CHAPTER 2

THE GEOMETRIC INITIAL VALUE PROBLEM

In this chapter and the next, I shall attempt to establish very general criteria that any acceptable theory of gravity must satisfy if it is to be compatible with the first principles (1.1.1) to (1.1.9). The present chapter is devoted to the investigation of vacuum space-times (on which the metric is the only fundamental field), with matter fields being added in Chapter 3. General relativity will emerge as the simplest possible acceptable theory in both the vacuum and general cases, and conjectural arguments will be given in Chapter 3 to support the claim that GR is the "correct" theory to use.

1. Space-time, Space, and Its Time Evolution

I shall begin, in this section, by developing a suitable mathematical formalism for describing the evolution of space through space-time. The conceptual picture I shall draw is not new, having been used by several authors [3],[6],[11] , but in the past a coordinate representation has always been used. My notation, here and throughout the thesis, is completely coordinate free. Topological and geometrical concepts are thus kept to the fore, and complete covariance is assured. For simplicity it is assumed that space and space-time are diffeomorphic with $\mathbb{R}^3$ and

$\mathbb{R}^4$, respectively. Global topological considerations are discussed, briefly, at the end of Chapter 3.

Let S be an open cell in 3-dimensional space; let M be the space-time manifold; and let $e:S \to M$ be a smooth embedding. Then it is always possible to construct a smooth map

$$E_D:S \to C^\infty(I_\delta, M) \quad , \tag{1.1}$$

from S into the set of C^∞ maps from the interval $I_\delta = (-\delta, \delta)$ into M , such that each of the maps

$$D_t:S \to M \quad ; \quad D_t(x) = (E_D(x))(t) \quad , t \in I_\delta \ , x \in S \ , \tag{1.2}$$

is a smooth embedding, and

$$D_0 = e \quad . \tag{1.3}$$

The hypersurfaces so defined are neighbouring in the sense that as t approaches t_0 the hypersurface $D_t(S)$ approaches $D_{t_0}(S)$ arbitrarily closely.

A differential description of the motion of $D_t(S)$ through space-time is provided by the vector valued maps

$$\vec{D}_t:S \to T(M) \quad ; \quad \vec{D}_t(x) \in T_{D_t(x)}(M)$$

which are defined by

$$\vec{D}_t f(x) = \frac{d}{dt}(f(D_t(x))) \tag{1.4}$$

for all differentiable test functions, $f:M \to R$, and all $x \in S$.

As t is increased, the hypersurface $D_t(S)$ is deformed continuously through M, with each point $D_t(x)$ following the trajectory of the vector $\vec{D}_t(x)$.

In general, the hypersurfaces $D_t(S)$ need not be disjoint surfaces in M. The image space, $D_t^{\vdash}(S)$, could even be the same for all of them, each map D_t being obtained from e by composing it with a diffeomorphism of S. I shall consider, however, only those maps E_D for which $D_t^{\vdash}(S) \cap D_s^{\vdash}(S) = \emptyset$ for all $t \in I_\delta$ and $s \neq t$. This means that the hypersurfaces $D_t(S)$ provide a foliation of some open region $M' \subset M$ which has the topology $S \times I_\delta \simeq \mathbb{R}^4$. The vector fields $\vec{D}_t$ may now all be combined to form a smooth <u>deformation vector field</u>, $\vec{D}: M \to T(M)$, defined on M':

$$\vec{D}(X) = \vec{D}_t(D_t^{-1}(X)) \quad , \quad X \in D_t(S) \quad . \tag{1.5}$$

Specification of $\vec{D}$ and e uniquely determines E_D, so in that which follows no direct reference will be made to E_D. For convenience, I shall assume that a particular choice of e and $\vec{D}$ has been made.

Consider now, a vector field $\vec{u}$ on S. Each of the maps D_t may be used to push $\vec{u}$ forward onto M, yielding parallel vector fields $D_{t*}\vec{u}$ defined on the surfaces $D_t(S)$. Because the hypersurfaces provide a foliation of M', each point in M' will have associated with it exactly one vector, thus yielding a smooth vector field on M', also denoted by $\vec{u}$, which is everywhere parallel to the hypersurfaces $D_t(S)$ and satisfies

$$\pounds_{\vec{D}}\vec{u} = 0 \quad . \tag{1.6}$$

(Wherever it is practical in that which follows, I shall use the symbols $\vec{u},\vec{v},\vec{w},\ldots$ to denote vector fields on S or parallel vector fields on M, and the symbols $\vec{U},\vec{V},\vec{W},\ldots$ to denote general (not necessarily parallel) vector fields on M. Other symbols will be treated individually.)

Now, if M is the space-time manifold, then in accordance with (1.1.8) it must have defined on it a set of smooth tensor fields, F_j, $j \in \omega$, such that each universe U may be realized as a submanifold of M, with the fields of U determined by the fields on M. In particular, since I want to consider $e(S)$ as space at some time, there must be a symmetric field g of type (0,2) defined on a region of M containing $e(S)$, such that the pullback e^*g is a Riemannian metric on S. For the purposes of this chapter, I shall assume that g is the only fundamental physical field defined on M'.

