CHAPTER 3

MATTER FIELDS AND THE GEOMETRY OF SPACE-TIME

While much of the universe appears to be vacuous (or very nearly so), we are unable to make direct observations of the vacuum. Instead, we observe the matter that is contained in the universe, and deduce from the distribution of matter the geometry of both the vacuous and non-vacuous regions. It is thus essential to include matter in any complete discussion of the geometrical structure of space and space-time.

1. Initial Value Problem

The distribution of matter in space and its evolution in time is characterized by a set of smooth tensor fields F_j, $j \in \omega$, on space-time (cf. (1.1.8)). Since the space-time metric g may always be used to lower indices, I shall assume (without loss of generality) that the fields F_j are all covariant tensors.

Let F be a typical representative of the matter fields defined on M'. (The notation here is the same as in Chapter 2 - i.e. M' is an open cell in the space-time manifold, M.) Then, on the initial surface, $e(S)$, the instantaneous configuration of F is described by the hypersurface tensor fields

$$\Pi F = F(\Pi(\_),\ldots,\Pi(\_))$$
$$F(\vec{n},\Pi(\_),\ldots,\Pi(\_)) \; , \quad F(\Pi(\_),\vec{n},\Pi(\_),\ldots,\Pi(\_)) \; , \; \ldots$$
$$F(\vec{n},\vec{n},\Pi(\_),\ldots,\Pi(\_)) \; , \quad F(\vec{n},\Pi(\_),\vec{n},\Pi(\_),\ldots,\Pi(\_)) \; , \; \ldots$$
etc.

The instantaneous rate of change of F is described by analogous hypersurface fields constructed from $\nabla_{\vec{n}}F$; the accelerations are constructed from $\{\nabla_{\vec{n}}\nabla_{\vec{n}}F - \nabla_{\nabla_{\vec{n}}\vec{n}}F\}$; and so on.

Although all of the fields that are induced on $e(S)$ in this way are a priori independent of each other, only a small subset of them need be specified (along with the geometrical initial data) in order to determine the complete set. For a given matter field, F_i , only a finite number of the induced hypersurface fields may be considered as initial data; and, as with the metric, all of the time derivatives of F_i beyond some given order, say m_i , can be obtained as explicit functionals of the lower derivatives of F_i , the initial data for the other matter fields, F_j , and the geometrical initial data (discussed in Chapter 2).

Now suppose, once again, that the geometrical initial data is given by ^3g and K , corresponding to $m = 1$. In the vacuum case the dynamical equations for g took the form

$$\Pi S = \Pi S[{}^3g, K, {}^3R, {}^3\nabla K, \ldots] \quad ; \qquad (2.4.1)$$

but when matter fields are present there is much more initial data upon which ΠS can depend. To avoid confusion between the field, ΠS , and the functional, I shall write

$$\Pi S = {}^{3}E[\ I.D.\ ] \tag{1.1}$$

with I.D. representing the complete set of initial data fields and their hypersurface derivatives.

Once the functional ${}^{3}E$ has been chosen, equation (1.1) may be substituted into (2.2.25) to find $\nabla_{\vec{n}}K$ in terms of the initial data and the deformation vector field, $\vec{D}$; and this, together with the (as yet undetermined) dynamical equations for the matter fields, allows us to compute $\nabla_{\vec{n}}{}^{3}E$ as a functional of the initial data and $\vec{D}$. We know from (2.4.2), however, that this new functional must take the form

$$\nabla_{\vec{n}}{}^{3}E(\vec{u},\vec{v}) = {}^{3}E'(\vec{u},\vec{v}) + \frac{1}{D^{\perp}}\vec{u}D^{\perp}\,{}^{3}P(\vec{v}) + \frac{1}{D^{\perp}}\vec{v}D^{\perp}\,{}^{3}P(\vec{u}) \quad , \tag{1.2}$$

where ${}^{3}E'$ and ${}^{3}P$ are again explicit functionals of the initial data. Further comparison with (2.4.2) gives

$$\nabla_{\vec{n}}S(\vec{u},\vec{v}) = {}^{3}E'(\vec{u},\vec{v})[\ I.D.\ ] \quad , \tag{1.3}$$

