APPENDIX I

DIFFERENTIAL MANIFOLDS

Topology, Geometry, and Submanifolds

The purpose of this appendix is to provide an introduction to the mathematical structures used in the thesis, and also to establish notational conventions. While I recognize that most readers will be totally familiar with the items discussed, I have attempted to make the presentation fairly complete for the benefit of those who do not have a background in topology or differential geometry. The emphasis is on definitions and an understanding of the basic concepts. Most results are stated without proof.

Numerous comprehensible references exist for this material, but I shall mention here only those monographs that I have found particularly useful. 'General Topology' by Lipschutz [38] is a good primer for the basics of point set topology, leading up to but not including the definition of a manifold. Munkres [39] provides clear definitions of manifold, differential structure, and differential manifold, but does not indicate that it is the field structure, rather than the topology, of the real numbers which gives rise to the possibility of defining derivatives. The fundamental definition of derivative is contained in Porteous [40] along with the basics of algebra, topology and a great wealth of other information that should be of interest to many physicists, but which cannot be treated here. Guillemin and Pollack, in their well illustrated text 'Differential Topology' [33] , explore many aspects of that vaguely defined field. They develop both differential and integral calculus on manifolds and show how these relate to the global structure of differential manifolds. The properties of differentiable maps from one manifold into another are shown to depend significantly on the global topologies of the two manifolds.

Differential geometry, considered in its broadest sense as the mathematics of differentiable fields on manifolds, is perhaps the branch of modern mathematics most familiar to physicists. An excellent classical text is 'Ricci Calculus' by Schouten [21], but I much prefer the more modern treatment and notation of Kobayashi and Nomizu [41]. Throughout this thesis I will employ a coordinate free notation which is similar to that used in reference [41], however the elegant fibre bundle picture which they develop will not be used here because it would be overkill for the simple geometries to be considered.

1. Topology

Topology is the study of sets and their subsets. Let X be a non-empty set and let Sub X be the class consisting of all subsets of X . A subset $\mathcal{T}$ of Sub X is a topology on X iff $\mathcal{T}$ satisfies

(i) $\emptyset, X \in \mathcal{T}$;

(ii) for all $A,B \in \mathcal{T}$, $A \cap B \in \mathcal{T}$;

(iii) for all $A_i \in \mathcal{S} \subset \mathcal{T}$, $\bigcup_i A_i \in \mathcal{T}$,

where $\emptyset$ is the null set. The elements of $\mathcal{T}$ are called the open sets of the topology. A set X , together with a topology $\mathcal{T}$ on X , is called a topological space, $(X,\mathcal{T})$. Normally this will be denoted by X alone, with the topology $\mathcal{T}$ assumed to be known. It is important to recognize that the open subsets of X may generally be chosen in more than one way, each different choice giving rise to a different topology on X .

A base for the topology $\mathcal{T}$ on X is a subset $\mathcal{B} \subset \mathcal{T}$ such that every open set $A \in \mathcal{T}$ is the union of members of $\mathcal{B}$. An open cover or cover for a topological space $(X,\mathcal{T})$ is a subset $\mathcal{S}$ of $\mathcal{T}$ such that $\bigcup\mathcal{S} = X$. Every base is a cover, but not every cover is a base. If for each cover $\mathcal{S}$ for X a finite subset $\mathcal{S}'$ of $\mathcal{S}$ covers X then X is said to be compact.

A topological space X is a Hausdorff space iff for each pair of distinct points $a,b \in X$ there exist open sets $A,B \in \mathcal{T}$ such that $a \in A$, $b \in B$, and $A \cap B = \emptyset$.

We are often concerned with maps from one topological space to another. Let $(X,\mathcal{T})$ and $(X',\mathcal{T}')$ be topological spaces. A function $f:X \to X'$ is said to be continuous iff the inverse image of every open set of X' is an open set of X, that is, iff

$$f^{-1}[A] \in \mathcal{T} \quad \text{for all} \quad A \in \mathcal{T}' .$$

Two topological spaces X and X' are called homeomorphic or topologically equivalent if there exists a bijective map $f:X \to X'$ such that both f and f^{-1} are continuous. The map f is called a homeomorphism.

Let $g:W \to X$ be a continuous map with domain a subset of the topological space W and let $a \in W\backslash\text{dom } g$ (i.e. a is an element of the complement of $\text{dom } g$ in W). Then g has a limit b at a if there exists a continuous map $f:W \to X$ such that $f(a) = b$ and $f(w) = g(w)$ for all $w \in \text{dom } g$. If $\text{dom } g$ is a proper subset of W, a is an element of the closure of $\text{dom } g$, and X is a Hausdorff space, then b is unique. Porteous notes that this is one of the most important features of a

Hausdorff space [40].

If $Y \subset X$ and $\mathcal{T}_Y = \{A_Y : A_Y = A \cap Y$ for some $A \in \mathcal{T}\}$ then $(Y, \mathcal{T}_Y)$ is a topological <u>subspace</u> of $(X, \mathcal{T})$ with the <u>induced</u> topology. Any subspace of a Hausdorff space is a Hausdorff space.

Let X and Y be topological spaces and let $W = X \times Y$ be the cartesian product of the sets X and Y. The <u>product topology</u> induced on W from X and Y consists of all those subsets of W that can be constructed as the union of sets of the form $A \times B$ where A is open in X and B is open in Y. Unless specified otherwise, the product $X \times Y$ of two topological spaces will be assumed to have the product topology.

Until now I have avoided reference to numbers. However, the number systems with which we are so familiar play an important role in topology. Porteous [40] starts with the null set and builds up the <u>natural</u> numbers through a constructive process. The non-existence of a largest natural number is the

> Archimedian Order Axiom: The set $\omega = \{0, 1, 2, \ldots\}$ of natural numbers is not bounded from above.

Addition, multiplication, and exponentiation are defined in a set theoretic fashion. Further constructions yield the <u>integers</u> $\mathbb{Z}$ and the <u>rational</u> numbers $\mathbb{Q}$. The <u>real</u> numbers $\mathbb{R}$ are then defined to be the elements of an ordered field with the usual operations of addition and multiplication, containing $\mathbb{Q}$ as an ordered subfield, such that

> (Least Upper Bound Axiom): If A is a subset of $\mathbb{R}$ bounded

from above, then A has a least upper bound.
This is equivalent to the statement that the real numbers are <u>complete</u>, that is, that every Cauchy sequence of real numbers converges to a point in $\mathbb{R}$.

The topology of $\mathbb{R}$ is defined with the use of the open intervals $S = \{x : a < x < b \; ; \; x,a,b \in \mathbb{R}\}$. Let $A \subset \mathbb{R}$. A point $p \in A$ is an <u>interior point</u> of A iff p belongs to some open interval S_p which is contained in A . The set A is <u>open</u> iff each of its points is an interior point [38]. Note that the topology of $\mathbb{R}$ does not depend explicitly on the operations of addition or multiplication, but only on the well ordered property of the real numbers. As is usual, $\mathbb{R}^n$ will be used to denote the topological product $\mathbb{R}\times\mathbb{R}\times\ldots\times\mathbb{R}$ with n factors.

A point $x \in \mathbb{R}^n$ may be represented by the ordered n-tuple of its components $(x^1, \ldots ,x^n)$. If we exploit the field structure of the real numbers then we may use these components to define a <u>norm</u> on $\mathbb{R}^n$:

$$|x| = \sqrt{(x^1)^2 + \ldots + (x^n)^2} \quad . \tag{1.1}$$

For any $\delta>0$ the sets $\{x \in \mathbb{R}^n : |x|<\delta\}$, $\{x \in \mathbb{R}^n : |x|\leq\delta\}$, and $\{x \in \mathbb{R}^n : |x|=\delta\}$ are called respectively the <u>open ball</u>, the <u>closed ball</u>, and the <u>sphere</u> in $\mathbb{R}^n$, centred on the origin, with radius δ . The origin or additive identity of $\mathbb{R}^n$ need not play such a fundamental role, however. Instead we can make use of the <u>metric</u> or <u>distance function</u> :

$$d(x,y) = |x - y| \quad , \tag{1.2}$$

which specifies the euclidean distance between any two points x and y in $\mathbb{R}^n$. An open ball with centre x and radius δ is then the set $B(x,\delta) = \{y \in \mathbb{R}^n : d(x,y) < \delta\}$. The class of all such open balls provides a base for a <u>metric topology</u> $\mathcal{T}_d$ on the set $\mathbb{R}^n$. For finite dimension n this metric topology coincides with the usual topology on $\mathbb{R}^n$. When no particular point of $\mathbb{R}^n$ is singled out as the origin, the space $(\mathbb{R}^n, \mathcal{T}_d)$ is called an <u>affine space</u> and denoted by $\mathbf{A}^n$.

One more word must be added to our mathematical vocabulary before we can define manifold. A set X is said to be <u>countable</u> iff there exists an injective map from X into ω , that is, iff the natural numbers may be used to uniquely label the elements of X .