Knowing e^*g, it is always possible to construct a triad field of orthonormal basis vectors, $\vec{h}_i$, on S:

$$e^*g(\vec{h}_i,\vec{h}_j) = \delta_{ij} \quad , \quad i,j = 1,2,3 \quad . \tag{1.7}$$

These can be pushed forward onto $e(S)$, and used there to construct an operator, Π, which projects space-time vector fields $\vec{U},\vec{V},\ldots,\vec{W}$ onto the hypersurface:

$$\Pi(\vec{U}) = \sum_{i=1}^{3} g(\vec{U}, e_*\vec{h}_i)\, e_*\vec{h}_i \quad . \tag{1.8}$$

Space-time vectors defined at points of $e(S)$ may thus be decomposed into <u>parallel</u> and <u>perpendicular</u> parts, even if g is singular:

$$\vec{U} = \vec{U}^{\parallel} + \vec{U}^{\perp} \quad , \qquad \vec{U}^{\parallel} = \Pi(\vec{U}) \quad . \tag{1.9}$$

The vector field $\vec{u}$ defined above is, of course, already parallel: $\vec{u} = \Pi(\vec{u})$. We may also use Π to construct, on $e(S)$, a hypersurface metric, ${}^3g$, which will prove to be much more convenient than e^*g . It is defined by

$${}^3g(\vec{U},\vec{V}) = g(\vec{U}^{\parallel},\vec{V}^{\parallel}) = g(\Pi(\vec{U}),\Pi(\vec{V})) \quad . \tag{1.10}$$

The definition of space-like given in (1.1.3) guarantees that if $e(S)$ is space-like (which I assume to be the case), then any hypersurface obtained from $e(S)$ by a smooth infinitesimal deformation is also space-like; but this property does not necessarily hold for finite deformations. I shall assume, however, that $\vec{D}$ has been carefully chosen so that all of the hypersurfaces $D_t(S)$ are space-like, deferring until later a discussion of the constraints imposed on $\vec{D}$ in order to assure this. With this assumption, it is clear that the pullback, ${}^3g_t = D_{t*}g$, of g onto S , corresponding to each of the hypersurfaces $D_t(S)$, is positive definite; and both Π and ${}^3g$ may be extended smoothly to all of M' . Of course, this extension is not unique, depending as it does on the particular choice of $\vec{D}$.

2. Initial Data

The tensor field ^3g_t provides a complete description of the Riemannian geometry of space at the time labelled by t . As t increases, the field changes continuously, its rate of change being given by

$$\frac{d}{dt}({}^3g_t(\vec{u},\vec{v}))(x) = \mathcal{L}_{\vec{D}}g(\vec{u},\vec{v})(D_t(x))$$

$$= \mathcal{L}_{\vec{D}^{\parallel}}{}^3g(\vec{u},\vec{v})(\dot{D}_t(x)) + \mathcal{L}_{\vec{D}^{\perp}}g(\vec{u},\vec{v})(D_t(x)) \quad , \tag{2.1}$$

where $\vec{D}^{\parallel} = \Pi(\vec{D})$, $\vec{D}^{\perp} = \vec{D} - \vec{D}^{\parallel}$, and $\vec{u},\vec{v}$ on the left are arbitrary vector fields on S , and on the right they are the corresponding induced fields on M , which satisfy $\mathcal{L}_{\vec{D}}\vec{u} = \mathcal{L}_{\vec{D}}\vec{v} = 0$. Considering just the "initial surface", $e(S)$, we see that once $\vec{D}^{\parallel}$ has been chosen, the first term on the right is uniquely determined by the hypersurface metric, ^3g , which is equivalent to e^*g . The second term, however, is not determined by known data, nor is it invariantly defined, depending as it does on the component, $\vec{D}^{\perp}$, of the deformation vector field that cannot (yet) be specified from within the initial surface. Higher order derivatives, constructed by iterating (2.1), depend on $\vec{D}$ in still more complicated ways.

All by itself, the field ^3g , restricted to $e(S)$, provides insufficient data to determine the geometry of space at any other time; but the only other tensor fields available on $e(S)$ are the time derivatives discussed above. If we are to be able to construct the dynamical theory that is demanded by (1.1.9), then we must find some way of extracting the deformation dependence from these time derivatives to leave behind invariantly defined physical data. It is not difficult to see that this separation can only be achieved if $\vec{D}^{\perp}$ can be expressed in the form

$$\vec{D}^{\perp} = D^{\perp}\vec{n} \quad , \tag{2.2}$$

where $D^{\perp}$ is a scalar field (which <u>can</u> be specified from within $e(S)$) and $\vec{n}$ is a perpendicular vector field that is completely determined on $e(S)$ by g and the embedding, e. (In Newtonian gravity $\vec{n}$ would be an absolute time-like vector field, but that is excluded here by the assumption that all fields are dynamic.) Assuming such an $\vec{n}$, the offending last term in (2.1) can be written in the form

$$\begin{aligned} \mathcal{L}_{\vec{D}^{\perp}} g(\vec{u},\vec{v}) &= \vec{D}^{\perp}(g(\vec{u},\vec{v})) - g(\mathcal{L}_{\vec{D}^{\perp}}\vec{u},\vec{v}) - g(\vec{u},\mathcal{L}_{\vec{D}^{\perp}}\vec{v}) \\ &= D^{\perp}\vec{n}(g(\vec{u},\vec{v})) - g(D^{\perp}\mathcal{L}_{\vec{n}}\vec{u} - \vec{u}D^{\perp}\vec{n},\vec{v}) - g(\vec{u},D^{\perp}\mathcal{L}_{\vec{n}}\vec{v} - \vec{v}D^{\perp}\vec{n}) \\ &= D^{\perp}\mathcal{L}_{\vec{n}} g(\vec{u},\vec{v}) \quad . \end{aligned} \tag{2.3}$$

The tensor ${}^{3}g'$, defined by

$$ {}^{3}g'(\vec{U},\vec{V}) = \mathcal{L}_{\vec{n}} g(\Pi(\vec{U}),\Pi(\vec{V})) \tag{2.4}$$

is thus the invariantly defined piece of initial data that allows us to complete the specification, from within $e(S)$, of the rate of change of ^3g :

$$\frac{d}{dt}(^3g_t(\vec{u},\vec{v}))\Big|_{t=0} = \mathcal{L}_{\vec{D}^{\parallel}}\,{}^3g(\vec{u},\vec{v}) + D^{\perp}\,{}^3g'(\vec{u},\vec{v}) \quad . \tag{2.5}$$

The fields $\vec{n}$ and $^3g'$, just like Π and ^3g , may be extended to all of M' by constructing them on each of the hypersurfaces $D_t(S)$. Equation (2.4) remains the defining equation for $^3g'$, while $\vec{n}$ always satisfies $\Pi(\vec{n}) = 0$.

