Orientations, Extensors, and the General Form of Stokes' Theorem

Just writing down Stokes' Theorem as a formula is easy enough; the whole difficulty is in coming up with a convenient and consistent set of conventions for the orientations, signs and normalizations involved. The conventions made below are chosen to agree with all the standard rules of vector analysis when one specializes to that case (for instance, an element of the surface bounding a compact region in flat $\mathbb{R}^3$ will be representable by the vector $d\vec{S} = \vec{n}\,dA$ where $\vec{n}$ is the outward unit normal).

In the following, $\mathfrak{X}$ will be a compact manifold embedded as a region, a surface, a curve, etc. within the n-dimensional manifold M. The boundary of $\mathfrak{X}$ (which may be empty) will be denoted by $\partial\mathfrak{X}$. For abstract indices, we will, unless otherwise specified, use the lower case letters $a, b, c, \cdots$; for numerical ("concrete") indices we will use $\alpha, \beta, \gamma, \cdots$.

The symbol '$\mathcal{O}$' should really be the the ideogram for orientation, but I don't know how to get it in TeX.

Orientations

A space $\mathcal{O}$ of orientations has only two elements, but it does have some structure. We can multiply an orientation by ± 1 and also take the product of two orientations. Specifically, if Ω' and Ω'' are the two elements of some orientation-space $\mathcal{O}$ then we have $\Omega'\Omega' = 1$, $\Omega'\Omega'' = -1$, $-\Omega' = \Omega''$, etc. Similarly, one can form the "tensor product" $\otimes$ of two orientation-spaces, giving a third orientation-space.

For vector spaces $V \supseteq W$ we define

$$\mathcal{O}(V) \equiv \mathcal{O}(V, int)$$

as the space of *internal orientations* of V, and

$$\mathcal{O}(V/W) \equiv \mathcal{O}(W, ext)$$

as the space of *external orientations* of W in V. Notice that $\mathcal{O}(V) = \mathcal{O}(V/\{0\})$.

We can regard $\mathcal{O}(V)$ as the product of $\mathcal{O}(V/W)$ with $\mathcal{O}(W)$, and *we always take the factors in that order*:

$$\mathcal{O}(V) = \mathcal{O}(V/W) \otimes \mathcal{O}(W). \tag{1a}$$

Similarly, for $U \supseteq V \supseteq W$ we write

$$\mathcal{O}(U/W) = \mathcal{O}(U/V) \otimes \mathcal{O}(V/W), \tag{1b}$$

$$\mathcal{O}(U) = \mathcal{O}(U/V) \otimes \mathcal{O}(V/W) \otimes \mathcal{O}(W), \tag{1c}$$

etcetera.

An internal orientation for W can be specified by an ordered basis $(w_1, w_2, \cdots w_K)$ for W (where $K = \dim W$), or equivalently by an ordered basis $(\omega_1 \cdots \omega_K)$ for W^*. The orientations specified thereby are *equal* iff $\det(\omega^i \cdot w_j)$ is positive, i.e. iff the inner product of $\omega_a^1 \wedge \cdots \wedge \omega_b^K$ with $w_1^a \wedge \cdots \wedge w_K^b$ is positive. An external orientation of W in V can be specified by an ordered basis for the quotient-space V/W, or equivalently by an ordered set of K linearly independent covectors $\omega^1 \cdots \omega^K$ annihilating W, where $K = \mathrm{codim}(W) := \dim V - \dim W$. (Recall here the sign nuisance that n^a and $n_a = g_{aa} n^a$ specify *opposite* orientations for a spacelike hypersurface in a Lorentzian manifold, because $n_a n^a < 0$ in this case.)

So far everything was for vector spaces. For manifolds we just apply the above definitions to tangent spaces in order to define local orientations, and we take a global orientation to be any continuous choice of local orientation, assuming such exists. Notationally, if $\mathfrak{X}$ is (internally) orientable then $\mathcal{O}(\mathfrak{X})$ will denote the space of its internal orientations; if it is externally orientable in M then $\mathcal{O}(M/\mathfrak{X})$ will denote the space of its external orientations. Note that an external orientation for $\mathfrak{X}$ in M can be specified by giving K=codim($\mathfrak{X}$) independent real functions $y^1, \cdots, y^K$ vanishing on $\mathfrak{X}$, because their gradients $\partial_a y^1, \cdots, \partial_a y^K$ furnish K linearly independent covectors annihilating $T_x\mathfrak{X}$. (For example, for K=1 and t a time-coordinate in Minkowski space, the function $y^1 = \pm t$ gives a future/past orientation to the t=0 hypersurface.)