A <u>topological manifold</u> M is a Hausdorff space with a countable basis, satisfying the following condition: There is a number $n \in \omega$ such that each point of M has a neighbourhood homeomorphic with an open set of $\mathbb{R}^n$. The number n is the <u>dimension</u> of M . If A is an open proper subset of M then $M \backslash A$ is a <u>manifold with boundary</u>. The set $\bar{A} \backslash A$, where $\bar{A}$ is the closure of A in M , is the common boundary of $\bar{A}$ and $M \backslash A$ and is an (n-1)-dimensional manifold. The notation ∂M is commonly used to denote the boundary of a manifold-with-boundary M . For any M , $\partial\partial M = \emptyset$.

The spaces A and $\partial\bar{A}$ are examples respectively of n and (n-1)-dimensional submanifolds of M . An m-dimensional <u>submanifold</u> M' of a manifold M , with $0 \leq m \leq n$, is a topological subspace of M which (with the induced topology) is an m-dimensional manifold. A submanifold of dimension zero has the discrete topology, and consists of isolated points in M .

The product space $M \times M'$ of two manifolds M and M' is a manifold with $\dim(M \times M') = \dim(M) + \dim(M')$.

It is possible for a manifold to be made up of several disjoint pieces. A topological space (manifold) is <u>connected</u> iff it is not the union of two non-empty disjoint open sets. Unless specified otherwise a manifold will be assumed to be connected.

Every connected manifold M is metrizable, that is, M admits a distance function $d: M \times M \to \mathbb{R}$ which is <u>compatible</u> with the topology of M and which satisfies, for all $x,y,z \in M$:

$$\left.\begin{array}{ll} \text{(i)} & d(x,y) \geq 0 \quad \text{and} \quad d(x,x) = 0 \;; \\ \text{(ii)} & d(x,y) = d(y,x) \;; \\ \text{(iii)} & d(x,z) \leq d(x,y) + d(y,z) \;; \\ \text{(iv)} & \text{If } x \neq y \text{ then } d(x,y) > 0 \;. \end{array}\right\} \quad (1.3)$$

The open balls defined with the use of d provide a base for the topology of M . However the topology of M does not uniquely determine d , nor is every distance function necessarily compatible with the topology of M . This will be elaborated in the next section.

Let M be an n-dimensional manifold with $n \geq 1$. An <u>open n-cell</u> in M is an n-dimensional submanifold of M which is homeomorphic to an open ball in $\mathbb{R}^n$. A <u>closed n-cell</u> is a subspace $S \subset M$ which is homeomorphic to a closed ball in $\mathbb{R}^n$. The boundary ∂S of a closed n-cell S is an (n-1)-sphere. In general, an <u>(n-1)-sphere</u> S^{n-1} is a manifold homeomorphic with a sphere in $\mathbb{R}^n$.

All manifolds of a given dimension are <u>locally</u> the same: if M is an n-dimensional manifold, then every point $x \in M$ is contained in the interior of some closed n-cell in M . When considered in their entirety, however, two manifolds of the same dimension can be very different. It is thus the <u>global</u> structure of M which serves to differentiate it from other n-dimensional manifolds. This is best illustrated with specific examples in one and two dimensions:

(i) There are only two distinct (connected) 1-dimensional manifolds (without boundary). These are the line $A^1 = \mathbb{R}$ and the circle or 1-sphere S^1 (Figure 1.1). They are distinguished by the fact that S^1 is compact while A^1 is not. If we choose a point $x \in S^1$ and a closed 1-cell $C \subset S^1$ which contains x in its interior, then $S^1 \backslash C$ is an open 1-cell which is homeomorphic with A^1 . The line can thus be considered as a 1-sphere which has had a closed 1-cell removed or "cut out".

(ii) There is a countable infinity of distinct 2-manifolds. They fall into two natural classes, orientable and non-orientable, and may all be constructed from the 2-sphere S^2 by a process of "cutting and pasting" [42]. Although it is technically a purely

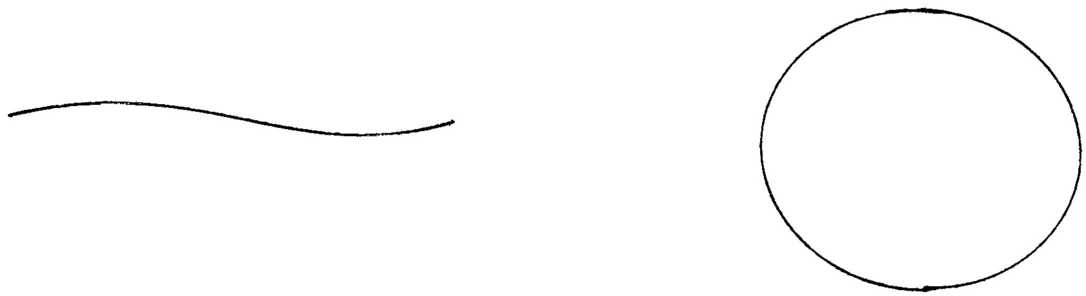

Figure 1.1 The line and circle are, up to homeomorphism, the only 1-dimensional manifolds.

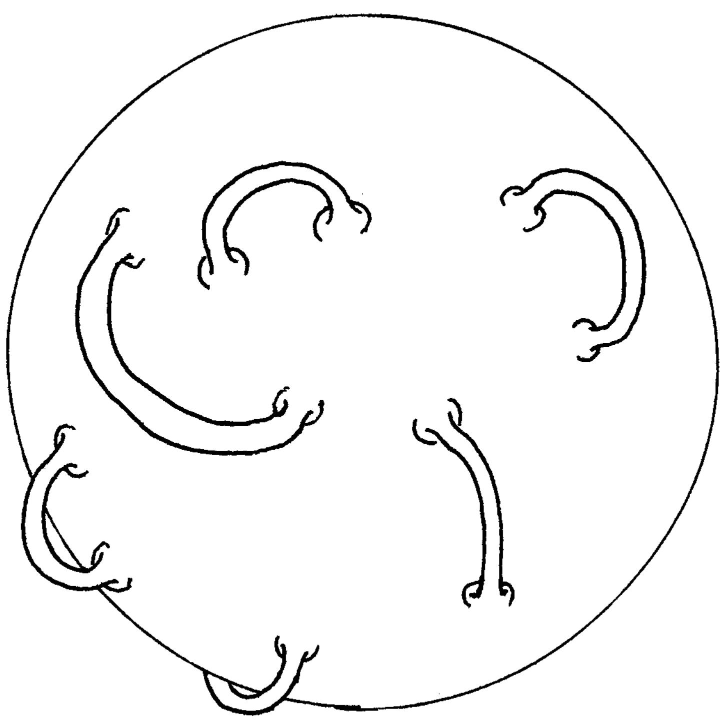

Figure 1.2 A sphere with (at least) six handles.

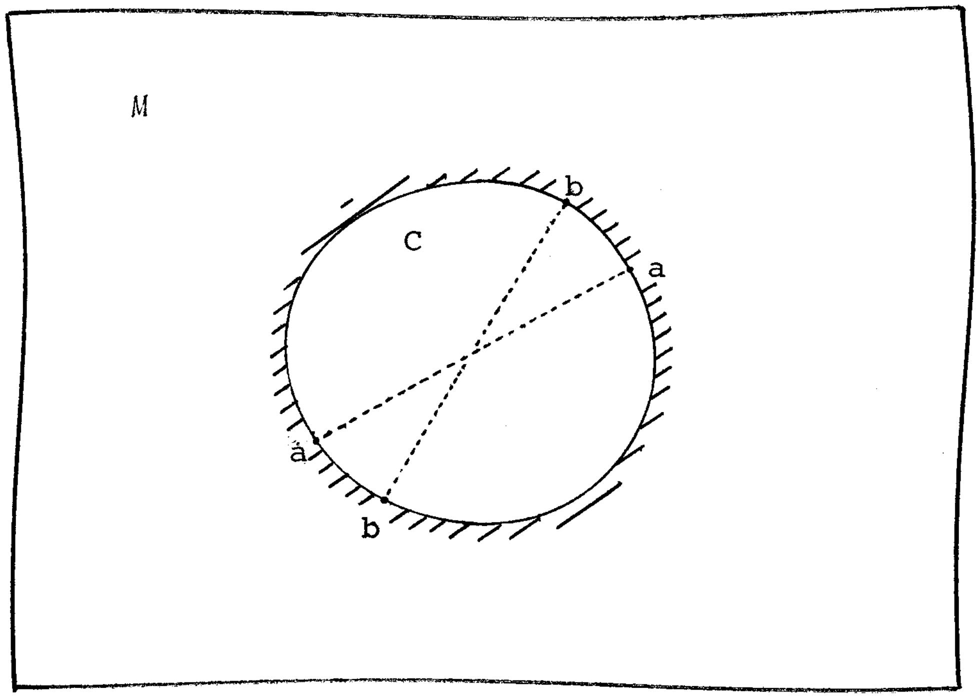

Figure 1.3 A crosscap is formed by removing from M the interior of the 2-cell, C , and then identifying opposite points on the resulting boundary.