As stated above, the field $\vec{n}$ must be completely determined by the space-time field g and the set of embedding maps, D_t . Since space is of co-dimension one in space-time, it is possible, using only the properties of g assumed in the previous section, to uniquely determine the direction of $\vec{n}$ at each point of M' . However, in order to fix the magnitudes of these vectors we must make the additional assumption that g is everywhere non-singular. With this condition, $\vec{n}$ may be chosen to be a field of <u>unit</u> vectors that are everywhere <u>normal</u> to the hypersurfaces $D_t(S)$:

$$\left.\begin{aligned} &g(\vec{n},\vec{n}) = \varepsilon \quad , \quad \varepsilon = \pm 1 \quad ; \\ &g(\vec{n},\Pi(\vec{U})) = 0 \quad \text{for all } \vec{U} \in \mathcal{T}(M) \quad . \end{aligned}\right\} \tag{2.6}$$

The sign, ε , of $g(\vec{n},\vec{n})$ need not be specified at this point, but must eventually be set to -1 in order to accommodate Dirac spinors and to prevent space-like events from being causally related.

Being non-singular, g is a (pseudo-)Riemannian metric on M'. It uniquely determines a covariant derivative, ∇, while the hypersurface metric, ${}^3g$, determines a covariant derivative, ${}^3\nabla$. These satisfy

$$\left.\begin{aligned} &\vec{U}(g(\vec{V},\vec{W})) = g(\nabla_{\vec{U}}\vec{V},\vec{W}) + g(\vec{V},\nabla_{\vec{U}}\vec{W}) \quad , \\ &\vec{u}({}^3g(\vec{v},\vec{w})) = {}^3g({}^3\nabla_{\vec{u}}\vec{v},\vec{w}) + {}^3g(\vec{v},{}^3\nabla_{\vec{u}}\vec{w}) \quad , \\ &{}^3\nabla_{\vec{u}}\vec{v} = \Pi({}^3\nabla_{\vec{u}}\vec{v}) \quad , \end{aligned}\right\} \tag{2.7}$$

and are uniquely determined if their torsions are set to zero:

$$[\vec{U},\vec{V}] = \nabla_{\vec{U}}\vec{V} - \nabla_{\vec{V}}\vec{U} \quad , \qquad [\vec{u},\vec{v}] = {}^3\nabla_{\vec{u}}\vec{v} - {}^3\nabla_{\vec{v}}\vec{u} \quad . \tag{2.8}$$

They are related by

$$\nabla_{\vec{u}}\vec{v} = {}^3\nabla_{\vec{u}}\vec{v} + K(\vec{u},\vec{v})\vec{n} \quad , \tag{2.9}$$

where $K(\vec{U},\vec{V}) = K(\vec{V},\vec{U}) = K(\Pi(\vec{U}),\Pi(\vec{V}))$ is the <u>extrinsic</u> <u>curvature</u> tensor. K is actually not new, since

$$\begin{aligned} {}^3g'(\vec{U},\vec{V}) &= \pounds_{\vec{n}}g(\Pi(\vec{U}),\Pi(\vec{V})) \\ &= \vec{n}(g(\Pi(\vec{U}),\Pi(\vec{V})) - g(\pounds_{\vec{n}}(\Pi(\vec{U})),\Pi(\vec{V})) - g(\Pi(\vec{U}),\pounds_{\vec{n}}(\Pi(\vec{V}))) \\ &= g(\nabla_{\Pi(\vec{U})}\vec{n},\Pi(\vec{V})) + g(\Pi(\vec{U}),\nabla_{\Pi(\vec{V})}\vec{n}) \\ &= -\,g(\vec{n},\nabla_{\Pi(\vec{U})}\Pi(\vec{V})) - g(\nabla_{\Pi(\vec{V})}\Pi(\vec{U}),\vec{n}) \\ &= -\,2\varepsilon K(\vec{U},\vec{V}) \quad . \end{aligned} \tag{2.10}$$

The second line, here, is obtained with the use of Leibniz' rule for $\pounds$; the third line uses Leibniz' rule for ∇ and the first half of (2.8); and the fourth line again uses Leibniz' rule for ∇ and the second half of (2.6). This repeated use of Leibniz' rule and the free interchange of Lie and covariant derivatives, made possible by equations (2.8), will characterize many of the calculations that follow.

Also used extensively are the following relations. Let $\vec{u}$ be a parallel vector field: $\Pi(\vec{u}) = \vec{u}$. Then †

$$[\vec{n},\vec{u}] = \frac{1}{D^{\perp}}(\pounds_{\vec{D}^{\perp}}\vec{u} + \vec{u}D^{\perp}\vec{n}) \quad . \tag{2.11}$$

It is obvious that both $\pounds_{\vec{D}}\vec{u}$ and $\pounds_{\vec{D}^{\parallel}}\vec{u}$ are parallel; but this implies that $\pounds_{\vec{D}^{\perp}}\vec{u} = \pounds_{\vec{D}}\vec{u} - \pounds_{\vec{D}^{\parallel}}\vec{u}$ is also a parallel vector field. Thus

$$\Pi([\vec{n},\vec{u}]) = \frac{1}{D^{\perp}}\pounds_{\vec{D}^{\perp}}\vec{u} \quad . \tag{2.12}$$

Since $\vec{n}$ is a unit vector it must satisfy

$$g(\vec{n},\nabla_{\vec{n}}\vec{n}) = g(\vec{n},\nabla_{\vec{u}}\vec{n}) = 0 \quad ; \tag{2.13}$$

so the perpendicular part of (2.11), taken with (2.8), gives

$$g(\vec{n},\nabla_{\vec{n}}\vec{u}) = \varepsilon\frac{1}{D^{\perp}}\vec{u}D^{\perp} \quad . \tag{2.14}$$

