

Conventions, Definitions, Identities, and Other Useful Formulae

1. Curvature tensors

Consider a $d + 1$ dimensional manifold \mathcal{M} with metric $g_{\mu\nu}$. The covariant derivative on \mathcal{M} that is metric-compatible with $g_{\mu\nu}$ is ∇_μ .

Christoffel

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \quad (1)$$

Riemann Tensor

$$R^\lambda_{\mu\sigma\nu} = \partial_\sigma \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\sigma}^\lambda + \Gamma_{\mu\nu}^\kappa \Gamma_{\kappa\sigma}^\lambda - \Gamma_{\mu\sigma}^\kappa \Gamma_{\kappa\nu}^\lambda \quad (2)$$

Ricci Tensor

$$R_{\mu\nu} = \delta^\sigma_\lambda R^\lambda_{\mu\sigma\nu} \quad (3)$$

Schouten Tensor

$$S_{\mu\nu} = \frac{1}{d-1} \left(R_{\mu\nu} - \frac{1}{2d} g_{\mu\nu} R \right) \quad (4)$$

$$\nabla^\nu S_{\mu\nu} = \nabla_\mu S^\nu_\nu \quad (5)$$

Weyl Tensor

$$C^\lambda_{\mu\sigma\nu} = R^\lambda_{\mu\sigma\nu} + g^\lambda_\nu S_{\mu\sigma} - g^\lambda_\sigma S_{\mu\nu} + g_{\mu\sigma} S^\lambda_\nu - g_{\mu\nu} S^\lambda_\sigma \quad (6)$$

Commutators of Covariant Derivatives

$$[\nabla_\mu, \nabla_\nu] A_\lambda = R_{\lambda\sigma\mu\nu} A^\sigma \quad (7)$$

$$[\nabla_\mu, \nabla_\nu] A^\lambda = R^\lambda_{\sigma\mu\nu} A^\sigma \quad (8)$$

Bianchi Identity

$$\nabla_\kappa R_{\lambda\mu\sigma\nu} - \nabla_\lambda R_{\kappa\mu\sigma\nu} + \nabla_\mu R_{\kappa\lambda\sigma\nu} = 0 \quad (9)$$

$$\nabla^\nu R_{\lambda\mu\sigma\nu} = \nabla_\mu R_{\lambda\sigma} - \nabla_\lambda R_{\mu\sigma} \quad (10)$$

$$\nabla^\nu R_{\mu\nu} = \frac{1}{2} \nabla_\mu R \quad (11)$$

Bianchi Identity for Weyl

$$\nabla^\nu C_{\lambda\mu\sigma\nu} = (d-2) \left(\nabla_\mu S_{\lambda\sigma} - \nabla_\lambda S_{\mu\sigma} \right) \quad (12)$$

$$\begin{aligned} \nabla^\lambda \nabla^\sigma C_{\lambda\mu\sigma\nu} = \frac{d-2}{d-1} & \left[\nabla^2 R_{\mu\nu} - \frac{1}{2d} g_{\mu\nu} \nabla^2 R - \frac{d-1}{2d} \nabla_\mu \nabla_\nu R - \left(\frac{d+1}{d-1} \right) R_{\mu}{}^\lambda R_{\nu\lambda} \right. \\ & \left. + C_{\lambda\mu\sigma\nu} R^{\lambda\sigma} + \frac{(d+1)}{d(d-1)} R R_{\mu\nu} + \frac{1}{d-1} g_{\mu\nu} \left(R^{\lambda\sigma} R_{\lambda\sigma} - \frac{1}{d} R^2 \right) \right] \end{aligned} \quad (13)$$

2. Euler Densities

Let \mathcal{M} be a manifold with dimension $d + 1 = 2n$ an even number. Normalized so that $\chi(S^{2n}) = 2$.

Euler Number

$$\chi(\mathcal{M}) = \int_{\mathcal{M}} d^{2n}x \sqrt{g} \mathcal{E}_{2n} \quad (14)$$

$$= \int_{\mathcal{M}} \mathbf{e}_{2n} \quad (15)$$

Euler Density

$$\mathcal{E}_{2n} = \frac{1}{(8\pi)^n \Gamma(n+1)} \epsilon^{\mu_1 \dots \mu_{2n}} \epsilon^{\nu_1 \dots \nu_{2n}} R^{\mu_1 \mu_2 \nu_1 \nu_2} \dots R^{\mu_{2n-1} \mu_{2n} \nu_{2n-1} \nu_{2n}} \quad (16)$$

$$\mathbf{e}_{2n} = \frac{1}{(4\pi)^n \Gamma(n+1)} \epsilon_{a_1 \dots a_{2n}} \mathbf{R}^{a_1 a_2} \wedge \dots \wedge \mathbf{R}^{a_{2n-1} a_{2n}} \quad (17)$$