topological concept we shall delay the definition of orientability until the next section, relying for the time being on the intuitive notion that an orientable surface has two sides (e.g. the inside and outside of the unit sphere in $\mathbb{R}^3$) while a non-orientable surface has only one side. Examples of compact orientable 2-manifolds are the sphere S^2 , the torus $S^1 \times S^1$ which may also be considered as a sphere with one handle, and more generally a surface of genus h or sphere with h handles (Figure 1.2). The simplest compact non-orientable 2-manifold is a sphere with one crosscap. Being the non-orientable analogue of a handle, a crosscap is constructed by removing from a 2-dimensional manifold M an open 2-cell C and then identifying (with the use of an orientation reversing homeomorphism $f:\partial\overline{C} \to \partial\overline{C}$) "opposite" points of the resulting circular boundary (Figure 1.3). The non-orientable analogue of a sphere with h handles is a sphere with q crosscaps. A sphere with 2 crosscaps is topologically equivalent to the well known Klein bottle. Non-compact 2-manifolds may be constructed from a compact 2-manifold M by removing from M any number r of closed 2-cells. The resulting manifold is then said to have r contours. The general result which emerges is that any 2-dimensional manifold (without boundary) is characterized topologically by its orientability class, its contour number r , and its number of handles or crosscaps, h or q . The Moebius band, for example, is a non-orientable surface with one contour and one crosscap.

For manifolds of dimension greater than two the problem of analyzing the global structure becomes much more complex. The vast field of algebraic topology [42],[43] provides general procedures for describing some aspects of the global structure of a given manifold M, but no scheme is known for uniquely identifying each distinct manifold. It is thought, in fact, that for $n \geq 4$ it is impossible to find a classification scheme which will uniquely label each and every n-dimensional manifold (up to homeomorphism) [32]. The 3-dimensional problem, on the other hand, may be solvable. Much progress has been made toward a partial solution and I believe that some of the results may have a direct and profound application in physics. This is discussed in detail in Chapter 4.

2. Differential Manifolds

We return now to the local properties of manifolds. Let U be an open set of an n-dimensional manifold M and let $\phi : U \to \mathbb{R}^n$ be a homeomorphism of U onto an open set in $\mathbb{R}^n$. The pair (U,ϕ) is called a chart on M and the n component functions of ϕ determine a local coordinate system in U. An atlas of M is a family of charts (U_i,ϕ_i) on M such that the open sets U_i cover M.

A mapping f of an open set of $\mathbb{R}^n$ into $\mathbb{R}^m$ is said to be of class C^r, $r \in \omega$, if its m component functions $f^1, \ldots, f^m$ are r times continuously differentiable. If f is real analytic then it is said to be of class C^ω. By C^0 we mean that f is continuous.

A <u>differential structure</u> $\mathcal{D}$ of class C^r on an n-manifold M is an atlas (U_i,ϕ_i) on M such that

(i) If (U_i,ϕ_i) and (U_j,ϕ_j) belong to $\mathcal{D}$, then

$$\phi_i\phi_j^{-1} : \phi_j(U_i \cap U_j) \to \mathbb{R}^n \tag{2.1}$$

is differentiable of class C^r; and

(ii) The atlas $\mathcal{D}$ is maximal with respect to property (i); i.e., if any chart not in $\mathcal{D}$ is adjoined to $\mathcal{D}$, then (i) fails.

The manifold M together with the differential structure $\mathcal{D}$ is called an n-dimensional <u>differential manifold</u> of class C^r. A differential manifold of class C^∞ is called a <u>smooth</u> manifold. Although it does not appear explicitly, we have exploited for the first time the full field structure of the real numbers in the definition of a differential structure.

Let M be of class C^r, $r \geq 1$, and let $x \in U_i \cap U_j$. Denote by $a_{ij}(x)$ the $n\times n$-matrix of first partial derivatives of the functions (2.1) evaluated at $\phi_j(x)$, i.e. the Jacobian matrix of (2.1). The atlas $\mathcal{D}$ is called <u>oriented</u> if the determinant of $a_{ij}(x)$ is positive for all i,j and $x \in U_i \cap U_j$. If $\mathcal{D}$, $\mathcal{D}'$ are two distinct oriented atlases of M then the Jacobian matrices of $\phi_i'\phi_j^{-1}$ have determinants which, for all i,j and $x \in U_i' \cap U_j$, are either always positive or always negative. Accordingly, the orientations of $\mathcal{D}$ and $\mathcal{D}'$ are said to be either the same or opposite. The class of all atlases which have the same orientation as $\mathcal{D}$ is called an orientation of M. If M admits an oriented atlas then it is orientable, and has exactly two orientations. If no oriented atlas exists for M then M is non-orientable.

Given two differential manifolds M and M' of dimensions n and n' and of class C^r, a map $f:M \to M'$ is said to be of class C^k, $k \leqslant r$, if for all i,j the map

$$\phi_i' f \phi_j^{-1} : \phi_j(U_j \cap f^{-1}(U_i')) \to \mathbb{R}^{n'} \qquad (2.2)$$

is of class C^k. The rank of f at $x \in U_j$ is defined to be the rank of the Jacobian matrix of the map (2.2) at $\phi_j(x)$. If rank $f = n$ at each point $x \in M$, f is said to be an immersion. The map f is proper if the preimage of every compact set in M' is compact in M. An immersion that is injective and proper is called an embedding. If f is a homeomorphism of M and M' and an immersion then it is called a diffeomorphism. In this case M and M' are said to be diffeomorphic. An embedding $f:M \to M'$ maps M diffeomorphically onto a submanifold of M'.

Not every manifold admits a differential structure [44], nor do two different differential structures on the same manifold always give rise to diffeomorphic differential manifolds. However, if $\dim M \leqslant 3$ or if M is homeomorphic to $\mathbb{R}^n$, $n \neq 4$, then M admits a differential structure [32] and all differential manifolds $(M,\mathcal{D})$ are diffeomorphic [39].

Two important theorems [39],[45] allow us to restrict our attention to differential manifolds of class C^∞ and to smooth (C^∞) maps between C^∞ manifolds. The first states that if M and M' are C^∞ manifolds and $f:M \to M'$ is a C^1 immersion, embedding, or diffeomorphism, then f may be approximated by a C^∞ immersion, embedding, or diffeomorphism, respectively. The second theorem states that every differential structure of class C^1 on a manifold M contains a C^∞ structure. From now on, unless indicated otherwise by the context, the term manifold will be taken to mean smooth differential manifold, and all maps between manifolds will be assumed to be smooth.

Differential manifolds are of special interest to physicists because they serve as the substrate for all of the geometrical structures with which we deal. The simplest such structure is a function or <u>scalar field</u> $f:M \to R$ on the manifold M. We shall denote the algebra of all such (smooth) functions by $\mathcal{F}(M)$. A <u>differentiable curve</u>, or simply <u>curve</u>, in M is a mapping of a closed interval $[a,b] \subset R$ into M. If $x(t)$, $t \in [a,b]$ is a curve in M, then the <u>vector</u> tangent to $x(t)$ at $p=x(t_0)$ is the derivative operator $\vec{X}_p$ defined by

$$\vec{X}_p f = (df(x(t))/dt)_{t_0} \qquad \text{for all} \quad f \in \mathcal{F}(M) \; . \tag{2.3}$$

The set of all vectors that can be constructed at p is, in a natural way, an n-dimensional vector space (where $n = \dim M$) called the <u>tangent space</u> to M at p, and is denoted by $T_p(M)$.

If $(U,\phi) \in \mathcal{D}$ is a chart on M such that $p \in U$, and the quantities $u^i = \phi^i(x)$, $i=1,\ldots,n$, are the local coordinates of a point x, then the curve $x(t)$ has the coordinate representation $x^i(t) = \phi^i(x(t))$. Within this coordinate picture we find that

$$(df(x(t))/dt)_{t_0} = \sum_i (\partial f/\partial u^i)_p \cdot (dx^i(t)/dt)_{t_0} \quad , \tag{2.4}$$

so that the vector $\vec{X}_p$ may be represented by the differential operator

$$\vec{X}_p = \sum_i (dx^i(t)/dt)_{t_0} (\partial /\partial u^i)_p \quad . \tag{2.5}$$

The partial derivative operators $(\partial /\partial u^i)_p$, or simply $(\partial_i)_p$, are linearly independent vectors at p and constitute a basis, called a <u>coordinate basis</u>, for $T_p(M)$. The numbers

$$X^i_p = (dx^i(t)/dt)_{t_0} \tag{2.6}$$

are called the <u>components</u> of the vector $\vec{X}_p$ in the coordinate basis. Although it is always possible to represent a vector in this fashion it is certainly not necessary, nor even desirable in many situations. In that which follows, a <u>coordinate free</u> formalism will be used almost exclusively so that the basic geometrical concepts being investigated will remain in the fore.