Now let $\vec{h}_i$, $i = 1,2,3,$ be a triad of orthonormal parallel

† I will use the notations $\pounds_{\vec{U}}\vec{V}$ and $[\vec{U},\vec{V}]$ interchangeably.

vector fields on M' :

$$\Pi(\vec{h}_i) = \vec{h}_i \quad , \qquad {}^3g(\vec{h}_i,\vec{h}_j) = g(\vec{h}_i,\vec{h}_j) = \delta_{ij} \quad . \tag{2.15}$$

The quadruple $(\vec{n},\vec{h}_i)$ is then a vierbein for g , in terms of which any vector field may be expanded. In particular

$$\begin{aligned}\nabla_{\vec{u}}\vec{n} &= \sum_i g(\nabla_{\vec{u}}\vec{n},\vec{h}_i)\vec{h}_i + \varepsilon g(\nabla_{\vec{u}}\vec{n},\vec{n})\vec{n} \\ &= -\sum_i g(\vec{n},\nabla_{\vec{u}}\vec{h}_i)\vec{h}_i \\ &= -\sum_i K(\vec{u},\vec{h}_i)\vec{h}_i \quad ,\end{aligned} \tag{2.16}$$

$$\begin{aligned}\nabla_{\vec{n}}\vec{n} &= \sum_i g(\nabla_{\vec{n}}\vec{n},\vec{h}_i)\vec{h}_i + \varepsilon g(\nabla_{\vec{n}}\vec{n},\vec{n})\vec{n} \\ &= -\sum_i g(\vec{n},\nabla_{\vec{n}}\vec{h}_i)\vec{h}_i \\ &= -\varepsilon\frac{1}{D^{\perp}} \sum_i \vec{h}_i D^{\perp}\vec{h}_i \quad .\end{aligned} \tag{2.17}$$

A feature of these relations that will arise repeatedly is that any derivative along $\vec{n}$, if it is not determined by some identity (as in (2.13)), must depend explicitly on the deformation vector field.

Associated with ∇ is the <u>Riemann curvature</u> tensor, R , of space-time:

$$R(\vec{U},\vec{V})\vec{W} = \nabla_{\vec{U}}\nabla_{\vec{V}}\vec{W} - \nabla_{\vec{V}}\nabla_{\vec{U}}\vec{W} - \nabla_{[\vec{U},\vec{V}]}\vec{W} \quad . \tag{2.18}$$

Similarly, the <u>hypersurface curvature</u> tensor, ${}^3R$, is constructed with the use of ${}^3\nabla$. Its non-zero components are given by

$${}^3R(\vec{u},\vec{v})\vec{w} = {}^3\nabla_{\vec{u}}{}^3\nabla_{\vec{v}}\vec{w} - {}^3\nabla_{\vec{v}}{}^3\nabla_{\vec{u}}\vec{w} - {}^3\nabla_{[\vec{u},\vec{v}]}\vec{w} \quad . \tag{2.19}$$

If we maintain the convention that $\vec{u},\vec{v},\vec{w},\ldots$ are hypersurface (parallel) vector fields, then we find that there are four distinct kinds of terms into which R may be decomposed: $R(\vec{u},\vec{v})\vec{w}$, $R(\vec{u},\vec{v})\vec{n}$, $R(\vec{n},\vec{v})\vec{w}$, and $R(\vec{n},\vec{v})\vec{n}$. These may be expanded, in terms of hypersurface fields, as follows:

$$R(\vec{u},\vec{v})\vec{w} = {}^3R(\vec{u},\vec{v})\vec{w} + \{{}^3\nabla_{\vec{u}}K(\vec{v},\vec{w}) - {}^3\nabla_{\vec{v}}K(\vec{u},\vec{w})\}\vec{n}$$
$$+ \varepsilon\sum_i\{K(\vec{u},\vec{w})K(\vec{v},\vec{h}_i) - K(\vec{v},\vec{w})K(\vec{u},\vec{h}_i)\}\vec{h}_i \quad , \tag{2.20}$$

$$R(\vec{u},\vec{v})\vec{n} = -\varepsilon\sum_i\{{}^3\nabla_{\vec{u}}K(\vec{v},\vec{h}_i) - {}^3\nabla_{\vec{v}}K(\vec{u},\vec{h}_i)\}\vec{h}_i \quad , \tag{2.21}$$

$$R(\vec{n},\vec{v})\vec{n} = \nabla_{\vec{n}}\nabla_{\vec{v}}\vec{n} - \nabla_{\vec{v}}\nabla_{\vec{n}}\vec{n} - \nabla_{[\vec{n},\vec{v}]}\vec{n}$$
$$= \nabla_{\vec{n}}(-\varepsilon\sum_i K(\vec{v},\vec{h}_i)\vec{h}_i) - \nabla_{\vec{v}}(-\varepsilon\frac{1}{D^{\perp}}\sum_i \vec{h}_i D^{\perp}\vec{h}_i)$$
$$+ \varepsilon\frac{1}{D^{\perp}}\sum_i K(\pounds_{\vec{D}^{\perp}}\vec{v},\vec{h}_i)\vec{h}_i + \varepsilon(\frac{1}{D^{\perp}})^2\ \vec{v}D^{\perp}\sum_i\vec{h}_i D^{\perp}\vec{h}_i$$
$$= -\varepsilon\sum_i\{\nabla_{\vec{n}}K(\vec{v},\vec{h}_i) - \frac{1}{D^{\perp}}(\vec{h}_i\vec{v}D^{\perp} - {}^3\nabla_{\vec{h}_i}\vec{v}D^{\perp})$$
$$- \varepsilon\sum_j K(\vec{v},\vec{h}_j)K(\vec{h}_i,\vec{h}_j)\}\vec{h}_i \quad , \tag{2.22}$$