and the primary constraint equations

$$S(\vec{u},\vec{n}) = {}^{3}P(\vec{u})[\ I.D.\ ] \quad . \tag{1.4}$$

These constraints on the initial data must always hold on $e(S)$. But since $e(S)$ is arbitrarily chosen they must also be satisfied on any other space-like hypersurface. It immediately follows that

$$\vec{n}(S(\vec{u},\vec{n}) - {}^3P(\vec{u})) = 0 \quad . \tag{1.5}$$

Expanding this and using (1.1) and (1.4) we find that (with ${}^3P(\vec{n})=0$)

$$\nabla_{\vec{n}}{}^3P(\vec{u}) = \nabla_{\vec{n}}S(\vec{u},\vec{n}) + \frac{1}{D^\perp}\vec{u}D^\perp S(\vec{n},\vec{n}) - \varepsilon\frac{1}{D^\perp}\sum_i \vec{h}_i D^\perp\, {}^3E(\vec{u},\vec{h}_i) \quad . \tag{1.6}$$

But $\nabla_{\vec{n}}S$ must be independent of $\vec{D}$, so when $\nabla_{\vec{n}}{}^3P(\vec{u})$ is computed directly it must take the form

$$\nabla_{\vec{n}}{}^3P(\vec{u}) = {}^3P'(\vec{u}) + \frac{1}{D^\perp}\vec{u}D^\perp F - \varepsilon\frac{1}{D^\perp}\sum_i \vec{h}_i D^\perp\, {}^3E(\vec{u},\vec{h}_i) \quad , \tag{1.7}$$

where F and ${}^3P'$ are functionals of the initial data. Comparison with (1.6) then gives the secondary constraint

$$S(\vec{n},\vec{n}) = F \quad . \tag{1.8}$$

The equations (1.1),(1.4), and (1.8) are easily recognized as the Einstein field equations:

$$S(\vec{U},\vec{V}) = E(\vec{U},\vec{V}) \quad , \tag{1.9}$$

where E is the symmetric space-time tensor defined by

$$\begin{aligned} E(\vec{u},\vec{v}) &= {}^3E(\vec{u},\vec{v}) \quad , \\ E(\vec{u},\vec{n}) &= {}^3P(\vec{u}) \quad , \\ E(\vec{n},\vec{n}) &= F \quad . \end{aligned} \tag{1.10}$$

By defining the Einstein tensor

$$G(\vec{U},\vec{V}) = S(\vec{U},\vec{V}) - \tfrac{1}{2}g(\vec{U},\vec{V})S \quad , \tag{1.11}$$

and the stress-energy tensor (in natural units)

$$T(\vec{U},\vec{V}) = (1/8\pi)(E(\vec{U},\vec{V}) - \tfrac{1}{2}g(\vec{U},\vec{V})\{\sum_i E(\vec{h}_i,\vec{h}_i) + \varepsilon E(\vec{n},\vec{n})\} \quad , \quad (1.12)$$

which, like E , is an explicit functional of the initial data, the field equations may be recast into their standard form:

$$G(\vec{U},\vec{V}) = 8\pi T(\vec{U},\vec{V}) \quad . \qquad (1.13)$$

The cosmological term appearing in the vacuum equations, (2.4.15), has here been absorbed into T .

The role of the stress-energy as the source of the gravitational (metric) field is now manifest. But we are not finished yet. If the tensor S is to be a genuine Ricci tensor, then it must satisfy the contracted Bianchi identities:

$$\sum_i \nabla_{\vec{h}_i} S(\vec{h}_i,\vec{V}) + \varepsilon \nabla_{\vec{n}} S(\vec{n},\vec{V}) = \tfrac{1}{2}\vec{V}S \quad . \qquad (1.14)$$