A <u>vector field</u> $\vec{X}$ on M is an assignment of a vector $\vec{X}_p$ to each point p of M. Acting pointwise on a function $f \in \mathcal{F}(M)$, $\vec{X}$ generates a new function $\vec{X}f$ defined by

$$(\vec{X}f)(p) = \vec{X}_p f \quad . \tag{2.7}$$

If $\vec{X}f$ is a smooth function on M for each $f \in \mathcal{F}(M)$ then $\vec{X}$ is a smooth vector field. Just as we are restricting our attention to smooth maps we shall consider only smooth vector fields, and we shall use the terms vector field and smooth vector field interchangeably. The (infinite dimensional) space of all vector fields on M will be denoted by $\mathcal{T}(M)$.

Let $T_p^*(M)$ denote the vector space dual to $T_p(M)$, that is the space of all linear maps $\omega_p : T_p(M) \to \mathbb{R}$. A 1-form $\underline{\omega}$ on M is an assignment to each point $p \in M$ of an element of $T_p^*(M)$. If $\vec{X} \in \mathcal{T}(M)$ and $\underline{\omega}$ is a 1-form on M , then $\underline{\omega}(\vec{X})$ is the function defined by

$$\underline{\omega}(\vec{X})(p) = \omega_p(\vec{X}_p) \quad , \quad p \in M \quad . \tag{2.8}$$

If $\underline{\omega}(\vec{X}) \in \mathcal{F}(M)$ for all $\vec{X} \in \mathcal{T}(M)$, then $\underline{\omega}$ is a C^∞ 1-form (differential form of degree 1). As usual, we shall consider only smooth 1-forms, denoting the space of all C^∞ 1-forms by $\mathcal{T}_1(M)$. The 1-form $\underline{du}^i$, $i \in \{1,\ldots,n\}$, dual to the coordinate basis vector field $\vec{\partial}_i$ in the neighbourhood $U \subset M$ is defined by

$$\underline{du}^i(\vec{\partial}_j) = \delta^i_j \quad , \text{ for all } j \in \{1,\ldots,n\} \tag{2.9}$$

where δ^i_j is the Kronecker δ function. Accordingly, $\underline{du}^1_p, \ldots ,$ $\underline{du}^n_p$ are linearly independent in $T_p^*(M)$ and are called the coordinate (basis) 1-forms. In the neighbourhood U , any $\underline{\omega} \in \mathcal{T}_1(M)$ may always be written in the form

$$\underline{\omega} = \sum_i \omega_i \underline{du}^i \tag{2.10}$$

with the components ω_i functions in $\mathcal{F}(M)$.

The second dual V^{**} of a finite dimensional vector space V is naturally isomorphic to V. Thus, just as a 1-form $\underline{\omega}$ maps $\mathcal{T}(M)$ into $\mathcal{F}(M)$, we may think of a vector field $\vec{X}$ as a map from $\mathcal{T}_1(M)$ into $\mathcal{F}(M)$. As a natural generalization of this picture we define a <u>tensor</u> T_p <u>of type</u> (r,s) at $p \in M$ to be an $(r+s)$-linear map from $T_p^*(M)\times \dots \times T_p^*(M)\times T_p(M)\times \dots \times T_p(M)$ into the real numbers, where $T_p^*(M)$ appears r times and $T_p(M)$ appears s times. The space of all such maps is an n^{r+s}-dimensional vector space $(T_s^r)_p(M)$ called the tensor space of type (r,s) at p. A (smooth) <u>tensor field</u> T <u>of type</u> (r,s) on M is an assignment, to each point $p \in M$, of a tensor T_p of type (r,s) such that the function $T(\underline{\omega}_1, \dots, \underline{\omega}_r, \vec{X}_1, \dots, \vec{X}_s)$ given by

$$T(\underline{\omega}_1, \dots, \underline{\omega}_r, \vec{X}_1, \dots, \vec{X}_s)(p) = T_p(\underline{\omega}_{1p}, \dots, \underline{\omega}_{rp}, \vec{X}_{1p}, \dots, \vec{X}_{sp}) \tag{2.11}$$

is in $\mathcal{F}(M)$ for all $\underline{\omega}_1, \dots, \underline{\omega}_r \in \mathcal{T}_1(M)$ and $\vec{X}_1, \dots, \vec{X}_s \in \mathcal{T}(M)$. The space of all such tensor fields on M is denoted $\mathcal{T}_s^r(M)$. The components of T relative to the coordinate bases $\vec{\partial}_i$ and $\underline{du}^i$ defined on U are the functions

$$T^{i\dots j}{}_{k\dots l} = T(\underline{du}^i, \dots, \underline{du}^j, \vec{\partial}_k, \dots, \vec{\partial}_l) \;, \tag{2.12}$$

where $i,\dots,j;k,\dots,l = 1,\dots,n$. Tensors of type $(r,0)$ are said to be <u>contravariant</u> of degree r and tensors of type $(0,s)$ are said to be <u>covariant</u> of degree s. A tensor of type (r,s) $(r,s \neq 0)$

is mixed of degree (r+s).

Although the definition of a tensor given above is perhaps the simplest, it is not the only way in which we can think of tensors. It is easy to see that a tensor T_p of type (r,s) may also be thought of as a (u+v)-linear map (with $u \leqslant r$, $v \leqslant s$) from $T_p^*(M) \times \ldots \times T_p^*(M) \times T_p(M) \times \ldots \times T_p(M)$ into $(T_{s-v}^{r-u})_p(M)$, where $T_p^*(M)$ now appears u times and $T_p(M)$ appears v times. With this interpretation, the tensor field T , now written T' , has the explicit action

$$T'(\underline{\omega}_1,\ldots,\underline{\omega}_u,\vec{X}_1,\ldots,\vec{X}_v) =$$

$$= \sum_{i\ldots j} \sum_{k\ldots l} T(\underline{\omega}_1,\ldots,\underline{\omega}_u,\underline{du}^i,\ldots,\underline{du}^j,\vec{X}_1,\ldots,\vec{X}_v,\vec{\partial}_k,\ldots,\vec{\partial}_l) \cdot \vec{\partial}_i \otimes \ldots \otimes \vec{\partial}_j \otimes \underline{du}^k \otimes \ldots \otimes \underline{du}^l \qquad (2.13)$$

where the sums range from 1 to n and the quantities

$$(E_{i\ldots j}{}^{k\ldots l})_p = (\vec{\partial}_i \otimes \ldots \otimes \vec{\partial}_j \otimes \underline{du}^k \otimes \ldots \otimes \underline{du}^l)_p \quad , \qquad (2.14)$$

with (r-u) lower and (s-v) upper indices, are the basis vectors for $(T_{s-v}^{r-u})_p(M)$ defined by

$$E_{i\ldots j}{}^{k\ldots l}(\underline{du}^{i'},\ldots,\underline{du}^{j'},\vec{\partial}_{k'},\ldots,\vec{\partial}_{l'}) = \delta_i^{i'} \cdot \ldots \cdot \delta_j^{j'} \cdot \delta_{k'}^{k} \cdot \ldots \cdot \delta_{l'}^{l} \quad . \qquad (2.15)$$

The tensor product $\otimes$ used in (2.13) is an associative bilinear product used to create, from two given tensors of types (r,s) and (u,v), a new tensor of type (r+u,s+v). For example, if $\underline{\theta}$ and $\underline{\omega}$ are two 1-forms on M , then $\underline{\theta}\otimes\underline{\omega}$ is the tensor field of type (0,2) defined by

$$\underline{\theta}\otimes\underline{\omega}(\vec{U},\vec{V}) = \underline{\theta}(\vec{U})\cdot\underline{\omega}(\vec{V}) \quad . \tag{2.16}$$

The <u>exterior</u> or <u>wedge</u> <u>product</u> of $\underline{\theta}$ and $\underline{\omega}$ is the <u>2-form</u>

$$\underline{\theta}\wedge\underline{\omega} = (\ \underline{\theta}\otimes\underline{\omega} - \underline{\omega}\otimes\underline{\theta}\ ) \quad , \tag{2.17}$$

which is an <u>antisymmetric</u> covariant tensor field of degree 2. If M is an n-manifold, then a <u>p-form</u>, $p \leqslant n$, on M is a completely antisymmetric covariant tensor field of degree p. In order to generalize the wedge product to products of p- and q-forms ($p+q \leqslant n$) we require that $\wedge$ be associative and bilinear, and we define the p-form $\underline{\omega}_1\wedge \dots \wedge\underline{\omega}_p$ constructed from p 1-forms to be the tensor that satisfies

$$\underline{\omega}_1\wedge \dots \wedge\underline{\omega}_p(\vec{V}_1, \dots ,\vec{V}_p) = \det \| \underline{\omega}_i(\vec{V}_j) \| \tag{2.18}$$

for all $\vec{V}_1, \dots ,\vec{V}_p \in \mathcal{T}(M)$. If $\underline{\theta}$ is a p-form and $\underline{\phi}$ a q-form, then it follows that

$$\underline{\theta}\wedge\underline{\phi} = (-1)^{pq}\ \underline{\phi}\wedge\underline{\theta} \quad . \tag{2.19}$$

If $p+q > n$ then $\underline{\theta}\wedge\underline{\phi} = 0$. By convention, a function $f \in \mathcal{F}(M)$ is a 0-form and $f\wedge\underline{\theta} = f\cdot\underline{\theta}$.