$$R(\vec{n},\vec{v})\vec{w} = \sum_i g(\vec{h}_i,R(\vec{n},\vec{v})\vec{w})\vec{h}_i + \varepsilon g(\vec{n},R(\vec{n},\vec{v})\vec{w})\vec{n}$$
$$= \sum_i g(\vec{n},R(\vec{h}_i,\vec{w})\vec{v})\vec{h}_i - \varepsilon g(\vec{w},R(\vec{n},\vec{v})\vec{n})\vec{n}$$
$$= \varepsilon\sum_i\{{}^3\nabla_{\vec{h}_i}K(\vec{w},\vec{v}) - {}^3\nabla_{\vec{w}}K(\vec{h}_i,\vec{v})\}\vec{h}_i + \{\nabla_{\vec{n}}K(\vec{w},\vec{v})$$
$$- \frac{1}{D^{\perp}}(\vec{v}\vec{w}D^{\perp} - {}^3\nabla_{\vec{v}}\vec{w}D^{\perp}) - \varepsilon\sum_i K(\vec{v},\vec{h}_i)K(\vec{w},\vec{h}_i)\}\vec{n} \quad . \tag{2.23}$$

Equations (2.20) and (2.21) are the classical equations of Gauss and Codazzi, and are derived in Appendix I. Equation (2.22), on the other hand, is a new result made possible by the introduction of the deformation vector field. In it, the implicit deformation dependence of $\nabla_{\vec{n}}K$ is balanced by the terms that depend explicitly on $\vec{D}$.

The Ricci tensor, S , of space-time is obtained by contracting the curvature, R :

$$S(\vec{U},\vec{V}) = \sum_i g(\vec{h}_i,R(\vec{h}_i,\vec{U})\vec{V}) + \varepsilon g(\vec{n},R(\vec{n},\vec{U})\vec{V}) \quad ; \tag{2.24}$$

and its components may be written in the expanded forms

$$S(\vec{u},\vec{v}) = {}^3S(\vec{u},\vec{v}) - \varepsilon K(\vec{u},\vec{v})\sum_i K(\vec{h}_i,\vec{h}_i) + \nabla_{\vec{n}}K(\vec{u},\vec{v}) - \frac{1}{D^\perp}(\vec{u}\vec{v}D^\perp - {}^3\nabla_{\vec{u}}\vec{v}D^\perp) \quad , \tag{2.25}$$

$$S(\vec{u},\vec{n}) = \varepsilon\sum_i\{ {}^3\nabla_{\vec{u}}K(\vec{h}_i,\vec{h}_i) - {}^3\nabla_{\vec{h}_i} K(\vec{u},\vec{h}_i)\} \quad , \tag{2.26}$$

$$S(\vec{n},\vec{n}) = -\sum_{i,j} K(\vec{h}_i,\vec{h}_j)K(\vec{h}_i,\vec{h}_j) + \varepsilon\sum_i\{\nabla_{\vec{n}}K(\vec{h}_i,\vec{h}_i) - \frac{1}{D^\perp}(\vec{h}_i\vec{h}_iD^\perp - {}^3\nabla_{\vec{h}_i}\vec{h}_iD^\perp)\} \quad , \tag{2.27}$$

where ${}^3S(\vec{u},\vec{v})$ is the Ricci tensor associated with ${}^3R$. Contracting again yields the Ricci scalar:

$$\begin{aligned} S &= \sum_i S(\vec{h}_i,\vec{h}_i) + \varepsilon S(\vec{n},\vec{n}) \\ &= {}^3S - \varepsilon\sum_{i,j}\{K(\vec{h}_i,\vec{h}_j)K(\vec{h}_i,\vec{h}_j) + K(\vec{h}_i,\vec{h}_i)K(\vec{h}_j,\vec{h}_j)\} \\ &\quad + 2\sum_i\{\nabla_{\vec{n}}K(\vec{h}_i,\vec{h}_i) - \frac{1}{D^\perp}(\vec{h}_i\vec{h}_iD^\perp - {}^3\nabla_{\vec{h}_i}\vec{h}_iD^\perp)\} \quad , \end{aligned} \tag{2.28}$$

in which ${}^3S = \sum_i {}^3S(\vec{h}_i,\vec{h}_i)$.

Substituting (2.25) into (2.22) and (2.23) gives, respectively

$$R(\vec{n},\vec{v})\vec{n} = -\varepsilon\sum_i\{S(\vec{v},\vec{h}_i) - {}^3S(\vec{v},\vec{h}_i)$$

$$+ \varepsilon\sum_j(K(\vec{v},\vec{h}_i)K(\vec{h}_j,\vec{h}_j) - K(\vec{v},\vec{h}_j)K(\vec{h}_i,\vec{h}_j))\}\vec{h}_i \quad ; \tag{2.29}$$

$$R(\vec{n},\vec{v})\vec{w} = \varepsilon\sum_i\{{}^3\nabla_{\vec{h}_i}K(\vec{w},\vec{v}) - {}^3\nabla_{\vec{w}}K(\vec{h}_i,\vec{v})\}\vec{h}_i + \{S(\vec{w},\vec{v}) - {}^3S(\vec{w},\vec{v})$$

$$+ \varepsilon\sum_i(K(\vec{w},\vec{v})K(\vec{h}_i,\vec{h}_i) - K(\vec{w},\vec{h}_i)K(\vec{v},\vec{h}_i))\}\vec{n} \quad . \tag{2.30}$$

Thus, once the hypersurface metric, ${}^3g$, and the extrinsic curvature, K , have been determined, the only new (independent) data needed to complete the specification of R from within a given hypersurface are the hypersurface components, $S(\vec{u},\vec{v})$, of the space-time Ricci tensor.

