u_n = \sup_{n\leq i\leq n+q}(t_i), \tag{37}$$

so that

$$\begin{aligned}(t_n^{(q)})^{1/n} = \left(\sum_{j=0}^{q}\begin{pmatrix} q \\ j\end{pmatrix} t_{n+j}\right)^{1/n} \quad &\leq \quad u_n^{1/n}2^{q/n} \quad \sim \quad u_n^{1/n}\ , \\ &\geq \quad u_n^{1/n}\begin{pmatrix} q \\ q/2\end{pmatrix}^{1/n} \sim u_n^{1/n}\,.\end{aligned} \tag{38}$$

The latter inequality is because at least one of the terms t_{n+j} in the sum is equal to u_n and the binomial coefficient $\begin{pmatrix} q \\ q/2\end{pmatrix}$ is the largest of the lot. The inequalities (38) show in particular that

$$(t_n^{(q)})^{1/n} \sim u_n^{1/n}. \tag{39}$$

Now, to study the convergence of the sequence $u_n^{1/n}$, we split it into $q+1$ subsequences with indices $k(q+1)+i$, $0 \leq i \leq q$, which cover the whole sequence. Thus, reordering the elements of the supremum in (37), we have

$$u_{k(q+1)+i}^{1/(k(q+1)+i)} = \sup_{0\leq j\leq q}[t_{(k+H(i-j))(q+1)+j}^{1/(k(q+1)+i)}],$$

where $H(p)$ is an Heaviside–like function defined by

$$\begin{aligned} H(n) \quad &= \quad 1 \text{ if } x > 0 \\ &= \quad 0 \text{ otherwise.}\end{aligned}$$

Then from the ansatz (35),

$$t_{(k+H(i-j))(q+1)+j}^{1/(k(q+1)+i)} = (t_{(k+H(i-j))(q+1)+j}^{1/[(k+H(i-j))(q+1)+j]})^{[(k+H(i-j))(q+1)+j]/[k(q+1)+i]} \sim T_j,$$

and therefore the subsequence $u_{k(q+1)+i}$ converges:

$$\lim_{k\to\infty}(u_{k(q+1)+i}^{1/(k(q+1)+i)}) = \sup_{0\le j\le q}(T_j) = t.$$

Since each of the subsequences converges to the same limit t and since they cover the whole sequence u_n, we deduce

$$\lim_{n\to\infty} u_n^{1/n} = t\,,$$

and hence the result (36) from eq. (39).

In more general cases, it seems to be impossible to decide the question of convergence with our present methods. However, since the sequences we have not studied will have converging subsequences in stretches of diverging size, it seems that, as the number p of renormalizations goes to infinity, longer and longer initial stretches of the renormalized t_n will, for almost all p, look like percolation for some t (which will in general vary with p). However, this same idea also gives a way to construct sequences which will not flow to any single fixed point, but alternate between various ones, and we describe such a counterexample in the next section.

7 A counterexample and a conjecture

We have seen that a large number of sequences (t_n) yield trajectories $(\tilde{t}_n^{(p)})$ which converge to fixed points of M, including all those (t_n) for which $\lim\limits_{n\to\infty} t_n^{1/n}$ exists. In this section, we will describe a counterexample to the supposition that all trajectories whatsoever approach limits. However, our counterexample does not contradict the weaker supposition that every trajectory, in some sense "spends most of its time near to the fixed point set as a whole", and we conclude this section with a sample conjecture to that effect.