The primary use of differential forms lies in the theory of integration on manifolds. Let M be an n-manifold with differential structure $\mathcal{D}$; let $\{(U_I,\phi_I)\}$ be a subset of $\mathcal{D}$ such that $\{U_I\}$ is a locally finite cover for M ; and let $\{\rho_\alpha\}$ be a partition of unity on M subordinate to the cover $\{U_I\}$ [33]. If $\underline{\omega}$ is an n-form on M then for any given α

$$\underline{\omega}_\alpha = \rho_\alpha \cdot \underline{\omega} \tag{2.20}$$

is an n-form on M which is non-zero only in some neighbourhood $U_J \in \{U_I\}$. Denoting by u^i , $i = 1,\ldots,n$, the local coordinates induced in this neighbourhood by the map ϕ_J , one can quickly verify that on U_J

$$\underline{\omega}_\alpha = f_\alpha \cdot \underline{du}^1 \wedge \ldots \wedge \underline{du}^n \tag{2.21}$$

for some unique $f_\alpha \in \mathcal{F}(U_J)$. The integral of $\underline{\omega}_\alpha$ on M is then defined to be

$$\int_M \underline{\omega}_\alpha = \int_{\phi_J(U_J)} f_\alpha(\phi_J^{-1}(x))\, d^n x \tag{2.22}$$

where the integral on the right hand side is the usual integral in Euclidean n-space. Noting that $\sum_\alpha \underline{\omega}_\alpha = \underline{\omega}$, we can now define the integral of $\underline{\omega}$ over M to be

$$\int_M \underline{\omega} = \sum_\alpha \int_M \underline{\omega}_\alpha \quad . \tag{2.23}$$

If $\underline{\omega}$ has compact support then this sum will converge and will be independent of the choice of cover $\{U_I\}$ or the partition of unity $\{\rho_\alpha\}$.

A very useful generalization of the classical theorems of Stokes and Gauss to an arbitrary manifold M may also be derived.

But before stating (without proof) this generalized Stokes' theorem we must define the exterior derivative operator $\underline{d}$. If M is an n-dimensional manifold and $f \in \mathcal{F}(M)$, then we define the 1-form $\underline{df} = \underline{d}f$ by

$$\underline{df}(\vec{V}) = \vec{V}f \qquad \text{for all} \quad \vec{V} \in \mathcal{T}(M) \quad . \tag{2.24}$$

More generally, if $\underline{\omega}$ is a p-form on M , then the exterior derivative $\underline{d\omega}$ of $\underline{\omega}$ is a (p+1)-form that is uniquely determined by the following properties of $\underline{d}$ [33]:

$$\text{(i)} \qquad \underline{d}(\underline{\theta}_1 + \underline{\theta}_2) = \underline{d\theta}_1 + \underline{d\theta}_2 \quad ; \tag{2.25}$$

$$\text{(ii)} \qquad \underline{d}(\underline{\theta} \wedge \underline{\phi}) = (\underline{d\theta}) \wedge \underline{\phi} + (-1)^p \underline{\theta} \wedge \underline{d\phi} \quad ; \tag{2.26}$$

$$\text{(iii)} \qquad \underline{d}(\underline{d\theta}) = 0 \quad ; \tag{2.27}$$

which must hold for all p-forms $\underline{\theta}, \underline{\theta}_1, \underline{\theta}_2$ and q-forms $\underline{\phi}$ on M . In particular, $\underline{d}(\underline{du}^i) = 0$, where $\underline{du}^i$ are the coordinate 1-forms in some neighbourhood U . Thus, when $\underline{\omega}$ is expanded in the form

$$\underline{\omega} = \sum_{i_1 < \ldots < i_p} \omega_{i_1 \ldots i_p} \cdot \underline{du}^{i_1} \wedge \ldots \wedge \underline{du}^{i_p} \quad , \tag{2.28}$$

its exterior derivative is just

$$\underline{d\omega} = \sum_{i_1 < \ldots < i_p} \underline{d}(\omega_{i_1 \ldots i_p}) \wedge \underline{du}^{i_1} \wedge \ldots \wedge \underline{du}^{i_p} \quad . \tag{2.29}$$

Now let $\underline{\omega}$ be an (n-1)-form on M and let M' be an n-dimensional submanifold of M with boundary $\partial M'$. A curve in $\partial M'$ is, in a natural way, also a curve in M ; so we may always think of a vector field $\vec{V} \in \mathcal{T}(\partial M')$ as the restriction to $\partial M'$ of some $\vec{V} \in \mathcal{T}(M)$. Similarly, the (n-1)-form $\underline{\omega}$ on M restricts to an (n-1)-form, also denoted by $\underline{\omega}$, on $\partial M'$. Since $\partial M'$ is an (n-1)-dimensional manifold, the integral of $\underline{\omega}$ over this boundary may be constructed as in (2.23). If M' is compact and oriented and $\partial M'$ is assigned the boundary orientation [33], then the <u>generalized Stokes' theorem</u> may be concisely expressed in the form

$$\int_{\partial M'} \underline{\omega} = \int_{M'} \underline{d\omega} \quad . \tag{2.30}$$

The exterior derivative operator $\underline{d}$ is intrinsic to the manifold M on which it is defined. It acts only on differential forms, raising the degree of a p-form to (p+1). In view of (2.24), it may be considered as a generalization of the gradient operator of vector calculus. Also intrinsic to each manifold is a second kind of differential operator $\pounds$ called the <u>Lie derivative</u>. If $\vec{X}$ and $\vec{Y}$ are vector fields on M , then the Lie derivative of $\vec{Y}$ along $\vec{X}$ is the vector field $\pounds_{\vec{X}}\vec{Y}$ defined by

$$(\pounds_{\vec{X}}\vec{Y})f = [\vec{X},\vec{Y}]f = \vec{X}(\vec{Y}f) - \vec{Y}(\vec{X}f) \tag{2.31}$$

for all $f \in \mathcal{F}(M)$. The Lie derivative of a function f along $\vec{X}$ is defined to be the function

$$\pounds_{\vec{X}}f = \vec{X}f \quad , \tag{2.32}$$

and if $\underline{\omega}$ is a 1-form, then $\pounds_{\vec{X}}\underline{\omega}$ is the 1-form which satisfies

$$(\pounds_{\vec{X}}\underline{\omega})(\vec{V}) = \vec{X}(\underline{\omega}(\vec{V})) - \underline{\omega}(\pounds_{\vec{X}}\vec{V}) \tag{2.33}$$

for all $\vec{V} \in \mathcal{T}(M)$. By requiring that $\pounds$ satisfy Leibniz' rule for the derivative of a product, we can define the Lie derivative of an arbitrary tensor field. For example, if T is a tensor field of type (1,1), then $\pounds_{\vec{X}}T$ is defined by

$$\pounds_{\vec{X}}T(\underline{\omega},\vec{V}) = \vec{X}(T(\underline{\omega},\vec{V})) - T(\pounds_{\vec{X}}\underline{\omega},\vec{V}) - T(\underline{\omega},\pounds_{\vec{X}}\vec{V}) \tag{2.34}$$

for all $\underline{\omega} \in \mathcal{T}_1(M)$ and $\vec{V} \in \mathcal{T}(M)$.