Curvature Two-Form

$$\mathbf{R}^a{}_b = \frac{1}{2} R^a{}_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \quad (18)$$

Examples

$$\begin{aligned} \mathcal{E}_2 &= \frac{1}{8\pi} \epsilon_{\mu\nu} \epsilon_{\lambda\rho} R^{\mu\nu\lambda\rho} \\ &= \frac{1}{4\pi} R \end{aligned} \quad (19)$$

$$\begin{aligned} \mathcal{E}_4 &= \frac{1}{128\pi^2} \epsilon_{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} R^{\mu\nu\alpha\beta} R^{\lambda\rho\gamma\delta} \\ &= \frac{1}{32\pi^2} \left(R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} - 4 R^{\mu\nu} R_{\mu\nu} + R^2 \right) \\ &= \frac{1}{32\pi^2} C^{\mu\nu\lambda\rho} C_{\mu\nu\lambda\rho} - \frac{1}{8\pi^2} \left(\frac{d-2}{d-1} \right) \left(R^{\mu\nu} R_{\mu\nu} - \frac{d+1}{4d} R^2 \right) \end{aligned} \quad (20)$$

3. Hypersurfaces

Let $\Sigma \subset \mathcal{M}$ be a d dimensional hypersurface whose embedding is described locally by an outward-pointing, unit normal vector n^μ . Rather than keeping track of the signs associated with n^μ being either spacelike or timelike, we will just assume that n^μ is spacelike. Indices are lowered and raised using $g_{\mu\nu}$ and $g^{\mu\nu}$, and symmetrization of indices is implied when appropriate.

First Fundamental Form / Induced Metric on Σ

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \quad (21)$$

Projection onto Σ

$$\perp T^{\mu\dots}{}_{\nu\dots} = h^\mu{}_\lambda \dots h^\sigma{}_\nu \dots T^{\lambda\dots}{}_{\sigma\dots} \quad (22)$$

Second Fundamental Form / Extrinsic Curvature of Σ

$$K_{\mu\nu} = \perp(\nabla_\mu n_\nu) = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\lambda n_\sigma = \frac{1}{2} \mathcal{L}_n h_{\mu\nu} \quad (23)$$

Trace of Extrinsic Curvature

$$K = \nabla_\mu n^\mu \quad (24)$$

‘Acceleration’ Vector

$$a^\mu = n^\nu \nabla_\nu n^\mu \quad (25)$$

Surface-Forming Normal Vectors

$$n_\mu = \frac{1}{\sqrt{g^{\nu\lambda} \partial_\nu \alpha \partial_\lambda \alpha}} \partial_\mu \alpha \quad \Rightarrow \quad \perp \nabla_{[\mu} n_{\nu]} = 0 \quad (26)$$

Covariant Derivative on Σ compatible with $h_{\mu\nu}$

$$\mathcal{D}_\mu T^{\alpha\dots}_{\beta\dots} = \perp \nabla_\mu T^{\alpha\dots}_{\beta\dots} \quad \forall \quad T = \perp T \quad (27)$$

Intrinsic Curvature of (Σ, h)

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] A^\lambda = \mathcal{R}^\lambda_{\sigma\mu\nu} A^\sigma \quad \forall \quad A^\lambda = \perp A^\lambda \quad (28)$$

Gauss-Codazzi

$$\perp R_{\lambda\mu\sigma\nu} = \mathcal{R}_{\lambda\mu\sigma\nu} - K_{\lambda\sigma} K_{\mu\nu} + K_{\mu\sigma} K_{\nu\lambda} \quad (29)$$

$$\perp \left(R_{\lambda\mu\sigma\nu} n^\lambda \right) = \mathcal{D}_\nu K_{\mu\sigma} - \mathcal{D}_\sigma K_{\mu\nu} \quad (30)$$

$$\perp \left(R_{\lambda\mu\sigma\nu} n^\lambda n^\sigma \right) = -\mathcal{L}_n K_{\mu\nu} + K_\mu{}^\lambda K_{\lambda\nu} + \mathcal{D}_\mu a_\nu - a_\mu a_\nu \quad (31)$$