Now let (t_n) be any starting sequence of t's. If the corresponding trajectory $(\tilde{t}_n^{(p)})$ of renormalized t's converged to some limit $(\tilde{t}_n^{(\infty)})$ then in particular the ratio $\tilde{t}_2^{(p)}/\tilde{t}_1^{(p)} \equiv f(p)$ would have to converge to $\tilde{t}_2^{(\infty)}/\tilde{t}_1^{(\infty)}$. To construct our counterexample, then, it suffices to contrive (t_n) so that $f(p)$ has no limit. But we claim that, if $a_1, a_2, a_3 \cdots$ is any sequence whatsoever of positive reals, and if $\epsilon > 0$ is arbitrary, then we can find a starting sequence (t_n) and a subsidiary sequence of integers $p_1 < p_2 < p_3 \cdots$ such that $|f(p_n) - a_n| < \epsilon$ for all n. For example, let us try

$$\begin{aligned}
t_n &= (a_1)^n \quad \text{for} \quad 1 \le n \le p_1 + 2\,,\\
t_n &= (a_2)^n \quad \text{for} \quad p_1 + 2 < n \le p_2 + 2\,,\\
t_n &= (a_3)^n \quad \text{for} \quad p_2 + 2 < n \le p_3 + 2\,,\\
&\ \ \text{etcetera.}
\end{aligned}$$

If, for any k, we were to send p_k to infinity, then obviously $\lim_{n\to\infty}(t_n)^{1/n}$ would be a_k and our main result above would assure us that $f(n) \stackrel{n\to\infty}{\longrightarrow} a_k$; hence we can certainly find p_k great enough that $|f(p_k) - a_k| < \epsilon$. Moreover, we can subsequently alter the values of t_n for $n > p_k + 2$ without affecting $f(p_k)$, since the latter clearly depends only on $\widetilde{t}_1^{(p_k)}$ and $\widetilde{t}_2^{(p_k)}$, and these in turn depend only on t_n for $n \le p_k + 1$ and $n \le p_k + 2$, respectively. Hence, we can always select the p_k to verify our claim. Figure 1 illustrates this technique with a closely related example in which $\widetilde{t}_2^{(p)}/\widetilde{t}_1^{(p)}$ oscillates between (near to) 1/2 and (near to) 2, the specific choice of (t_n) in that case being: $t_n = n$ if n is a power of 2, and $t_n = 0$ otherwise, i.e. $(t_n) = (1, 2, 0, 4, 0, 0, 0, 8, 0, 0, 0, \cdots)$.