If all the components of the space-time curvature are known, then they can be used to change the order of multiple covariant derivatives, allowing the calculation of quantities that would be otherwise inaccessible. I shall give a few examples which are of later importance. The simplest example is $\nabla_{\vec{n}}{}^3\nabla_{\vec{u}}\vec{v}$:

$$\nabla_{\vec{n}}{}^3\nabla_{\vec{u}}\vec{v} = \nabla_{\vec{n}}(\nabla_{\vec{u}}\vec{v} - K(\vec{u},\vec{v})\vec{n})$$

$$= R(\vec{n},\vec{u})\vec{v} + \nabla_{\vec{u}}\nabla_{\vec{n}}\vec{v} + \nabla_{[\vec{n},\vec{u}]}\vec{v} - \nabla_{\vec{n}}K(\vec{u},\vec{v})\vec{n} - K(\nabla_{\vec{n}}\vec{u},\vec{v})\vec{n}$$

$$- K(\vec{u},\nabla_{\vec{n}}\vec{v})\vec{n} - K(\vec{u},\vec{v})\nabla_{\vec{n}}\vec{n}$$

$$= \varepsilon\sum_i\{{}^3\nabla_{\vec{h}_i}K(\vec{v},\vec{u}) - {}^3\nabla_{\vec{v}}K(\vec{h}_i,\vec{u})\}\vec{h}_i - \varepsilon\sum_i K(\vec{u},\vec{h}_i)K(\vec{v},\vec{h}_i)\vec{n}$$

$$+ \nabla_{\vec{u}}\left(\frac{1}{D^\perp}\pounds_{\vec{D}^\perp}\vec{v} + \frac{1}{D^\perp}\vec{v}D^\perp\vec{n} + \nabla_{\vec{v}}\vec{n}\right) + \frac{1}{D^\perp}\nabla_{\pounds_{\vec{D}^\perp}\vec{u}}\vec{v} + \frac{1}{D^\perp}\vec{u}D^\perp\nabla_{\vec{n}}\vec{v}$$

$$-\{\frac{1}{D^{\perp}}(\vec{u}\vec{v}D^{\perp} - {}^3\nabla_{\vec{u}}\vec{v}D^{\perp}) + \frac{1}{D^{\perp}}K(\pounds_{\vec{D}^{\perp}}\vec{u},\vec{v}) + K(\nabla_{\vec{u}}\vec{n},\vec{v}) + \frac{1}{D^{\perp}}K(\vec{u},\pounds_{\vec{D}^{\perp}}\vec{v})$$

$$+ K(\vec{u},\nabla_{\vec{v}}\vec{n})\}\vec{n} + \varepsilon\frac{1}{D^{\perp}}K(\vec{u},\vec{v})\sum_i \vec{h}_i D^{\perp}\vec{h}_i$$

$$= \varepsilon\sum_i\{{}^3\nabla_{\vec{h}_i}K(\vec{v},\vec{u}) - {}^3\nabla_{\vec{v}}K(\vec{h}_i,\vec{u})\}\vec{h}_i + \frac{1}{D^{\perp}}{}^3\nabla_{\vec{u}}\pounds_{\vec{D}^{\perp}}\vec{v} - \varepsilon\frac{1}{D^{\perp}}\vec{v}D^{\perp}\sum_i K(\vec{u},\vec{h}_i)\vec{h}_i$$

$$- \varepsilon\nabla_{\vec{u}}(\sum_i K(\vec{v},\vec{h}_i)\vec{h}_i) + \frac{1}{D^{\perp}}{}^3\nabla_{\pounds_{\vec{D}^{\perp}}\vec{u}}\vec{v} - \varepsilon\frac{1}{D^{\perp}}\vec{u}D^{\perp}\sum_i K(\vec{v},\vec{h}_i)\vec{h}_i + \frac{1}{D^{\perp}}{}^3\nabla_{\vec{u}}\vec{v}D^{\perp}\vec{n}$$

$$+ \varepsilon\sum_i K(\vec{u},\vec{h}_i)K(\vec{v},\vec{h}_i)\vec{n} + \varepsilon\frac{1}{D^{\perp}}K(\vec{u},\vec{v})\sum_i \vec{h}_i D^{\perp}\vec{h}_i$$

$$= \varepsilon\sum_i\{{}^3\nabla_{\vec{h}_i}K(\vec{v},\vec{u}) - {}^3\nabla_{\vec{v}}K(\vec{h}_i,\vec{u}) - {}^3\nabla_{\vec{u}}K(\vec{v},\vec{h}_i) - K({}^3\nabla_{\vec{u}}\vec{v},\vec{h}_i)\}\vec{h}_i$$

$$+ \frac{1}{D^{\perp}}\{{}^3\nabla_{\pounds_{\vec{D}^{\perp}}\vec{u}}\vec{v} + {}^3\nabla_{\vec{u}}\pounds_{\vec{D}^{\perp}}\vec{v} + {}^3\nabla_{\vec{u}}\vec{v}D^{\perp}\vec{n}\}$$

$$+ \varepsilon\frac{1}{D^{\perp}}\sum_i\{\vec{h}_i D^{\perp}K(\vec{u},\vec{v}) - \vec{u}D^{\perp}K(\vec{h}_i,\vec{v}) - \vec{v}D^{\perp}K(\vec{u},\vec{h}_i)\}\vec{h}_i \quad . \tag{2.31}$$

Before proceeding to more complicated examples, I must digress for a moment to establish new notation. Let T be an arbitrary tensor field on M', and let T_{co} be the associated covariant tensor. I shall denote by ΠT the tensor field on M' that has the same type as T and whose covariant components are defined by

$$(\Pi T)_{co}(\vec{U},\vec{V},\ldots) = T_{co}(\Pi(\vec{U}),\Pi(\vec{V}),\ldots) \quad . \tag{2.32}$$

We already have $\Pi\,{}^3g = {}^3g$ and $\Pi K = K$. By assuming that ${}^3\nabla$ has the generalized action

$${}^3\nabla_{\vec{U}}\vec{V} = {}^3\nabla_{\Pi(\vec{U})}\Pi(\vec{V}) \quad , \tag{2.33}$$

we obtain, in addition, $\Pi\,{}^3R = {}^3R$ and $\Pi\,{}^3S = {}^3S$. This assumption is necessary in order to avoid ambiguities.