Finally recall that an *axial tensor* of type $\mathcal{T}$ at $x \in M$ lives in $\mathcal{T} \otimes \mathcal{O}(T_x M)$, i.e. it is the product of an ordinary tensor (possibly density-weighted) with an orientation of $T_x M$. Similarly, a $\mathfrak{X}$-externally axial tensor at $x \in \mathfrak{X}$ lives in $\mathcal{T} \otimes \mathcal{O}(T_x M/T_x\mathfrak{X})$, etc.

Multi-indices

$$A = ab \cdots c$$

is a totally skew multi-index, with $|A|$ being the number of individual indices $a, b \cdots, c$.

Contraction of multi-indices:

$$v_A w^A := v_{ab\cdots c} w^{ab\cdots c}/|A|! \tag{2}$$

The ε-tensor

ε^{abcd} (in 4D as an example) is an *axial* tensor of *density-weight* $+1$. Its components *in any basis* are $\varepsilon^{0123} = 1$, $\varepsilon^{1023} = -1$, etc. The covariant tensor ε_{abcd} is axial of weight -1 with the identical components; it is normalized by $\varepsilon_{abcd}\varepsilon^{abcd}/4! = \varepsilon_A \varepsilon^A = 1$. (Caution: when a Lorentzian metric is involved there is the sign nuisance that $\sqrt{-g}\varepsilon_{abcd} = -g_{aa}g_{bb}g_{cc}g_{dd}\varepsilon^{abcd}/\sqrt{-g}$. On the other hand, you can always rely on $\varepsilon^{a\cdots b} g_{aa} \cdots g_{bb} = \varepsilon_{a\cdots b} \det g$.) Both $\varepsilon^{a\cdots b}$ and $\varepsilon_{a\cdots b}$ exist on any manifold, orientable or not.

Note that $\varepsilon^A_B = \varepsilon^{AC}\varepsilon_{BC}$ acts as the identity (a generalized Kronecker δ) in contractions with multi-indices A or B.

Extensors (tensors of extension)

An infinitesimal portion $d\mathfrak{X}$ of a submanifold $\mathfrak{X}$ can be represented by a tensor in two dual ways. In the first way, $d\mathfrak{X}$ is represented by a K-index skew-tensor $d\sigma_A$ of density-weight -1, where K is the *codimension* of $\mathfrak{X}$. We may call $d\sigma_A$ a "tensor of extension" or "extensor" for short. For maximum generality, $d\sigma_A$ must be regarded as *externally axial*, where, as just described, an externally axial tensor at $x \in M$ is the product of an ordinary tensor at x with an external orientation of $T_x\mathfrak{X}$ in $T_x M$; in other words an externally axial tensor of type $\mathcal{T}$ lives in $\mathcal{T} \otimes \mathcal{O}(T_x M/T_x\mathfrak{X})$. However, if $\mathfrak{X}$ carries an *external orientation* Ω then $d\sigma_A$ may be construed as polar (non-axial) via multiplication with Ω. (Thus an externally oriented portion $d\mathfrak{X}$ of a submanifold $\mathfrak{X}$ can be represented by a polar extensor $d\sigma_A$.)

Consider the general case where $\mathfrak{X}$ is unoriented. If $y^1, \cdots y^n$ are n independent locally defined real functions of which the first K determine $\mathfrak{X}$ in the sense that (locally)

$$\mathfrak{X} = \{x \in M | y^1(x) = y^2(x) = \cdots = y^K(x) = 0\}$$

then we have the formula

$$d\sigma_{ab\cdots c} = \frac{\partial_a y^1 \wedge \partial_b y^2 \wedge \cdots \wedge \partial_c y^K \, dy^{K+1} \cdots dy^n}{|\varepsilon^{p\cdots q}\, \partial_p y^1 \cdots \partial_q y^n|} \otimes \Omega$$