Through (1.13), these give us conservation laws that the stress-energy must satisfy:

$$\sum_i \nabla_{\vec{h}_i} T(\vec{h}_i,\vec{V}) + \varepsilon \nabla_{\vec{n}} T(\vec{n},\vec{V}) = 0 \quad ; \qquad (1.15)$$

or, in terms of hypersurface fields,

$$\sum_i \{ {}^3\nabla_{\vec{h}_i} {}^3E(\vec{h}_i,\vec{v}) - \tfrac{1}{2}\, {}^3\nabla_{\vec{v}} {}^3E(\vec{h}_i,\vec{h}_i) - K(\vec{h}_i,\vec{h}_i)\, {}^3P(\vec{v})$$
$$- K(\vec{h}_i,\vec{v})\, {}^3P(\vec{h}_i) \} = \varepsilon(\tfrac{1}{2}\vec{v}F - {}^3P'(\vec{v})) \quad , \qquad (1.16)$$

and

$$\vec{n}F = -2\varepsilon\frac{1}{D^{\perp}}\sum_i \vec{h}_i D^{\perp}\,{}^3P(\vec{h}_i) + \varepsilon\sum_i\{{}^3E'(\vec{h}_i,\vec{h}_i) + 2K(\vec{h}_i,\vec{h}_i)F$$

$$- 2\,{}^3\nabla_{\vec{h}_i}\,{}^3P(\vec{h}_i) - 2\varepsilon\sum_j K(\vec{h}_i,\vec{h}_j)\,{}^3E(\vec{h}_i,\vec{h}_j)\} \quad . \tag{1.17}$$

Equations (1.16) and (1.17) are constraints on the form of the equations that govern the evolution of the matter fields, so through the Bianchi identities gravity exerts a back-reaction on matter.

<u>Example</u>. To illustrate this coupling of gravity and matter, I shall assume that the only matter field defined on space-time is a <u>real scalar field</u>, ϕ . On the initial hypersurface, $e(\mathcal{S})$, ϕ is characterized by the fields

$$\phi \ , \ \vec{n}\phi \ , \ \{\vec{n}\vec{n}\phi - \nabla_{\vec{n}}\vec{n}\phi\} \ , \text{ etc.}$$

For the functional ${}^3E = \Pi\,{}^3E$, I choose the form defined by

$${}^3E(\vec{u},\vec{v}) = \vec{u}\phi\vec{v}\phi + \tfrac{1}{2}\,{}^3g(\vec{u},\vec{v})\mu^2\,\phi^2 \quad , \tag{1.18}$$

where μ is a constant. Differentiating this along $\vec{n}$ gives

$$\nabla_{\vec{n}}\,{}^3E(\vec{u},\vec{v}) = \vec{n}({}^3E(\vec{u},\vec{v})) - {}^3E(\nabla_{\vec{n}}\vec{u},\vec{v}) - {}^3E(\vec{u},\nabla_{\vec{n}}\vec{v})$$

$$= \vec{n}\vec{u}\phi\vec{v}\phi + \vec{u}\phi\vec{n}\vec{v}\phi + \tfrac{1}{2}\vec{n}({}^3g(\vec{u},\vec{v}))\mu^2\,\phi^2 + {}^3g(\vec{u},\vec{v})\mu^2\,\phi\vec{n}\phi$$

$$- {}^3E(\tfrac{1}{D^{\perp}}\mathcal{L}_{\vec{D}^{\perp}}\vec{u} - \varepsilon\sum_i K(\vec{u},\vec{h}_i)\vec{h}_i,\vec{v}) - {}^3E(\vec{u},\tfrac{1}{D^{\perp}}\mathcal{L}_{\vec{D}^{\perp}}\vec{v} - \varepsilon\sum_i K(\vec{v},\vec{h}_i)\vec{h}_i)$$

$$= \tfrac{1}{D^{\perp}}\vec{u}D^{\perp}\vec{n}\phi\vec{v}\phi + \vec{u}\vec{n}\phi\vec{v}\phi + \tfrac{1}{D^{\perp}}\vec{v}D^{\perp}\vec{n}\phi\vec{u}\phi + \vec{u}\phi\vec{v}\vec{n}\phi + \tfrac{1}{2}\,{}^3g(\nabla_{\vec{n}}\vec{u},\vec{v})\mu^2\,\phi^2$$

$$+ \tfrac{1}{2}\,{}^3g(\vec{u},\nabla_{\vec{n}}\vec{v})\mu^2\,\phi^2 + {}^3g(\vec{u},\vec{v})\mu^2\,\phi\vec{n}\phi - \tfrac{1}{2}\,{}^3g(\tfrac{1}{D^{\perp}}\mathcal{L}_{\vec{D}^{\perp}}\vec{u},\vec{v})\mu^2\,\phi^2$$