Projections of the Ricci tensor

$$\perp (R_{\mu\nu}) = \mathcal{R}_{\mu\nu} + \mathcal{D}_\mu a_\nu - a_\mu a_\nu - \mathcal{L}_n K_{\mu\nu} - K K_{\mu\nu} + 2K_\mu{}^\lambda K_{\nu\lambda} \quad (32)$$

$$\perp (R_{\mu\nu} n^\mu) = \mathcal{D}^\mu K_{\mu\nu} - \mathcal{D}_\nu K \quad (33)$$

$$R_{\mu\nu} n^\mu n^\nu = -\mathcal{L}_n K - K^{\mu\nu} K_{\mu\nu} + \mathcal{D}_\mu a^\mu - a_\mu a^\mu \quad (34)$$

Decomposition of the Ricci scalar

$$R = \mathcal{R} - K^2 - K^{\mu\nu} K_{\mu\nu} - 2\mathcal{L}_n K + 2\mathcal{D}_\mu a^\mu - 2a_\mu a^\mu \quad (35)$$

Lie Derivatives along n^μ

$$\mathcal{L}_n K_{\mu\nu} = n^\lambda \nabla_\lambda K_{\mu\nu} + K_{\lambda\nu} \nabla_\mu n^\lambda + K_{\mu\lambda} \nabla_\nu n^\lambda \quad (36)$$

$$\perp (\mathcal{L}_n \mathcal{F}^{\mu\dots}_{\nu\dots}) = \mathcal{L}_n \mathcal{F}^{\mu\dots}_{\nu\dots} \quad \forall \quad \perp \mathcal{F} = \mathcal{F} \quad (37)$$

4. Sign Conventions for the Action

These conventions follow Weinberg, keeping in mind that he defines the Riemann tensor with a minus sign relative to our definition. They are appropriate when using signature $(-, +, \dots, +)$. The $d + 1$ -dimensional Newton’s constant is $2\kappa^2 = 16\pi G_{d+1}$. The sign on the boundary term follows from our definition of the extrinsic curvature.

Gravitational Action

$$I_G = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} (R - 2\Lambda) + \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma} K \quad (38)$$

$$= \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} (\mathcal{R} + K^2 - K^{\mu\nu} K_{\mu\nu} - 2\Lambda) \quad (39)$$

Gauge Field Coupled to Particles

$$I_M = -\frac{1}{4} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} F^{\mu\nu} F_{\mu\nu} \quad (40)$$

$$- \sum_n m_n \int dp \left(-g_{\mu\nu}(x_n(p)) \frac{dx_n^\mu(p)}{dp} \frac{dx_n^\nu(p)}{dp} \right)^{1/2} \quad (41)$$

$$+ \sum_n e_n \int dp \frac{dx_n^\mu(p)}{dp} A_\mu(x_n(p)) \quad (42)$$

Gravity Minimally Coupled to a Gauge Field

$$I = \int_{\mathcal{M}} d^{d+1}x \sqrt{g} \left[\frac{1}{2\kappa^2} (R - 2\Lambda) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] + \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma} K \quad (43)$$

5. Hamiltonian Formulation

The canonical variables are the metric $h_{\mu\nu}$ on Σ and its conjugate momenta $\pi^{\mu\nu}$. The momenta are defined with respect to evolution in the *spacelike* direction n^μ , so this is not the usual notion of the Hamiltonian as the generator of time translations.

Bulk Lagrangian Density

$$\mathcal{L}_{\mathcal{M}} = \frac{1}{2\kappa^2} \left(K^2 - K^{\mu\nu} K_{\mu\nu} + \mathcal{R} - 2\Lambda \right) \quad (44)$$

Momentum Conjugate to $h_{\mu\nu}$

$$\pi^{\mu\nu} = \frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial (\mathcal{L}_n h_{\mu\nu})} = \frac{1}{2\kappa^2} (h^{\mu\nu} K - K^{\mu\nu}) \quad (45)$$

Momentum Constraint

$$\mathcal{H}_\mu = \frac{1}{\kappa^2} \perp (n^\nu G_{\mu\nu}) = 2\mathcal{D}^\nu \pi_{\mu\nu} = 0 \quad (46)$$

Hamiltonian Constraint

$$\mathcal{H} = -\frac{1}{\kappa^2} n^\mu n^\nu G_{\mu\nu} = 2\kappa^2 \left(\pi^{\mu\nu} \pi_{\mu\nu} - \frac{1}{d-1} \pi^2 \right) + \frac{1}{2\kappa^2} (\mathcal{R} - 2\Lambda) = 0 \quad (47)$$