where $n = \dim(M)$ and Ω is the element of $\mathcal{O}(T_x M/T_x\mathfrak{X})$ specified by the functions $y^1, \cdots y^K$. For example, if $\mathfrak{X} = \{t = 0\}$ is a spacelike hyperplane in Minkowski space and Ω is an external orientation of $\mathfrak{X}$ then in t-x-y-z-coordinates, the only nonzero component of $\Omega d\sigma_a$ is $(\Omega d\sigma)_0 = \pm dxdydz$, where the sign $\pm$ is positive iff Ω agrees with the orientation specified by the function t. (It is instructive to think through in detail why this is true and why it is meaningful. For example, $d\sigma_a$ as such does not have well-defined numerical components, because a coordinate system cannot in general specify an external orientation for any submanifold.) Equivalently, $d\sigma_0 = +dx\,dy\,dz$ if $\mathfrak{X}$ is given a future orientation, and $d\sigma_0 = -dx\,dy\,dz$ if $\mathfrak{X}$ is given a past orientation. In general, $d\sigma_{\alpha\beta\cdots\gamma}$ is, up to sign, the magnitude of the coordinate projection of $d\mathfrak{X}$ onto the $\alpha = \beta = \cdots = \gamma = 0$ coordinate-plane.

When a metric is present, we may use instead of $d\sigma_a$ the ordinary tensor (of density-weight 0, though still axial in general)

$$dS_A = \sqrt{-g}\, d\sigma_A. \tag{3}$$

For the important special case of codimension 1 we have, up to orientation,

$$dS_a = n_a\, d^{n-1}V$$

where $d^{n-1}V = |d\mathfrak{X}|$ is the "volume" of $d\mathfrak{X}$ and n_a is the unit normal to $\mathfrak{X}$; or when $d\mathfrak{X}$ is null,

$$dS_a = dx_a\, d^{n-2}V$$

where $d^{n-2}V$ is the "sectional area" of $d\mathfrak{X}$ and $dx^a = g^{aa}dx_a$ is the extent of $d\mathfrak{X}$ along its null direction.

An infinitesimal element of extension $d\mathfrak{X}$ of dimension K also admits a representation in terms of a dual extensor $d\Sigma^A$, which is a K-index skew tensor (of density-weight 0). Notice that K now denotes the dimension of $\mathfrak{X}$, not its codimension. Most generally, $d\Sigma^A$ must be regarded as *internally axial*, where an internally axial tensor at $x \in M$ is the product of an ordinary tensor at x with an internal orientation of $T_x\mathfrak{X}$; in other words an internally axial tensor of type $\mathcal{T}$ lives in $\mathcal{T} \otimes \mathcal{O}(T_x\mathfrak{X})$. However, if $\mathfrak{X}$ carries an internal orientation Ω then $d\Sigma^A$ may be construed as polar (non-axial) via multiplication with Ω.

(Thus an internally oriented portion $d\mathfrak{X}$ of a submanifold $\mathfrak{X}$ can be represented by a polar extensor $d\Sigma^A$.)

In the general case of unoriented $d\mathfrak{X}$, if $x = f(y^1, \cdots, y^K)$ is a parametric representation of $\mathfrak{X}$, then

$$\begin{aligned} d\Sigma^{ab\cdots c} &= \frac{\partial x^a}{\partial y^1} \wedge \frac{\partial x^b}{\partial y^2} \wedge \cdots \wedge \frac{\partial x^c}{\partial y^K}\, dy^1 dy^2 \cdots dy^K \otimes \Omega \\ &= \frac{\partial x^a}{\partial y^j} \cdots \frac{\partial x^c}{\partial y^l}\, \varepsilon^{j\cdots l}\, dy^1 dy^2 \cdots dy^K \otimes \Omega, \end{aligned}$$

where Ω is the element of $\mathcal{O}(T_x\mathfrak{X})$ specified by the functions $y^1 \cdots y^K$, that is by the K vectors, $\frac{\partial x^a}{\partial y^1} \cdots \frac{\partial x^c}{\partial y^K}$. (For example, if $\mathfrak{X}$ is the t-axis in t-x-y-z-coordinates for Minkowski space, and Ω is an internal orientation for $\mathfrak{X}$, then the only nonzero component of $\Omega d\Sigma^a$ is $(\Omega d\Sigma)^0 = \pm dt$, where the sign $\pm$ is positive iff Ω agrees with the orientation specified by $\partial/\partial t$.) In general, $d\Sigma^{\alpha\beta\cdots\gamma}$ is, up to sign, the magnitude of the coordinate projection of $d\mathfrak{X}$ onto the $\alpha - \beta - \cdots - \gamma$ coordinate-plane.