$$+ \ \varepsilon\sum_i K(\vec{u},\vec{h}_i)\{\vec{h}_i\phi\vec{v}\phi + \tfrac{1}{2}\ {}^3g(\vec{h}_i,\vec{v})\mu^2\ \phi^2\} - \tfrac{1}{2}\ {}^3g(\vec{u},\tfrac{1}{D^{\perp}}t_{\vec{D}^{\perp}}\vec{v})\mu^2\ \phi^2$$

$$+ \ \varepsilon\sum_i K(\vec{v},\vec{h}_i)\{\vec{h}_i\phi\vec{u}\phi + \tfrac{1}{2}\ {}^3g(\vec{u},\vec{h}_i)\mu^2\ \phi^2\}$$

$$= \{\vec{u}(\vec{n}\phi)\vec{v}\phi + \vec{u}\phi\vec{v}(\vec{n}\phi) + {}^3g(\vec{u},\vec{v})\mu^2\ \phi\vec{n}\phi + \varepsilon\sum_i(K(\vec{u},\vec{h}_i)\vec{v}\phi$$

$$+ \ K(\vec{v},\vec{h}_i)\vec{u}\phi)\vec{h}_i\phi\} + \frac{1}{D^{\perp}}\vec{v}D^{\perp}\vec{n}\phi\vec{u}\phi + \frac{1}{D^{\perp}}\vec{u}D^{\perp}\vec{n}\phi\vec{v}\phi \quad , \qquad (1.19)$$

where extensive use has been made of (2.2.11) and (2.2.16). As required, this final expression takes the form stipulated in (1.2); and by identifying the appropriate terms we find

$${}^3E'(\vec{u},\vec{v}) = \vec{u}(\vec{n}\phi)\vec{v}\phi + \vec{u}\phi\vec{v}(\vec{n}\phi) + {}^3g(\vec{u},\vec{v})\mu^2\ \phi\vec{n}\phi$$

$$+ \ \varepsilon\sum_i\{K(\vec{u},\vec{h}_i)\vec{v}\phi + K(\vec{v},\vec{h}_i)\vec{u}\phi\}\vec{h}_i\phi \quad , \qquad (1.20)$$

$${}^3P(\vec{u}) = \vec{n}\phi\vec{u}\phi \quad . \qquad (1.21)$$

If we set ${}^3P(\vec{n}) = 0$, then (1.21) can be differentiated to give

$$\nabla_{\vec{n}}{}^3P(\vec{u}) = \vec{n}({}^3P(\vec{u})) - {}^3P(\nabla_{\vec{n}}\vec{u})$$

$$= \vec{u}\phi(\vec{n}\vec{n}\phi - \nabla_{\vec{n}}\vec{n}\phi) + \vec{n}\phi(\vec{u}\vec{n}\phi + \varepsilon\sum_i K(\vec{u},\vec{h}_i)\vec{h}_i\phi)$$

$$+ \ \frac{1}{D^{\perp}}\vec{u}D^{\perp}(\vec{n}\phi\vec{n}\phi + \tfrac{1}{2}\varepsilon\mu^2\ \phi^2) - \varepsilon\frac{1}{D^{\perp}}\sum_i\vec{h}_iD^{\perp}(\vec{u}\phi\vec{h}_i\phi + \tfrac{1}{2}\ {}^3g(\vec{u},\vec{h}_i)\mu^2\ \phi^2) \ . \qquad (1.22)$$

This takes the required form, (1.7), and again identifying terms we obtain

$$F = \vec{n}\phi\vec{n}\phi + \tfrac{1}{2}\varepsilon\mu^2\ \phi^2 \quad , \qquad (1.23)$$

$$^3P'(\vec{u}) = \vec{u}\phi(\vec{n}\vec{n}\phi - \nabla_{\vec{n}}\vec{n}\phi) + \vec{n}\phi(\vec{u}\vec{n}\phi + \varepsilon\sum_i K(\vec{u},\vec{h}_i)\vec{h}_i\phi) \quad . \qquad (1.24)$$