6. Conformal Transformations

The dimension of spacetime is $d+1$. Indices are raised and lowered using the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

Metric

$$\hat{g}_{\mu\nu} = e^{2\sigma} g_{\mu\nu} \quad (48)$$

Christoffel

$$\hat{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + \Theta_{\mu\nu}^\lambda \quad (49)$$

$$\Theta_{\mu\nu}^\lambda = \delta^\lambda_\mu \nabla_\nu \sigma + \delta^\lambda_\nu \nabla_\mu \sigma - g_{\mu\nu} \nabla^\lambda \sigma \quad (50)$$

Riemann Tensor

$$\widehat{R}^\lambda_{\mu\rho\nu} = R^\lambda_{\mu\rho\nu} + \delta^\lambda_\nu \nabla_\mu \nabla_\rho \sigma - \delta^\lambda_\rho \nabla_\mu \nabla_\nu \sigma + g_{\mu\rho} \nabla_\nu \nabla^\lambda \sigma - g_{\mu\nu} \nabla_\rho \nabla^\lambda \sigma \quad (51)$$

$$+ \delta^\lambda_\rho \nabla_\mu \sigma \nabla_\nu \sigma - \delta^\lambda_\nu \nabla_\mu \sigma \nabla_\rho \sigma + g_{\mu\nu} \nabla_\rho \sigma \nabla^\lambda \sigma - g_{\mu\rho} \nabla_\nu \sigma \nabla^\lambda \sigma \quad (52)$$

$$+ \left(g_{\mu\rho} \delta^\lambda_\nu - g_{\mu\nu} \delta^\lambda_\rho \right) \nabla^\alpha \sigma \nabla_\alpha \sigma \quad (53)$$

Ricci Tensor

$$\widehat{R}_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} \nabla^2 \sigma - (d-1) \nabla_\mu \nabla_\nu \sigma + (d-1) \nabla_\mu \sigma \nabla_\nu \sigma \quad (54)$$

$$- (d-1) g_{\mu\nu} \nabla^\lambda \sigma \nabla_\lambda \sigma \quad (55)$$

Ricci Scalar

$$\widehat{R} = e^{-2\sigma} \left(R - 2d \nabla^2 \sigma - d(d-1) \nabla^\mu \sigma \nabla_\mu \sigma \right) \quad (56)$$

Schouten Tensor

$$\widehat{S}_{\mu\nu} = S_{\mu\nu} - \nabla_\mu \nabla_\nu \sigma + \nabla_\mu \sigma \nabla_\nu \sigma - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \sigma \nabla_\lambda \sigma \quad (57)$$

Weyl Tensor

$$\widehat{C}^\lambda_{\mu\rho\nu} = C^\lambda_{\mu\rho\nu} \quad (58)$$

Normal Vector

$$\widehat{n}^\mu = e^{-\sigma} n^\mu \quad \widehat{n}_\mu = e^\sigma n_\mu \quad (59)$$

Extrinsic Curvature

$$\widehat{K}_{\mu\nu} = e^\sigma \left(K_{\mu\nu} + h_{\mu\nu} n^\lambda \nabla_\lambda \sigma \right) \quad (60)$$

$$\widehat{K} = e^{-\sigma} \left(K + d n^\lambda \nabla_\lambda \sigma \right) \quad (61)$$

7. Small Variations of the Metric

Consider a small perturbation to the metric of the form $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$. All indices are raised and lowered using the unperturbed metric $g_{\mu\nu}$ and its inverse. All quantities are expressed in terms of the perturbation to the metric with lower indices, and never in terms of the perturbation to the inverse metric. As in the previous sections, ∇_μ is the covariant derivative on \mathcal{M} compatible with $g_{\mu\nu}$ and \mathcal{D}_μ is the covariant derivative on a hypersurface Σ compatible with $h_{\mu\nu}$.