The extensors $d\Sigma_A$ and $d\sigma^A$ are related by duality. In the general case of unoriented $d\mathfrak{X}$ we have

$$d\sigma_A = \varepsilon_{AB}\, d\Sigma^B, \tag{4}$$

where the internal and external orientations entering respectively into $d\Sigma$ and $d\sigma$ are related with the overall orientation entering into ε_{AB} by eq. (1a), which in this case becomes

$$\mathcal{O}(T_x M) = \mathcal{O}(T_x M / T_x \mathfrak{X}) \otimes \mathcal{O}(T_x \mathfrak{X}). \tag{5}$$

Divergence and Curl

If $\mathfrak{A}^{ab\cdots c}$ is a (polar or axial) skew tensor of density-weight $+1$ then

$$(\operatorname{div} \mathfrak{A})^{a\cdots b} \equiv \partial_c \mathfrak{A}^{a\cdots bc} \tag{6}$$

is its *divergence* (with components $\partial \mathfrak{A}^{\alpha\cdots\beta\gamma}/\partial x^\gamma$, as the notation suggests). In the multi-index notation, the divergence appears as $\partial_b \mathfrak{A}^{Ab}$.

If $\omega_{b\cdots c}$ is a (polar or axial) form (skew covariant tensor) then

$$(\operatorname{curl} \omega)_{ab\cdots c} \equiv \partial_a \wedge \omega_{b\cdots c} \tag{7}$$

is its *curl* or "exterior derivative" (with components $\partial_\alpha \omega_{\beta\cdots\gamma} - \partial_\beta \omega_{\alpha\cdots\gamma} \cdots \pm \partial_\gamma \omega_{\beta\cdots}$).

Div and curl are dual to each other:

$$*_l \text{ div} = \text{curl } *_l \tag{8}$$

where the 'l' in '$*_l$' stands for 'left', the precise action of $*_l$ being

$$\mathfrak{A}^B \longrightarrow \omega_A = \varepsilon_{AB}\, \mathfrak{A}^B \tag{9}$$

which has ε_{AB} sitting on the left of $\mathfrak{A}^B$, as in (4).

Stokes' Theorem (first form)

$$\int_{\mathfrak{X}} d\sigma_A\, \partial_c\, \mathfrak{A}^{Ac} = \Omega \oint_{\partial\mathfrak{X}} d\sigma_B\, \mathfrak{A}^B \tag{$Stokes$ 1}$$

In this most general guise, the Stokes theorem does not require any of the manifolds involved to be oriented, or even orientable. However, one must be careful to specify the polar or axial character of the tensors involved. (We'll return to the definition of Ω in a moment.)

When the extensors $d\sigma_A$ and $d\sigma_B$ are defined without reference to any orientations, they are, as we have seen, externally axial with respect to $\mathfrak{X}$ and $\partial\mathfrak{X}$, respectively. For consistency then, $\mathfrak{A}^B$ must also be $\mathfrak{X}$-externally-axial. That is, $\mathfrak{A}^B(x)$ must be the product of a tensor-density of weight $+1$ with an element of $\mathcal{O}(T_xM/T_x\mathfrak{X})$ in order that the integrand on the left hand side of (*Stokes* 1) be an ordinary scalar. (Alternatively, $\mathfrak{A}^B$ can be $\partial\mathfrak{X}$-externally-axial, if we move Ω to the other side of the equation. I'm assuming here that one wants the value of the integral to be a true scalar, which normally one does, but if one is willing to have the result of the integration be an axial scalar, then one can sometimes relax the axiality assumption on $\mathfrak{A}$.)