The functional forms of ^3E, $^3E'$, ^3P, $^3P'$, and F can now be substituted into the conservation laws (1.16) and (1.17) to give, respectively

$$\{\sum_i(\vec{h}_i\vec{h}_i\phi - {}^3\nabla_{\vec{h}_i}\vec{h}_i\phi - K(\vec{h}_i,\vec{h}_i)\vec{n}\phi) + \varepsilon(\vec{n}\vec{n}\phi - \nabla_{\vec{n}}\vec{n}\phi) - \mu^2\phi\}\vec{u}\phi = 0 \quad , \qquad (1.25)$$

$$\{\sum_i(\vec{h}_i\vec{h}_i\phi - {}^3\nabla_{\vec{h}_i}\vec{h}_i\phi - K(\vec{h}_i,\vec{h}_i)\vec{n}\phi) + \varepsilon(\vec{n}\vec{n}\phi - \nabla_{\vec{n}}\vec{n}\phi) - \mu^2\phi\}\vec{n}\phi = 0 \quad . \qquad (1.26)$$

These equations must be satisfied everywhere on $e(S)$, but since at generic points $\vec{n}\phi$ and/or $\vec{u}\phi$ are non-zero we must set

$$\sum_i(\vec{h}_i\vec{h}_i\phi - \nabla_{\vec{h}_i}\vec{h}_i\phi) + \varepsilon(\vec{n}\vec{n}\phi - \nabla_{\vec{n}}\vec{n}\phi) - \mu^2\phi = 0 \quad . \qquad (1.27)$$

This last equation is easily recognized as the Klein Gordon equation, and from it we deduce that the initial data for ϕ consists of just the two fields, ϕ and $\vec{n}\phi$, on $e(S)$. The space-time metric must satisfy the field equations (1.9), with the explicit functionals obtained above being used in (1.10) to define E .

It is clear from the equations (1.13) (or (1.9)) that, as in the vacuum case, the dynamical equations governing the evolution of the metric are always supplemented by a set of primary and secondary constraint equations in such a way that the complete set is covariant in the space-time fields from which the initial data is constructed. This is equally true when it is supposed that time derivatives of g beyond the first are included in the initial data (i.e. $m>1$). Thus, for $m = 2$, the vacuum equations (2.3.5) generalize immediately to

$$\nabla S = \nabla S[g,R,F_j,\nabla F_j,\ldots,\nabla^{m_j}F_j] \tag{1.28}$$

when matter fields are present.

The back-reaction of the space-time geometry on the matter fields is also present, but less obvious, when $m > 1$. Once the geometrical field equations (i.e. equations (1.28) for $m = 2$) have been chosen, one must always check to see that they are compatible with the Bianchi identities (1.14); imposing restrictions on the matter field equations to assure this. These restrictions, when they are necessary, represent geometry's reaction on matter.

Before proceeding to the next section, a few brief remarks regarding gauge fields are in order. For convenience in the foregoing discussions, I have implicitly assumed that the distribution of matter in space-time is characterized by a <u>unique</u> configuration of the fields F_j. However, it is well known that many different configurations of the same set of fields (here more appropriately called gauge potentials) may actually provide equivalent, complete characterizations of the same matter distribution [12]. Moreover, it may be necessary to define the potentials, F_j, piecewise on overlapping neighbourhoods, in order to cover the entire space-time manifold [13]. (If $F_j^{(1)}$ and $F_j^{(2)}$ are the field configurations on the overlapping neighbourhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of M, respectively, then on the overlap region, $\mathcal{U}_1 \cap \mathcal{U}_2$, both $F_j^{(1)}$ and $F_j^{(2)}$ characterize the same matter distribution.)

The degrees of freedom in the fields F_j that are not needed to uniquely specify the matter distribution are called the gauge degrees of freedom, and the associated (gauge fixing) fields have no physical significance. When written in their four dimensional form in terms of space-time fields, the physical field equations make no reference to these non-physical fields. Nonetheless, in order to cast the field equations into an initial value form, specific gauge fixing conditions, which will have no ultimate effect on the physical predictions, must be chosen. In the first part of this section, no mention was made of these arbitrary gauge conditions, but since the gauge conditions have no influence on the physics, no generality was lost.

2. Alternative Geometries and Unified Field Theories

A great number of researchers have tried, during the past sixty-five years, to develop a new theory that maintains the philosophical and empirical successes of GR while either extending its domain of validity or else evading some of the philosophical problems that plague GR. The main premise of almost all such efforts is that the pseudo-Riemannian geometry of GR is too restrictive to provide a complete description of the world and, in particular, that the "physical" covariant derivative has non-vanishing torsion.