One more geometric structure deserves discussion here. A <u>Riemannian</u> <u>metric</u> on M is a (smooth) tensor field g of type (0,2) which is symmetric and positive definite. That is,

$$(g(\vec{U},\vec{V}))(x) = (g(\vec{V},\vec{U}))(x) \geqslant 0 \tag{2.35}$$

for all $\vec{U},\vec{V} \in \mathcal{T}(M)$ and $x \in M$, and $(g(\vec{U},\vec{V}))(x) = 0$ only if $\vec{U}_x = 0$ or $\vec{V}_x = 0$. Every differential manifold admits a Riemannian metric [45], and any Riemannian metric on M may be used to construct a distance function compatible with the topology of M. Let a and b be two points in M and let $x(t)$ be a curve such that $x(t_a) = a$, $x(t_b) = b$, and $t_a < t_b$. Then, if g is a Riemannian metric on M, the <u>length</u> of the curve $x(t)$ between a and b is

$$l(a,b)[x] = \int_{t_a}^{t_b} \sqrt{g(\vec{X}(t),\vec{X}(t))}\, dt \tag{2.36}$$

where $\vec{X}(t)$ is the vector tangent to the curve at $x(t)$ (defined as in equation (2.5)). Each such curve joining a and b has a well defined length which is greater than zero if a and b are distinct. The metric distance between a and b is the least upper bound of the lengths of all smooth curves connecting a and b :

$$d(a,b) = \underset{[x]}{\text{lub}} \; (l(a,b)[x]) \tag{2.37}$$

The distance function satisfies the conditions (1.3), and the open balls defined with the use of d provide a base for the topology of M . Each curve $x(t)$ joining a and b such that $l(a,b)[x] = d(a,b)$ is called a geodesic of the metric g on M .

If we relax the condition that g be a smooth tensor field, allowing it to be divergent at some point $x \in M$, then we may still be able to define a distance function as in (2.37), but d need no longer be compatible with the topology of M . If g fails to be positive definite, but is still symmetric and non-singular , then it is called a pseudo-Riemannian metric. In such a case the distance function d ceases to be well defined, and g may no longer be used to generate the open sets of M .

3. Affine and Riemannian Geometry

Let M be an n-manifold and let $\vec{U},\vec{V} \in \mathcal{T}(M)$ be any two vector fields. The Lie derivative $\pounds_{\vec{U}}\vec{V}$ is, in a natural way, also a vector field on M . However, this derivative cannot be considered as a derivative of $\vec{V}$ in the usual sense, since (when written in a coordinate representation) it depends on the derivatives of the components of $\vec{U}$ as well as on the components themselves. In fact

$$\pounds_{\vec{U}}\vec{V} = -\pounds_{\vec{V}}\vec{U} \quad , \tag{3.1}$$

so that $\vec{U}$ and $\vec{V}$ are really on an equal footing. In order to construct derivatives of $\vec{V}$ more akin to the directional derivatives of vectors in Euclidean space, we must define on M a new geometric structure ∇ called a <u>covariant derivative</u>. We define ∇ such that for any $\vec{U},\vec{V},\vec{W} \in \mathcal{T}(M)$ and $f \in \mathcal{F}(M)$, $\nabla_{\vec{U}}\vec{V} \in \mathcal{T}(M)$ and

$$\left.\begin{aligned}
\nabla_{(\vec{U}+\vec{V})}\vec{W} &= \nabla_{\vec{U}}\vec{W} + \nabla_{\vec{V}}\vec{W} \quad , \\
\nabla_{\vec{U}}(\vec{V}+\vec{W}) &= \nabla_{\vec{U}}\vec{V} + \nabla_{\vec{U}}\vec{W} \quad , \\
\nabla_{f\vec{U}}\vec{V} &= f\nabla_{\vec{U}}\vec{V} \quad , \\
\nabla_{\vec{U}}(f\vec{V}) &= (\vec{U}f)\vec{V} + f\nabla_{\vec{U}}\vec{V} \quad .
\end{aligned}\right\} \tag{3.2}$$

From ∇ we can construct two important tensors: a vector-valued 2-form $\underline{\Theta}$ called the <u>torsion</u>, which is defined by

$$\underline{\Theta}(\vec{U},\vec{V}) = \nabla_{\vec{U}}\vec{V} - \nabla_{\vec{V}}\vec{U} - [\vec{U},\vec{V}] \quad ; \tag{3.3}$$

and the <u>curvature</u> tensor R , which is of type (1,3) and has the action

$$R(\vec{U},\vec{V})\vec{W} = \nabla_{\vec{U}}\nabla_{\vec{V}}\vec{W} - \nabla_{\vec{V}}\nabla_{\vec{U}}\vec{W} - \nabla_{[\vec{U},\vec{V}]}\vec{W} \quad . \tag{3.4}$$

The operator $(R(\vec{U},\vec{V}))_x$, $x \in M$, is a linear transformation of $T_x(M)$. It is antisymmetric,

$$R(\vec{U},\vec{V}) = -R(\vec{V},\vec{U}) \quad , \tag{3.5}$$

and its trace is a symmetric tensor S called the Ricci tensor:

$$S(\vec{V},\vec{W})(x) = \mathrm{Trace}(\vec{U}_x \to (R(\vec{U},\vec{V})\vec{W})_x) \quad . \tag{3.6}$$

The curvature satisfies the cyclic Bianchi identity,

$$R(\vec{U},\vec{V})\vec{W} + R(\vec{W},\vec{U})\vec{V} + R(\vec{V},\vec{W})\vec{U} = \nabla_{\vec{U}}\underline{\Theta}(\vec{V},\vec{W}) + \nabla_{\vec{W}}\underline{\Theta}(\vec{U},\vec{V})$$

$$+ \nabla_{\vec{V}}\underline{\Theta}(\vec{W},\vec{U}) - \underline{\Theta}(\vec{U},\underline{\Theta}(\vec{V},\vec{W})) - \underline{\Theta}(\vec{W},\underline{\Theta}(\vec{U},\vec{V})) - \underline{\Theta}(\vec{V},\underline{\Theta}(\vec{W},\vec{U})) \quad , \tag{3.7}$$

which follows directly from the Jacobi identity

$$[[\vec{U},\vec{V}],\vec{W}] + [[\vec{W},\vec{U}],\vec{V}] + [[\vec{V},\vec{W}],\vec{U}] = 0 \quad . \tag{3.8}$$

By setting the covariant derivative of a function equal to its ordinary derivative:

$$\nabla_{\vec{U}} f = \vec{U} f \quad , \tag{3.9}$$

and requiring that ∇ satisfy Leibniz' rule for the derivative of a product, we can define the covariant derivative of an arbitrary tensor field. If $\underline{\omega}$ is a 1-form and T a tensor field of type (1,1), then the covariant analogues of (2.33) and (2.34) are

$$\nabla_{\vec{U}}\underline{\omega}(\vec{V}) = \vec{U}(\underline{\omega}(\vec{V})) - \underline{\omega}(\nabla_{\vec{U}}\vec{V}) \quad , \tag{3.10}$$

$$\nabla_{\vec{U}} T(\underline{\omega},\vec{V}) = \vec{U}(T(\underline{\omega},\vec{V})) - T(\nabla_{\vec{U}}\underline{\omega},\vec{V}) - T(\underline{\omega},\nabla_{\vec{U}}\vec{V}) \quad . \tag{3.11}$$

Bianchi's second identity,

$$\nabla_{\vec{U}} R(\vec{V},\vec{W}) + \nabla_{\vec{W}} R(\vec{U},\vec{V}) + \nabla_{\vec{V}} R(\vec{W},\vec{U}) = R(\vec{U},\underline{\Theta}(\vec{V},\vec{W}))$$
$$+ R(\vec{V},\underline{\Theta}(\vec{W},\vec{U})) + R(\vec{W},\underline{\Theta}(\vec{U},\vec{V})) \quad , \tag{3.12}$$

is obtained by noting that

$$\nabla_{\vec{U}} R(\vec{V},\vec{W})\vec{X} = \nabla_{\vec{U}}(R(\vec{V},\vec{W})\vec{X}) - R(\nabla_{\vec{U}}\vec{V},\vec{W})\vec{X} - R(\vec{V},\nabla_{\vec{U}}\vec{W})\vec{X}$$
$$- R(\vec{V},\vec{W})\nabla_{\vec{U}}\vec{X} \quad ,$$

expanding R in terms of the covariant derivative, and making use of (3.8) and the tensorial nature of the torsion.

In any coordinate neighbourhood $U \subset M$, with local coordinates u^i, the action of ∇ is completely determined by the functions Γ^i_{jk}, defined by

$$\Gamma^i_{jk} = \underline{du}^i(\nabla_{\vec{\partial}_j}\vec{\partial}_k) \quad . \tag{3.13}$$

These are called the components of the affine connection Γ associated with ∇, the name reflecting the fact that ∇ allows the comparison of vectors in the affine tangent spaces of distinct points along a curve. Let $x(t)$, $t \in \mathbb{R}$, be a curve in M and let $\vec{X}$ be a vector field on M such that $\vec{X}_{x(t_0)}$ is, for each t_0, the tangent vector to the curve at $x(t_0)$. A vector field $\vec{V} \in \mathcal{T}(M)$ is said to be <u>parallel along the curve</u> $x(t)$ if

$$(\nabla_{\vec{X}}\vec{V})_{x(t)} = f(t)\cdot\vec{V}_{x(t)} \tag{3.14}$$

for some function $f:\mathbb{R} \to \mathbb{R}$ and for each $t \in \mathbb{R}$. If the vector field $\vec{X}$ is itself parallel along the curve $x(t)$, then the curve is called a <u>path</u> of the affine connection Γ. A diffeomorphism $s:\mathbb{R} \to \mathbb{R}$ may be used to reparameterize the points of

$x(t)$, yielding the new curve

$$x'(t) = x(s(t)) \quad . \tag{3.15}$$

Moreover, if $x(t)$ is a path of Γ , then s may always be chosen so that

$$(\nabla_{\vec{X}'}\vec{X}')_{x'(t)} = 0 \qquad \text{for all} \quad t \in \mathbb{R} \quad , \tag{3.16}$$

where $\vec{X}'$ bears the same relation to $x'(t)$ as $\vec{X}$ bears to $x(t)$. The path $x'(t)$ is then called a <u>geodesic</u> of ∇ , and the parameter t is called an <u>affine parameter</u> for the geodesic.