Inverse Metric

$$g^{\mu\nu} \rightarrow g^{\mu\nu} - g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} g^{\lambda\rho} \delta g_{\alpha\lambda} \delta g_{\beta\rho} + \dots \quad (62)$$

Square Root of Determinant

$$\sqrt{g} \rightarrow \sqrt{g} \left(1 + \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} + \dots \right) \quad (63)$$

Variational Operator

$$\delta(g_{\mu\nu}) = \delta g_{\mu\nu} \quad \delta^2(g_{\mu\nu}) = \delta(\delta g_{\mu\nu}) = 0 \quad (64)$$

$$\delta(g^{\mu\nu}) = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \quad \delta^2(g^{\mu\nu}) = \delta\left(-g^{\mu\lambda} g^{\nu\rho} \delta g_{\lambda\rho}\right) = 2 g^{\mu\alpha} g^{\nu\beta} g^{\lambda\rho} \delta g_{\alpha\lambda} \delta g_{\beta\rho} \quad (65)$$

$$\mathcal{F}(g + \delta g) = \mathcal{F}(g) + \delta\mathcal{F}(g) + \frac{1}{2} \delta^2\mathcal{F}(g) + \dots + \frac{1}{n!} \delta^n\mathcal{F}(g) + \dots \quad (66)$$

Christoffel (All Orders)

$$\delta\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} \left(\nabla_\mu \delta g_{\rho\nu} + \nabla_\nu \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\nu} \right) \quad (67)$$

$$\delta^2\Gamma_{\mu\nu}^\lambda = -g^{\lambda\alpha} g^{\rho\beta} \delta g_{\alpha\beta} \left(\nabla_\mu \delta g_{\rho\nu} + \nabla_\nu \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\nu} \right) \quad (68)$$

$$\delta^n\Gamma_{\mu\nu}^\lambda = \frac{n}{2} \delta^{n-1} \left(g^{\lambda\rho} \right) \left(\nabla_\mu \delta g_{\rho\nu} + \nabla_\nu \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\nu} \right) \quad (69)$$

Riemann Tensor

$$\delta R^\lambda_{\mu\sigma\nu} = \nabla_\sigma \delta\Gamma_{\mu\nu}^\lambda - \nabla_\nu \delta\Gamma_{\mu\sigma}^\lambda \quad (70)$$

Ricci Tensor

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma_{\mu\nu}^\lambda - \nabla_\nu \delta\Gamma_{\mu\lambda}^\lambda \quad (71)$$

$$= \frac{1}{2} \left(\nabla^\lambda \nabla_\mu \delta g_{\lambda\nu} + \nabla^\lambda \nabla_\nu \delta g_{\mu\lambda} - g^{\lambda\rho} \nabla_\mu \nabla_\nu \delta g_{\lambda\rho} - \nabla^2 \delta g_{\mu\nu} \right) \quad (72)$$

Ricci Scalar

$$\delta R = -R^{\mu\nu} \delta g_{\mu\nu} + \nabla^\mu \left(\nabla_\nu \delta g_{\mu\nu} - g^{\lambda\rho} \nabla_\mu \delta g_{\lambda\rho} \right) \quad (73)$$

Surface Forming Normal Vector

$$\delta n_\mu = \frac{1}{2} n_\mu n^\nu n^\lambda \delta g_{\nu\lambda} = \frac{1}{2} \delta g_{\mu\nu} n^\nu + c_\mu \quad (74)$$

$$c_\mu = \frac{1}{2} n_\mu n^\nu n^\lambda \delta g_{\nu\lambda} - \frac{1}{2} \delta g_{\mu\nu} n^\nu = -\frac{1}{2} h_\mu^\lambda \delta g_{\lambda\nu} n^\nu \quad (75)$$

Extrinsic Curvatures

$$\begin{aligned} \delta K_{\mu\nu} &= \frac{1}{2} n^\alpha n^\beta \delta g_{\alpha\beta} K_{\mu\nu} - n_\mu c^\lambda h_{\lambda\nu} - n_\nu c^\lambda h_{\mu\lambda} + \frac{1}{2} \delta g_{\lambda\rho} n^\rho \left(n_\mu K^\lambda_\nu + n_\nu K_\mu^\lambda \right) \\ &\quad - \frac{1}{2} h_\mu^\lambda h_\nu^\rho n^\alpha \left(\nabla_\lambda \delta g_{\alpha\rho} + \nabla_\rho \delta g_{\lambda\alpha} - \nabla_\alpha \delta g_{\lambda\rho} \right) \end{aligned} \quad (76)$$

$$\delta K = -\frac{1}{2} K^{\mu\nu} \delta g_{\mu\nu} - \frac{1}{2} n^\mu \left(\nabla_\nu \delta g_{\mu\nu} - g^{\nu\lambda} \nabla_\mu \delta g_{\nu\lambda} \right) + \mathcal{D}_\mu c^\mu \quad (77)$$