Einstein himself was never completely happy with GR, primarily because of its singular solutions. Considering GR to be just a macroscopic theory, he hoped to be able to find a more complex geometric theory that would yield a singularity-free model for an elementary particle. As early as 1928 Einstein suggested a theory of gravity with non-vanishing torsion, but zero curvature [14]. His later efforts to construct a unified theory of gravity and electromagnetism [15] presumed a still more complicated geometry, with a non-symmetric fundamental tensor, the symmetric part of which was a locally Minkowskian metric, and again a non-trivial torsion tensor. Although Einstein never developed a completely acceptable model, it has been shown recently by Moffat and co-workers [16] that all of the phenomenology of gravitation and (classical) electromagnetism may be understood within the context of the (pseudo-)hermitian geometry of the Einstein-Schrodinger theory ([1],[17]). Moffat [18] has also shown that a variation on the Einstein-Strauss theory [15] can lead to particle-like solutions which are non-singular in the sense that they are world-line complete, even though there are singularities in some of the field invariants.

The desire to obtain a renormalizable quantum theory of gravity seems to be the main reason for renewed interest in

Einstein-Cartan type theories [19]. Several different models have been proposed [20], with spin being coupled to gravity, through the torsion, in a non-trivial way. However, all such efforts seem to lead to a torsion field which is algebraically related to spin, and which, therefore, does not propagate as an independent field.

On the surface, it may seem as though the formalism I have developed excludes from consideration any kind of geometric structure for space-time other than the pseudo-Riemannian geometry of GR, and thus all of the "generalized" or "unified" theories based on alternative kinds of geometry. This is not the case, however. All that I have done was to separate the metric defined on space-time from any other tensor fields that are pertinent to physics, and then determine what sorts of equations are capable of propagating the metric forward in time. Since, in each of the theories discussed above, the alternative geometries always include a metric tensor, and since the metric must always propagate, the results of Section 1 remain applicable even for theories with non-Riemannian geometry, provided additional geometric fields such as the torsion or the skew part of a non-symmetric fundamental tensor are treated as "matter" fields.

This general applicability of the (pseudo-)Riemannian results should not be surprising, and has actually been known for a long time [21]. It follows from the well known fact that the difference of any two affine connections is a tensor field.

What it implies is that any theory that is based on a non-Riemannian geometry (which includes a metric) may always be reformulated in terms of (pseudo-)Riemannian geometry plus tensor fields; and, in particular, any physical theory whose field equations include derivatives of the metric up to and including second order, but no higher, is mathematically equivalent to Einstein's general theory of relativity (with sources). Thus, while they cannot be dismissed altogether, the advantages of introducing alternative geometries seem limited to the motivation of field equations different from those that would normally be investigated, and of new interpretations for physical fields.

3. Already Unified Theory

Rather than probing new kinds of geometries, Misner and Wheeler [9] followed the early work of Rainich [8] and showed that the conventional (pseudo-)Riemannian space-time already provides a sufficiently rich structure to accommodate both gravity and electromagnetism.

For compactness in the exposition of their findings, I shall now adopt a component notation, with indices $i,j,\ldots$ and $\alpha,\beta,\ldots$ ranging from 0 to 3, and repeated indices being summed. The vectors $\vec{h}_i$, $i=0,1,2,3$, will now represent a vierbein field:

$$g(\vec{h}_i,\vec{h}_j) = \eta_{ij} \quad , \tag{3.1}$$

while $\vec{\partial}_\alpha$, $\alpha=0,1,2,3$, are the coordinate basis vectors for some implicit coordinate chart. Corresponding 1-forms $\underline{h}^i$ and $\underline{dx}^\alpha$ are defined by

$$\underline{h}^i(\vec{h}_j) = \delta^i{}_j \quad \text{and} \quad \underline{dx}^\alpha(\vec{\partial}_\beta) = \delta^\alpha{}_\beta \quad . \tag{3.2}$$