I shall also adopt a simplifying notation for the hypersurface derivatives of hypersurface tensors. Let $T = \Pi T$ be an arbitrary covariant hypersurface tensor field. Then the tensor field T_1 will be defined by

$$T_1(\vec{U};\vec{V},\vec{W},\ldots) = {}^3\nabla_{\vec{U}}T(\vec{V},\vec{W},\ldots) \quad , \tag{2.34}$$

the tensor field T_2 by

$$T_2(\vec{X},\vec{U};\vec{V},\vec{W},\ldots) = {}^3\nabla_{\vec{X}}{}^3\nabla_{\vec{U}}T(\vec{V},\vec{W},\ldots) - {}^3\nabla_{{}^3\nabla_{\vec{X}}\vec{U}}T(\vec{V},\vec{W},\ldots) \quad , \tag{2.35}$$

and so on for higher derivatives. By virtue of (2.33), all of these fields satisfy $\Pi T_i = T_i$, $i \in \omega$.

Repeated application of the techniques used in the derivation of (2.31) now yields the following additional examples:

$$\begin{aligned}
\nabla_{\vec{n}}K_1(\vec{w};\vec{u},\vec{v}) &= (\Pi S)_1(\vec{w};\vec{u},\vec{v}) - {}^3S_1(\vec{w};\vec{u},\vec{v}) \\
&+ \varepsilon\sum_i\{K(\vec{h}_i,\vec{h}_i)K_1(\vec{w};\vec{u},\vec{v}) + K(\vec{u},\vec{v})K_1(\vec{w};\vec{h}_i,\vec{h}_i) + K(\vec{w},\vec{h}_i)K_1(\vec{h}_i;\vec{u},\vec{v}) \\
&+ K(\vec{u},\vec{h}_i)[K_1(\vec{v};\vec{w},\vec{h}_i) - K_1(\vec{h}_i;\vec{w},\vec{v})] + K(\vec{v},\vec{h}_i)[K_1(\vec{u};\vec{w},\vec{h}_i) \\
&- K_1(\vec{h}_i;\vec{w},\vec{u})]\} + \frac{1}{D^\perp}\vec{w}D^\perp\{\Pi S(\vec{u},\vec{v}) - {}^3S(\vec{u},\vec{v}) + \varepsilon K(\vec{u},\vec{v})\sum_i K(\vec{h}_i,\vec{h}_i)\} \\
&+ \varepsilon\frac{1}{D^\perp}\sum_i\{\vec{u}D^\perp K(\vec{w},\vec{h}_i)K(\vec{v},\vec{h}_i) + \vec{v}D^\perp K(\vec{w},\vec{h}_i)K(\vec{u},\vec{h}_i) \\
&- \vec{h}_iD^\perp[K(\vec{w},\vec{u})K(\vec{v},\vec{h}_i) + K(\vec{w},\vec{v})K(\vec{u},\vec{h}_i)]\} + \frac{1}{D^\perp}\{\vec{w}\vec{u}\vec{v}D^\perp - \vec{w}\,{}^3\nabla_{\vec{u}}\vec{v}D^\perp \\
&- {}^3\nabla_{\vec{w}}\vec{u}\vec{v}D^\perp + {}^3\nabla_{{}^3\nabla_{\vec{w}}\vec{u}}\vec{v}D^\perp - {}^3\nabla_{\vec{w}}\vec{v}\vec{u}D^\perp + {}^3\nabla_{{}^3\nabla_{\vec{w}}\vec{v}}\vec{u}D^\perp\} \quad ,
\end{aligned} \tag{2.36}$$

$$\nabla_{\vec{n}}{}^3R(\vec{u},\vec{v})\vec{w} = \varepsilon\sum_i\{K(\vec{u},\vec{h}_i)\,{}^3R(\vec{h}_i,\vec{v})\vec{w} + K(\vec{v},\vec{h}_i)\,{}^3R(\vec{u},\vec{h}_i)\vec{w}$$

$$+ K(\vec{w},\vec{h}_i)\,{}^3R(\vec{u},\vec{v})\vec{h}_i\} + \varepsilon\sum_i\{-K_2(\vec{u},\vec{v};\vec{w},\vec{h}_i) - K_2(\vec{u},\vec{w};\vec{v},\vec{h}_i)$$

$$+ K_2(\vec{u},\vec{h}_i;\vec{v},\vec{w}) + K_2(\vec{v},\vec{u};\vec{w},\vec{h}_i) + K_2(\vec{v},\vec{w};\vec{u},\vec{h}_i) - K_2(\vec{v},\vec{h}_i;\vec{u},\vec{w})$$

$$+ \frac{1}{D^\perp}\vec{u}D^\perp[K_1(\vec{h}_i;\vec{v},\vec{w}) - K_1(\vec{w};\vec{v},\vec{h}_i)] - \frac{1}{D^\perp}\vec{v}D^\perp[K_1(\vec{h}_i;\vec{u},\vec{w})$$

$$- K_1(\vec{w};\vec{u},\vec{h}_i)] + \frac{1}{D^\perp}\vec{w}D^\perp[K_1(\vec{v};\vec{u},\vec{h}_i) - K_1(\vec{u};\vec{v},\vec{h}_i)]$$

$$+ \frac{1}{D^\perp}\vec{h}_iD^\perp[K_1(\vec{u};\vec{v},\vec{w}) - K_1(\vec{v};\vec{u},\vec{w})] + \frac{1}{D^\perp}(\vec{v}\vec{w}D^\perp - {}^3\nabla_{\vec{v}}\vec{w}D^\perp)K(\vec{u},\vec{h}_i)$$

$$- \frac{1}{D^\perp}(\vec{u}\vec{w}D^\perp - {}^3\nabla_{\vec{u}}\vec{w}D^\perp)K(\vec{v},\vec{h}_i) + \frac{1}{D^\perp}(\vec{u}\vec{h}_iD^\perp - {}^3\nabla_{\vec{u}}\vec{h}_iD^\perp)K(\vec{v},\vec{w})$$