The orientation Ω that stands on the right hand side of (*Stokes* 1), an element of $\mathcal{O}(\mathfrak{X}/\partial\mathfrak{X})$, is the customary *outward* orientation of $\partial\mathfrak{X}$ in $\mathfrak{X}$ (the orientation specified for example by a function f which vanishes on $\partial\mathfrak{X}$ and is negative within $\mathfrak{X}$). Without the presence of Ω, the integral would not make sense, since $d\sigma_B(x)\, \mathfrak{A}^B(x)$ is the product of a $\partial\mathfrak{X}$-externally axial quantity of density weight -1 with a $\mathfrak{X}$-externally axial quantity of density weight $+1$. Such a product is not a true scalar. Rather it lives in $\mathbb{R}\otimes\mathcal{O}(T_x\mathfrak{X}/T_x\partial\mathfrak{X})$, as follows from equations (1) and the rules for multiplication of orientations. Only when multiplied by an element Ω of $\mathcal{O}(T_x\mathfrak{X}/T_x\partial\mathfrak{X})$ — in our case by the particular Ω representing outward orientation — does the product become a valid integrand (assuming, once again, that one wants the integral to be an ordinary scalar.) Strictly speaking, it would thus be more correct to write the Ω inside the integral sign, rather than outside of it.

In the way that the Stokes theorem is ordinarily written, one does not see the orientation Ω explicitly, because tacit assumptions about orientablity and tacit rules for converting one kind of axiality to another are lurking in the background. Most commonly one imagines that orientations have been provided for both M and $\mathfrak{X}$, in which case all reference to axial quantities can be avoided by working in terms of suitable products of the given orientations with the quantities we have been using here. Because of this practice, one often gains the erroneous impression that the theorem is limited to orientable manifolds.

Stated as above, however, the theorem holds for *any* embedded submanifold $\mathfrak{X}$, even, for example, for a Möbius strip in $\mathbb{R}^3$. It thus has a much greater scope of application than is usually appreciated. On the other hand, the tensors $\mathfrak{A}^B$ which occur in practice are more likely to be overall-polar (like the electric field) or overall-axial (like the magnetic field) than $\mathfrak{X}$-externally axial, in which case one must have recourse to some reference orientation of $\mathfrak{X}$, internal or external as the case may be. Although a certain amount of orientablity is thus needed in some applications, the assumption that everything in sight is orientable is almost always much more than one needs.

When a metric is present, it is convenient to rewrite (*Stokes* 1) in terms of ordinary tensors, rather than density-weighted ones. It then appears as:

$$\int_{\mathfrak{X}} dS_A\, \nabla_c A^{Ac} = \Omega \oint_{\partial\mathfrak{X}} dS_B A^B \qquad (\textit{Stokes}\ 1')$$

Stokes' Theorem (second form)

The dual form of Stokes' Theorem is

$$\int\limits_{\mathfrak{X}} (\operatorname{curl}\omega)_B\, d\Sigma^B = \Omega \oint\limits_{\partial\mathfrak{X}} \omega_A\, d\Sigma^A, \qquad (\textit{Stokes}\ 2)$$

with $(\operatorname{curl}\omega)_{aA} = \partial_a \wedge \omega_A$ (also writable as $\nabla_a \wedge \omega_A$ when a metric is present). Here the form ω is $\mathfrak{X}$-internally axial, while the extensors $d\Sigma^B$ and $d\Sigma^A$ are repectively $\mathfrak{X}$-internally axial and $\partial\mathfrak{X}$-internally axial, as usual. And once again, $\Omega \in \mathcal{O}(\mathfrak{X}/\partial\mathfrak{X})$ is the *outward* orientation of $\partial\mathfrak{X}$ in $\mathfrak{X}$.

The full duality relating the two forms of Stokes' Theorem is

$$d\sigma_A = \varepsilon_{AB}\, d\sigma^B =: \ *_{\imath}\, d\Sigma$$
$$\mathfrak{A}^B = \varepsilon^{BC}\, \omega_C =: \ *_{\imath}\, \omega$$
$$*_{\imath}\ \mathrm{curl} = \mathrm{div}\ *_{\imath}$$

On making the substitutions given in the first two lines above, (*Stokes* 1) turns into its dual (*Stokes* 2). The brief computation needed to establish this is free of signs and combinatorial factors if one uses the identity pointed out at the end of the section on the ε-tensor.

Added Note For a "real life" instance of the axial vs. polar distinction, see R.D. Sorkin, "On the Relation Between Charge and Topology", *J. Phys.* **A10:** 717-725 (1977) and **A11:** 795 (1978); and "The Quantum Electromagnetic Field in Multiply-Connected Space", *J. Phys.* **A12:** 403-421 (1979).