The "already unified" theory of Misner and Wheeler is not a new theory, but just standard Einstein-Maxwell theory (with a source-free electromagnetic field) written in a purely geometrical form. Let $\underline{F} = \frac{1}{2}F_{\alpha\beta}\underline{dx}^\alpha \wedge \underline{dx}^\beta = \frac{1}{2}F_{ij}\underline{h}^i \wedge \underline{h}^j$ be the 2-form representing the electromagnetic field. Then its Poincaré dual is the 2-form $*F = \frac{1}{2}{*F}_{\alpha\beta}\underline{dx}^\alpha \wedge \underline{dx}^\beta$ whose components are defined by

$$*F_{\alpha\beta} = F^{\mu\nu}\underline{\text{Det}}(\vec{\partial}_\alpha,\vec{\partial}_\beta,\vec{\partial}_\mu,\vec{\partial}_\nu) \quad , \tag{3.3}$$

where $F^{\mu\nu} = g^{\mu\lambda}\, g^{\nu\sigma}\, F_{\lambda\sigma}$ and $\underline{\text{Det}}$ is the volume 4-form

$$\underline{\text{Det}} = \underline{h}^0 \wedge \underline{h}^1 \wedge \underline{h}^2 \wedge \underline{h}^3 \quad . \tag{3.4}$$

In terms of $\underline{F}$ and $*\underline{F}$, the source-free Maxwell equations take the simple form

$$\underline{dF} = 0 \quad \text{and} \quad \underline{d}*\underline{F} = 0 \quad , \tag{3.5}$$

and the Maxwell stress-energy tensor has components

$$T_{\alpha\beta} = F_{\alpha\mu}\, F_\beta{}^\mu + *F_{\alpha\mu}\, *F_\beta{}^\mu \quad . \tag{3.6}$$

The complete Einstein-Maxwell system thus consists of equations (3.5) and (1.13), with the stress-energy tensor in (1.13) being given by (3.6).

Rainich [8] showed that, quite independent of the Maxwell equations, (3.5), any Ricci tensor arising from (1.13) with the stress-energy tensor (3.6) must satisfy

$$S \equiv S_\alpha{}^\alpha = 0 \quad , \tag{3.7}$$

$$S_\alpha{}^\beta \, S_\beta{}^\gamma = \delta_\alpha{}^\gamma (S_{\sigma\tau} \, S^{\sigma\tau}/4) \quad , \tag{3.8}$$

$$S_{00} \geqslant 0 \quad . \tag{3.9}$$

Misner and Wheeler proved the converse, showing that any geometry whose Ricci tensor satisfies the Rainich conditions, (3.7), (3.8), and (3.9), can be represented as the "Maxwell square", (3.6), of some skew field $\underline{F}$. They showed, moreover, that the field $\underline{F}$ is uniquely determined by S (using equations (3.6) and (1.13)) up to a global <u>duality rotation</u>:

$$\underline{F} \to e^{*\alpha} \, \underline{F} = \underline{F} \operatorname{Cos} \alpha + {}^*\underline{F} \operatorname{Sin} \alpha \quad . \tag{3.10}$$

Defining the 1-form $\underline{\alpha} = \alpha_\mu \underline{dx}^\mu$ by the equation

$$\alpha_\tau = (\mathrm{Det})_{\tau\lambda\mu\nu} \, S^{\lambda\beta;\mu} \, S_\beta{}^\nu / (S_{\gamma\delta} S^{\gamma\delta}) \quad , \tag{3.11}$$

they then showed that if (3.7), (3.8), and (3.9) are satisfied and

$$\underline{d\alpha} = 0 \quad , \tag{3.12}$$

then the field $\underline{F}$ whose Maxwell square is S will automatically satisfy the Maxwell equations, (3.5); and they gave an explicit procedure (the details of which are not important here) for finding $\underline{F}$, given S , in the restricted case that S is not null ($S_{\sigma\tau}S^{\sigma\tau} \neq 0$).

The equations (3.12) are fourth order differential equations in the components of the space-time metric, g , and taken together with the Rainich conditions, (3.7), (3.8), and (3.9), and the Misner-Wheeler procedure for finding $\underline{F}$, given S , they constitute a purely geometrical, "already unified" way of representing the Einstein-Maxwell equations. The electromagnetic field, in this picture, is a derivative quantity and never enters on a fundamental level. The only fundamental field is g .