Now let g be a Riemannian or pseudo-Riemannian metric on M . The covariant derivative ∇ is said to be <u>metrical</u> if, for all $\vec{U},\vec{V},\vec{W} \in \mathcal{T}(M)$,

$$\nabla_{\vec{U}} g(\vec{V},\vec{W}) = 0 \quad , \tag{3.17}$$

or, more simply, $\nabla g = 0$. If ∇ is metrical, then the norm $g(\vec{X}_{x(t)},\vec{X}_{x(t)})$ of the tangent vector to a geodesic $x(t)$ is independent of t . The length (as defined in (2.36)) of the segment of $x(t)$ between $x(t_a)$ and $x(t_b)$ is thus directly proportional to the affine length $(t_b - t_a)$. In the Riemannian case this will always be positive, but if g is not positive-definite then there will also exist <u>null</u> geodesics, which have length zero, and <u>time-like</u> geodesics, whose lengths are pure imaginary.

A metrical covariant derivative which has vanishing torsion:

$$\nabla_{\vec{U}}\vec{V} - \nabla_{\vec{V}}\vec{U} = [\vec{U},\vec{V}] \quad , \tag{3.18}$$

is called a Riemannian covariant derivative. Each metric g on M uniquely determines a Riemannian covariant derivative ∇ on M, and from now on it is this covariant derivative with which we shall deal. The manifold M, together with g and ∇, is called a (pseudo-)Riemannian manifold. The Riemann curvature tensor of M has, in addition to (3.5) and (3.7), the symmetries

$$g(\vec{X},R(\vec{U},\vec{V})\vec{W}) = -g(\vec{W},R(\vec{U},\vec{V})\vec{X}) \quad , \tag{3.19}$$

$$g(\vec{X},R(\vec{U},\vec{V})\vec{W}) = g(\vec{V},R(\vec{W},\vec{X})\vec{U}) \quad . \tag{3.20}$$

Since g is non-degenerate, it is possible to find, in a neighbourhood U_x of each point $x \in M$, a set of vector fields $\vec{h}_i$, $i = 1,\ldots,n$, which are orthogonal,

$$g(\vec{h}_i,\vec{h}_j) = 0 \quad \text{for} \quad i \neq j \quad , \tag{3.21}$$

and which are normalized to plus or minus one,

$$g(\vec{h}_i,\vec{h}_i) = \varepsilon_i = \pm 1 \quad . \tag{3.22}$$

The sum $\sigma = \sum_i \varepsilon_i$ is the signature of the metric g and is an invariant quantity. In terms of the $\vec{h}_i$, the Ricci tensor of ∇ may be written

$$S(\vec{U},\vec{V}) = \sum_i \varepsilon_i g(\vec{h}_i,R(\vec{h}_i,\vec{U})\vec{V}) \quad . \tag{3.23}$$

It satisfies the contracted form,

$$\sum_i \varepsilon_i \nabla_{\vec{h}_i} S(\vec{h}_i,\vec{V}) = \tfrac{1}{2}\vec{V}S \quad , \tag{3.24}$$

of the Bianchi identity (3.12), where S on the right hand side is the Ricci scalar,

$$S = \sum_i \varepsilon_i S(\vec{h}_i,\vec{h}_i) \quad . \tag{3.25}$$

For any metrical connection Γ the Ricci tensor is symmetric:

$$S(\vec{U},\vec{V}) = S(\vec{V},\vec{U}) \quad . \tag{3.26}$$

4. Submanifold Geometry

Of especial importance in the physical discussions of chapters 2 and 3 is the relationship between the geometry of a (pseudo-) Riemannian manifold and that of its submanifolds. Let S and M be manifolds of dimensions n and $(n+p)$, respectively, and let $e:S \to M$ be an embedding. By identifying S with its image $e(S)$ in M, one can immediately see that e induces, for each $x \in S$, an injective map $de_x:T_x(S) \to T_{e(x)}(M)$ whose co-domain is the subspace $T^{\parallel}_{e(x)}(M)$ of $T_{e(x)}(M)$ consisting of those vectors which are tangent to curves in $e(S)$. The image of a vector field $\vec{v} \in \mathcal{T}(S)$ is denoted $e_*\vec{v}$, and is an assignment to each $e(x) \in e(S)$ of a vector in $T^{\parallel}_{e(x)}(M)$. Such a vector field is said to be parallel to the submanifold.

Now let M have defined on it a metric g with Riemannian covariant derivative ∇ . The pullback $\tilde{g} = e^*g$ of g onto S is the symmetric tensor field defined by

$$\tilde{g}(\vec{u},\vec{v})(x) = g(e_*\vec{u},e_*\vec{v})(e(x)) \qquad (4.1)$$

for all $x \in S$ and all $\vec{u},\vec{v} \in \mathcal{T}(S)$. If g is not positive definite, then $\tilde{g}$ need not be positive definite, nor even a metric on S . However, we shall consider here only those embeddings e for which $\tilde{g}$ is a Riemannian metric.

A general vector field $\vec{V}$ on $e(S)$ is a smooth assignment of a vector $\vec{V}_{e(x)} \in T_{e(x)}(M)$ to each point $e(x)$. If each $\vec{V}_{e(x)}$ is in $T^{\parallel}_{e(x)}(M)$, then $\vec{V}$ is a parallel vector field. (Note that this has nothing to do with "parallel along a curve".) On the other hand, a <u>perpendicular vector field</u> $\vec{V}$, is a vector field on $e(S)$ such that

$$g(\vec{V},\vec{u})(e(x)) = 0 \qquad (4.2)$$

for all $x \in S$ and all parallel vector fields $\vec{u}$. The space of all vector fields on $e(S)$ will be denoted by $\mathcal{T}_e(S)$, the space of parallel vector fields by $\mathcal{T}^{\parallel}_e(S)$, and the space of perpendicular vector fields by $\mathcal{T}^{\perp}_e(S)$. For convenience, the distinction between $\vec{v} \in \mathcal{T}(S)$ and $e_*\vec{v} \in \mathcal{T}^{\parallel}_e(S)$ will be dropped.

It is always possible to choose, in some neighbourhood $U \subset e(S)$ of each point $e(x)$ of the submanifold, a set $\vec{h}_i$, $i = 1,\ldots,n$, of orthonormal vector fields in $\mathcal{T}^{\parallel}_e(S)$:

$$g(\vec{h}_i,\vec{h}_j) = \delta_{ij} \quad , \qquad (4.3)$$

and a corresponding set $\vec{n}_\mu$, $\mu = 1,\ldots,p$, of orthogonal unit vectors in $\mathcal{T}_e^\perp(S)$:

$$g(\vec{n}_\mu,\vec{n}_\nu) = \varepsilon_\mu \delta_{\mu\nu} = \pm\delta_{\mu\nu} \quad . \tag{4.4}$$

The projection operator Π defined, on each such neighbourhood U , by

$$\Pi(\vec{V}) = \sum_i g(\vec{V},\vec{h}_i)\vec{h}_i \tag{4.5}$$

for all $\vec{V} \in \mathcal{T}_e(S)$,may then be used to project out the <u>parallel part</u> $\vec{V}^\parallel = \Pi(\vec{V})$ of the field $\vec{V}$. Similarly, the <u>perpendicular part</u> of $\vec{V}$ is $\vec{V}^\perp = \vec{V} - \vec{V}^\parallel = \sum_\mu \varepsilon_\mu g(\vec{V},\vec{n}_\mu)\vec{n}_\mu$. The metric $\tilde{g}$ can now be redefined by setting

$$\tilde{g}(\vec{U},\vec{V}) = g(\Pi(\vec{U}),\Pi(\vec{V})) \quad , \tag{4.6}$$

so that its arguments $\vec{U}$ and $\vec{V}$ need no longer be parallel vector fields.