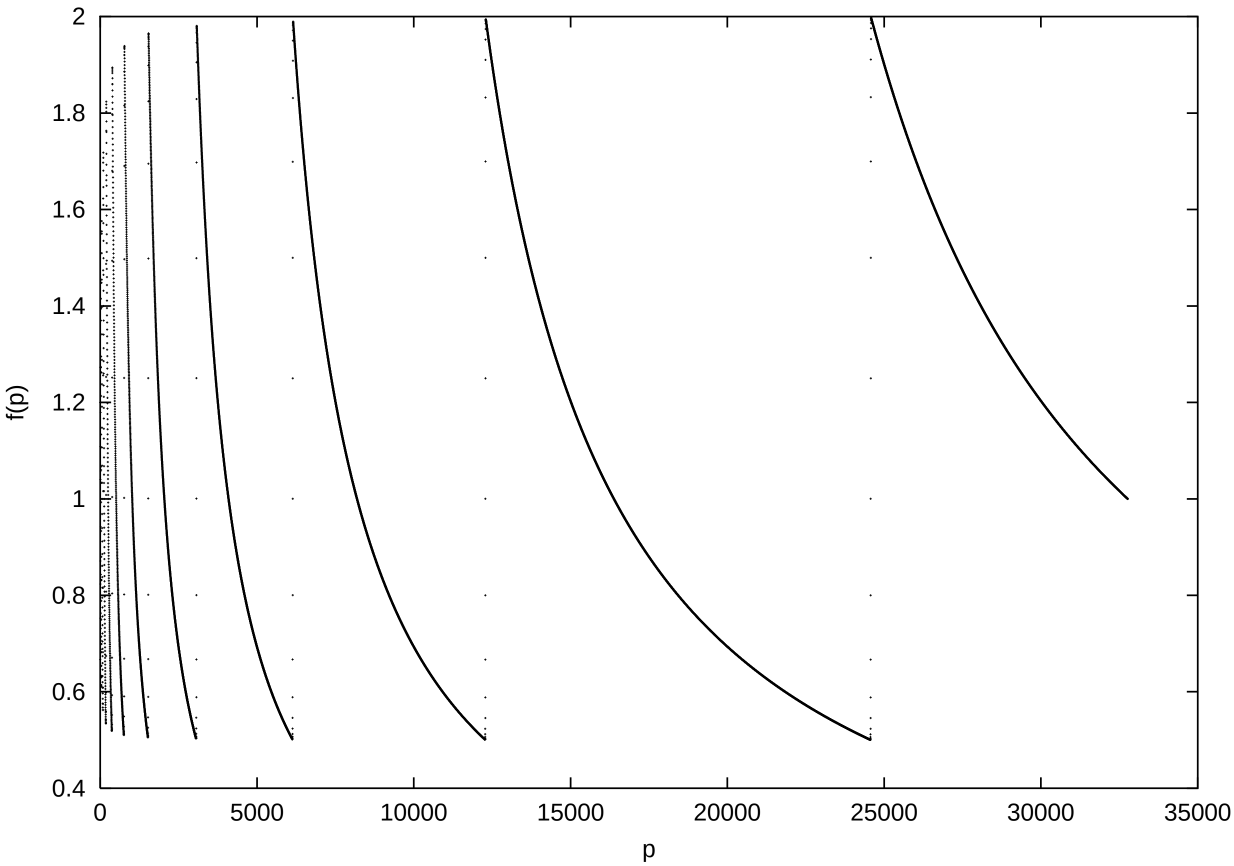

Figure 1: Flow of the ratio $\widetilde{t}_2^{(p)}/\widetilde{t}_1^{(p)}$ under renormalization

Closer inspection of this example, and in particular of Figure 1, reveals that the $\widetilde{t}_n^{(p)}$ are not behaving completely chaotically. Rather, the precipitous jumps are narrowly localized in "time", while between them, the ratios $\rho_n = t_{n+1}/t_n$ vary only gradually. Together with the evidence from other examples, this suggests that the "moments" when the renormalized t_n deviate from percolation-like values are few and far between. In some, yet to be specified sense, then, (t_n) would be spending most of the time near the "percolation submanifold", with but brief excursions to other regions. In order to render this idea somewhat more definite, let us cast it in the form of a conjecture. For brevity, let us say that a real number r is "within ϵ of ∞" when $r > 1/\epsilon$. Then we conjecture that: $(\forall\epsilon > 0)$ $(\forall m)$ (the fraction of $n < N$ for which the m ratios

$$\widetilde{t}_2^{(n)}/\widetilde{t}_1^{(n)},\ \widetilde{t}_3^{(n)}/\widetilde{t}_2^{(n)},\ \cdots,\widetilde{t}_{m+1}^{(n)}/\widetilde{t}_m^{(n)}$$

are all within ϵ of each other (or all within ϵ of ∞) tends to unity as $N\to\infty$).

8 Further reflections

The main theorem proved above guarantees pointwise convergence to percolation dynamics when $\lim\limits_{n\to\infty}(t_n)^{1/n}$ exists. However the topology of pointwise convergence is rather coarse, and using it, we could not even claim, for example, that the "basin of attraction" of the fixed point set was extensive enough to include any open neighborhood of the latter (because no open set in this topology can control more than a finite number of terms of the sequence (t_n)). On the contrary, the set of sequences *not* converging to the fixed-point set would be dense in the space $\mathcal{T}$ of all sequences. Thus this topology does not seem to provide a useful language for discussing the global features of our "RG flow".