While its development was a great achievement, the already unified theory of gravity and electromagnetism is not without problems. The first of these is that it is unable to cope with electromagnetic fields that are null on any set of measure greater than zero. This is not too severe a restriction, though, since in reasonable physical situations one would expect the electromagnetic field to have a coulomb component that is non-vanishing at generic points. More serious problems are the lack of a Lagrangian formulation for the theory, and the certainty that any linearized version or initial value formalism of the already unified theory would be indistinguishable from corresponding treatments of the Einstein-Maxwell theory.

Aside from its failings, and the obvious fact that no experiment can distinguish it from Einstein-Maxwell theory, I find the already unified theory very interesting for the following reason. Suppose that Einstein-Maxwell theory actually provides a "correct" description of the mutually interacting gravitational and electromagnetic fields, and that in the region M' of space-time there are no matter fields other than the electromagnetic. Then the work of Rainich, Misner, and Wheeler indicates that if we make thorough measurements of the metric on M', then we can <u>deduce</u> from those measurements the configuration of the electromagnetic field (up to an overall duality rotation). <u>There is no need to make measurements of the electromagnetic field independently of the measurements of the space-time geometry</u>.

Even if one wanted to make direct measurements of the electromagnetic field, how would one do it? The simplest procedure would be to take a known charged particle, say an electron; set it adrift with some initial velocity, $\vec{v}$; and make careful observations of its trajectory. But in charting its trajectory through space-time we would be measuring distances - that is, measuring the space-time geometry - so we really wouldn't be making direct measurements of the electromagnetic field.

I have not studied the problem sufficiently to make a definitive statement, but I suspect that all attempts to make direct measurements of the electromagnetic field would be similarly doomed. Jumping far beyond the domain of electromagnetism I shall adopt the following hypothesis:

Hypothesis: The only physical field that may be measured directly is the space-time metric. The configurations of all matter fields defined on space-time must be deduced from the space-time geometry.

If this is the case, and if we still want to think of the matter fields as being somehow fundamental (as they are in quantum mechanics), then should we not think of the metric as being just the messenger of the matter fields, shouting out their existence and their characteristics as clearly as possible without colouring or obscuring the message unnecessarily with its own idiosyncracies? But the metric would fulfil this task most readily if it were to couple to the matter fields in the simplest possible way, through the Einstein equations (1.13), leaving the determination of the space-time geometry completely up to the matter fields. In any higher order dynamical equations for the metric ($m > 1$), all of the components of the curvature would be included in the initial data (albeit constrained) making it potentially impossible to decide what portion of the curvature is "gravitational" in origin and what portion should be ascribed to the matter fields.

It also follows from my hypothesis, that if the various matter fields are to be perceived, and distinguished from each other, then they must each leave a distinctive "imprint", analogous to the imprint created by the electromagnetic field, on the space-time geometry; and accordingly, that there must exist a super already unified theory capable of providing a purely geometrical description of all forms of matter. Any reasonable theory of interacting metric and matter fields must therefore be able to be recast into an already unified form, and if it cannot be, then it must be rejected.

The last two paragraphs were, clearly, quite conjectural, and I do not intend that they be taken as more than that. Nonetheless, I believe that these conjectures deserve further investigation, firstly because of the remarkable Rainich, Misner, Wheeler results, and secondly because only by pursuing such ideas can we ever hope to gain an understanding, based on physical ideas rather than mathematical conveniences, of why the particular equations we use to describe the world should be more appropriate than any other set.

4. Global Considerations

Throughout the foregoing discussions no assumptions have been made about the global topology of space or space-time. Instead, I have restricted my attention to some open cell, M' , in space-time on which all physical fields are well defined and of class C^∞ . For each generic point, x , of space-time there exists such a cell containing x , so all of the conclusions I have drawn respecting field equations hold at all generic points.

The global topology of space-time becomes important, however, when one starts investigating solutions to the field equations. The interrelationship between the pseudo-Riemannian geometry and the global topology of space-time, and the topology of space-like hypersurfaces of space-time, is discussed extensively by Hawking and Ellis [22]. Here, I would just like to point out that since any manifold may be constructed by piecing together open cells, any solution of a set of field equations may be (and in practice is) constructed by piecing together solutions defined on open cells.