For all $\vec{u},\vec{v} \in \mathcal{T}_e^\parallel(S)$, the vector field $\nabla_{\vec{u}}\vec{v}$ may always be written in the form

$$\nabla_{\vec{u}}\vec{v} = \tilde{\nabla}_{\vec{u}}\vec{v} + \alpha(\vec{u},\vec{v}) \tag{4.7}$$

where $\tilde{\nabla}_{\vec{u}}\vec{v} \in \mathcal{T}_e^\parallel(S)$ and $\alpha(\vec{u},\vec{v}) \in \mathcal{T}_e^\perp(S)$. It is easy to check that $\tilde{\nabla}$ satisfies the conditions (3.2) for a covariant derivative. Moreover, $\tilde{\nabla}$ has vanishing torsion,

$$\tilde{\nabla}_{\vec{u}}\vec{v} - \tilde{\nabla}_{\vec{v}}\vec{u} = [\vec{u},\vec{v}] \quad , \tag{4.8}$$

because ∇ has vanishing torsion: If $\vec{u},\vec{v} \in \mathcal{T}_e^{\parallel}(S)$, then

$$[\vec{u},\vec{v}] = \tilde{\nabla}_{\vec{u}}\vec{v} - \tilde{\nabla}_{\vec{v}}\vec{u} + \alpha(\vec{u},\vec{v}) - \alpha(\vec{v},\vec{u}) \tag{4.9}$$

will also be a parallel vector field; but this implies that

$$\alpha(\vec{u},\vec{v}) = \alpha(\vec{v},\vec{u}) \quad , \tag{4.10}$$

reducing (4.9) to (4.8) . Finally, $\tilde{\nabla}$ is metrical, and hence Riemannian, since

$$\begin{aligned} \vec{u}(\tilde{g}(\vec{v},\vec{w})) &= \vec{u}(g(\vec{v},\vec{w})) \\ &= g(\nabla_{\vec{u}}\vec{v},\vec{w}) + g(\vec{v},\nabla_{\vec{u}}\vec{w}) \\ &= \tilde{g}(\tilde{\nabla}_{\vec{u}}\vec{v},\vec{w}) + \tilde{g}(\vec{v},\tilde{\nabla}_{\vec{u}}\vec{w}) \end{aligned} \tag{4.11}$$

for all $\vec{u},\vec{v},\vec{w} \in \mathcal{T}_e^{\parallel}(S)$.

The operator α defined by (4.7) is called the <u>second fundamental form</u> of S for the embedding e . It is linear in its first argument since $\nabla_{f\vec{u}}\vec{v} = f\nabla_{\vec{u}}\vec{v}$, and the symmetry condition (4.10) indicates that α must also be linear in the second argument. In analogy with $\tilde{g}$, we set

$$\alpha(\vec{U},\vec{V}) = \alpha(\Pi(\vec{U}),\Pi(\vec{V})) \tag{4.12}$$

so that α is a tensorial map from $\mathcal{T}_e(S)\times\mathcal{T}_e(S)$ into $\mathcal{T}_e^{\perp}(S)$. Making use of the unit vector fields $\vec{n}_\mu$, $\alpha(\vec{U},\vec{V})$ may be expanded in the form

$$\alpha(\vec{U},\vec{V}) = \sum_\mu K^\mu(\vec{U},\vec{V})\vec{n}_\mu \quad . \tag{4.13}$$

The geometric objects K^{μ} introduced here are p real-valued symmetric tensor fields on $e(S)$, called the <u>extrinsic curvatures</u> of the submanifold in the directions $\vec{n}_{\mu}$.

If $\vec{\xi} = \sum_{\mu} \xi^{\mu} \vec{n}_{\mu}$ is a perpendicular vector field and $\vec{u}, \vec{v} \in \mathcal{T}^{\parallel}_{e}(S)$, then

$$g(\vec{v}, \nabla_{\vec{u}}\vec{\xi}) = \vec{u}(g(\vec{v},\vec{\xi})) - g(\nabla_{\vec{u}}\vec{v},\vec{\xi})$$
$$= -g(\alpha(\vec{u},\vec{v}),\vec{\xi})$$
$$= -\sum_{\mu} \varepsilon_{\mu} K^{\mu}(\vec{u},\vec{v})\xi^{\mu} \quad , \tag{4.14a}$$

and

$$g(\vec{n}_{\mu}, \nabla_{\vec{u}}\vec{\xi}) = \vec{u}(g(\vec{n}_{\mu},\vec{\xi})) - g(\nabla_{\vec{u}}\vec{n}_{\mu},\vec{\xi})$$
$$= \varepsilon_{\mu}\vec{u}\xi^{\mu} - \sum_{\nu} \xi^{\nu} g(\nabla_{\vec{u}}\vec{n}_{\mu},\vec{n}_{\nu}) \quad . \tag{4.14b}$$

Equations (4.7) and (4.14) are known respectively as the formulas of Gauss and Weingarten [41]. If $e(S)$ is a <u>hypersurface</u> of M , that is, if $p = 1$, then there is only one unit normal vector field $\vec{n}$, and one extrinsic curvature K . In this case, which is the only case that we shall consider from now on, the last term in (4.14b) vanishes, because $g(\nabla_{\vec{u}}\vec{n},\vec{n}) = 0$.

The <u>hypersurface</u> <u>curvature</u> $\tilde{R}$ is defined to have the action

$$\tilde{R}(\vec{u},\vec{v})\vec{w} = \tilde{\nabla}_{\vec{u}}\tilde{\nabla}_{\vec{v}}\vec{w} - \tilde{\nabla}_{\vec{v}}\tilde{\nabla}_{\vec{u}}\vec{w} - \tilde{\nabla}_{[\vec{u},\vec{v}]}\vec{w} \tag{4.15}$$

for all $\vec{u},\vec{v},\vec{w} \in \mathcal{T}^{\parallel}_{e}(S)$, and to satisfy

$$\tilde{R}(\vec{U},\vec{V})\vec{W} = \Pi(\tilde{R}(\Pi(\vec{U}),\Pi(\vec{V}))\Pi(\vec{W})) \quad . \tag{4.16}$$

It has the usual symmetries (3.5) , (3.7) , (3.19), and (3.20) and satisfies the hypersurface analogue,

$$\tilde{\nabla}_{\vec{u}}\tilde{R}(\vec{v},\vec{w}) + \tilde{\nabla}_{\vec{w}}\tilde{R}(\vec{u},\vec{v}) + \tilde{\nabla}_{\vec{v}}\tilde{R}(\vec{w},\vec{u}) = 0 \quad , \qquad (4.17)$$

of the differential identity (3.12). The Ricci tensor of $\tilde{\nabla}$ is defined by

$$\tilde{S}(\vec{U},\vec{V}) = \sum_i \tilde{g}(\vec{h}_i,\tilde{R}(\vec{h}_i,\vec{U})\vec{V}) \quad , \qquad (4.18)$$

and the Ricci scalar is

$$\tilde{S} = \sum_i \tilde{S}(\vec{h}_i,\vec{h}_i) \quad . \qquad (4.19)$$

Any tensor which is left invariant by Π , as are $\tilde{R}$ and $\tilde{S}$, is called a <u>hypersurface tensor field</u>. Although in many cases, such as K , the tilde will be omitted, a superscript tilde indicates that the field under consideration is a hypersurface field.

The curvature R of ∇ is related to the hypersurface curvature and the extrinsic curvature through (4.7) . For all $\vec{u},\vec{v},\vec{w} \in \mathcal{T}^{\parallel}_e(S)$,

$$\begin{aligned} R(\vec{u},\vec{v})\vec{w} &= \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} - \nabla_{[\vec{u},\vec{v}]}\vec{w} \\ &= \nabla_{\vec{u}}(\tilde{\nabla}_{\vec{v}}\vec{w} + K(\vec{v},\vec{w})\vec{n}) - \nabla_{\vec{v}}(\tilde{\nabla}_{\vec{u}}\vec{w} + K(\vec{u},\vec{w})\vec{n}) \\ &\quad - \tilde{\nabla}_{[\vec{u},\vec{v}]}\vec{w} - K([\vec{u},\vec{v}],\vec{w})\vec{n} \\ &= \tilde{R}(\vec{u},\vec{v})\vec{w} + \varepsilon\sum_i\{K(\vec{u},\vec{w})K(\vec{v},\vec{h}_i) - K(\vec{v},\vec{w})K(\vec{u},\vec{h}_i)\}\vec{h}_i \\ &\quad + \{\tilde{\nabla}_{\vec{u}}K(\vec{v},\vec{w}) - \tilde{\nabla}_{\vec{v}}K(\vec{u},\vec{w})\}\vec{n} \quad , \end{aligned} \qquad (4.20)$$

and, as a result of the symmetry (3.19),

$$R(\vec{u},\vec{v})\vec{n} = -\varepsilon\sum_{i}\{\tilde{\nabla}_{\vec{u}}K(\vec{v},\vec{h}_i) - \tilde{\nabla}_{\vec{v}}K(\vec{u},\vec{h}_i)\}\vec{h}_i \quad . \tag{4.21}$$

Equation (4.20) is equivalent to the classical equations of Gauss and Codazzi [41].