From a physical point of view, pointwise convergence provides information with limited temporal validity. For example, pointwise convergence to $t_n = 0$ means that, following a cosmic cycle comprising a large number of causet elements, the ensuing expansion will be tree-like for a long, but in general finite period. To guarantee permanent tree-like behavior, one would need something like uniform convergence of the t_n, or perhaps more appropriately, uniform convergence of their ratios, t_{n+1}/t_n. An interesting question, therefore, is whether, by suitably strengthening its hypotheses, one could prove an analog of our main theorem for the topology of uniform convergence. Another natural extension of the present work, possibly of greater urgency, would be to explore, not just *which* trajectories approach the fixed point set, but also the *manner* in which they approach it; for this could help answer the question of how general is the phenomenon discovered in [6] according to which the "cosmic big number" associated with the so-called "flatness problem" could be explained by the hypothesis that the universe has undergone several previous cycles of expansion and collapse.

Finally, let us remark that the question of which topology (or topologies) is most suited to discuss the "renormalization group flow" with which we have been concerned in this paper is inseparable from the wider question of which sequences (t_n) represent dynamical laws that are genuinely conceivable from the physical point of view. One can imagine, for example, uncontrollably divergent sequences, and if $\mathcal{T}$ is really the set of *all* sequences, then one might expect its dimensionality as a projective space to exceed $\aleph_0$, the dimension of a separable Hilbert space. In that case, the normal tools of functional analysis would seem to be unavailing.

But does it really make sense that t_n could grow arbitrarily rapidly (or, for that matter, be a number whose precision required for its expression arbitrarily many significant digits)? Our mathematical framework allows this, but only — it would seem — as an artifact of the procedure by which one introduces the dynamical laws in a "non-material" manner, as if "from outside the universe". It would be more satisfactory if the laws could somehow be understood as embodied in the structure of material universe — in this case in the structure of the causal set. But then, the number of possible laws should itself be limited at any stage of the growth of the causal set, meaning that the number of possibilities for $t_0, \cdots, t_n$ (the parameters that determine the dynamics at stage n) would be bounded by something like the number of possibilities for a causet of n elements, a number which grows only as $2^{O(n^2)}$. It might be, therefore, that the space in which our renormalization transformation M acts is in reality not just of countable dimension but actually of countable cardinality. Important as this would be for the deeper understanding of the questions studied in this paper, it would be premature at present to speculate on how such a limitation on dynamical laws might work out in detail someday. Here, we wished only to raise the possibility, and in doing so to call to mind the idea that kinematics must ultimately fuse with (or absorb) dynamics as part of the further progress of fundamental physical theory.

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Appendix

In this appendix, we derive an identity used in Section 2 of the main text, namely

$$\lambda(\varpi, m \,|\, \widetilde{t}^{(N)}) = \lambda(\varpi + N, m \,|\, t) \tag{40}$$

where

$$\lambda(\varpi, m \,|\, t) = \sum_k \binom{\varpi - m}{k - m} t_k \tag{41}$$

and

$$\widetilde{t}_n^{(N)} = \sum_j \binom{N}{j} t_{n+j} \,. \tag{42}$$

The proof relies on a second (well known) identity,

$$\sum_{i+j=k} \binom{m}{i} \binom{n}{j} = \binom{m+n}{k} \,, \tag{43}$$

which itself follows from expanding out a third (trivial) identity,

$$(1+x)^m (1+x)^n = (1+x)^{m+n} \,.$$

From (41)-(43), we obtain

$$
\begin{aligned}
\lambda(\varpi, m \,|\, \tilde{t}^{(N)}) &= \sum_k \binom{\varpi - m}{k - m} \tilde{t}_k^{(N)} \\
&= \sum_k \binom{\varpi - m}{k - m} \sum_j \binom{N}{j} t_{k+j} \\
&= \sum_k \sum_j \binom{\varpi - m}{k - m} \binom{N}{j} t_{k+j} \\
&= \sum_l \left(\sum_{j+k=l} \binom{\varpi - m}{k - m} \binom{N}{j} \right) t_l \\
&= \sum_l \binom{\varpi - m + N}{l - m} t_l = \lambda(\varpi + N, m \,|\, t) \,,
\end{aligned}
$$

which is (40).

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