$$- \frac{1}{D^\perp}(\vec{v}\vec{h}_iD^\perp - {}^3\nabla_{\vec{v}}\vec{h}_iD^\perp)K(\vec{u},\vec{w}) - K({}^3R(\vec{u},\vec{v})\vec{w},\vec{h}_i)\}\vec{h}_i$$

$$+ \frac{1}{D^\perp}{}^3R(\vec{u},\vec{v})\vec{w}D^\perp\vec{n} \qquad , \qquad (2.37)$$

$$\nabla_{\vec{n}}{}^3S(\vec{u},\vec{v}) = \varepsilon\sum_i\{K(\vec{u},\vec{h}_i)\,{}^3S(\vec{h}_i,\vec{v}) + K(\vec{v},\vec{h}_i)\,{}^3S(\vec{u},\vec{h}_i) + K_2(\vec{h}_i,\vec{h}_i;\vec{u},\vec{v})$$

$$- K_2(\vec{h}_i,\vec{u};\vec{h}_i,\vec{v}) - K_2(\vec{h}_i,\vec{v};\vec{u},\vec{h}_i) + K_2(\vec{u},\vec{v};\vec{h}_i,\vec{h}_i)\}$$

$$+ \varepsilon\frac{1}{D^\perp}\sum_i\{\vec{u}D^\perp[K_1(\vec{v};\vec{h}_i,\vec{h}_i) - K_1(\vec{h}_i;\vec{h}_i,\vec{v})] + \vec{v}D^\perp[K_1(\vec{u};\vec{h}_i,\vec{h}_i)$$

$$- K_1(\vec{h}_i;\vec{u},\vec{h}_i)] + \vec{h}_iD^\perp[2K_1(\vec{h}_i;\vec{u},\vec{v}) - K_1(\vec{u};\vec{h}_i,\vec{v}) - K_1(\vec{v};\vec{u},\vec{h}_i)]$$

$$+ (\vec{u}\vec{v}D^\perp - {}^3\nabla_{\vec{u}}\vec{v}D^\perp)K(\vec{h}_i,\vec{h}_i) - (\vec{u}\vec{h}_iD^\perp - {}^3\nabla_{\vec{u}}\vec{h}_iD^\perp)K(\vec{h}_i,\vec{v})$$

$$- (\vec{v}\vec{h}_iD^\perp - {}^3\nabla_{\vec{v}}\vec{h}_iD^\perp)K(\vec{u},\vec{h}_i) + (\vec{h}_i\vec{h}_iD^\perp - {}^3\nabla_{\vec{h}_i}\vec{h}_iD^\perp)K(\vec{u},\vec{v})\} \ . \qquad (2.38)$$

We saw above (cf. Equations (2.20),(2.21),(2.29), and (2.30)) that R may be expressed completely in terms of ${}^3g$, K , and ΠS , and hypersurface derivatives (i.e. derivatives along hypersurface vector fields $\vec{u},\vec{v},\ldots$) of these quantities. The components $\nabla_{\vec{u}}R(\vec{V},\vec{W})\vec{X}$ and (with the use of Bianchi's second set of identities) $\nabla_{\vec{n}}R(\vec{v},\vec{w})\vec{X}$ of ∇R may also be expressed in terms of this same data. However, whenever there are two or more derivatives in the <u>normal</u> direction, as in $\nabla_{\vec{n}}R(\vec{n},\vec{v})\vec{X}$, more data is required. Direct calculations, in which Equations (2.36), (2.37), and (2.38) are used, give the following results:

$$\begin{aligned}
\nabla_{\vec{n}}R(\vec{n},\vec{v})\vec{x} &= \varepsilon\sum_i\{(\Pi S)_1(\vec{h}_i;\vec{x},\vec{v}) - {}^3S_1(\vec{h}_i;\vec{x},\vec{v}) - (\Pi S)_1(\vec{x};\vec{h}_i,\vec{v}) \\
&+ {}^3S_1(\vec{x};\vec{h}_i,\vec{v})\}\vec{h}_i + \sum_{i,j}\{K(\vec{h}_j,\vec{h}_j)[K_1(\vec{h}_i;\vec{x},\vec{v}) - K_1(\vec{x};\vec{h}_i,\vec{v})] \\
&+ K(\vec{x},\vec{v})K_1(\vec{h}_i;\vec{h}_j,\vec{h}_j) - K(\vec{h}_i,\vec{v})K_1(\vec{x};\vec{h}_j,\vec{h}_j) + K(\vec{x},\vec{h}_j)[K_1(\vec{v};\vec{h}_i,\vec{h}_j) \\
&- 2K_1(\vec{h}_j;\vec{h}_i,\vec{v})] + K(\vec{v},\vec{h}_j)[K_1(\vec{x};\vec{h}_i,\vec{h}_j) - K_1(\vec{h}_i;\vec{x},\vec{h}_j)] \\
&+ K(\vec{h}_i,\vec{h}_j)[2K_1(\vec{h}_j;\vec{x},\vec{v}) - K_1(\vec{v};\vec{x},\vec{h}_j)]\}\vec{h}_i \\
&+ \varepsilon\sum_i\{K(\vec{x},\vec{v})[S(\vec{h}_i,\vec{h}_i) - {}^3S(\vec{h}_i,\vec{h}_i) + \varepsilon K(\vec{h}_i,\vec{h}_i)\sum_j K(\vec{h}_j,\vec{h}_j)] \\
&\quad + K(\vec{h}_i,\vec{h}_i)[S(\vec{x},\vec{v}) - {}^3S(\vec{x},\vec{v}) + \varepsilon K(\vec{x},\vec{v})\sum_j K(\vec{h}_j,\vec{h}_j)] \\
&\quad - K(\vec{x},\vec{h}_i)[S(\vec{v},\vec{h}_i) - {}^3S(\vec{v},\vec{h}_i) + \varepsilon K(\vec{v},\vec{h}_i)\sum_j K(\vec{h}_j,\vec{h}_j)] \\
&\quad - K(\vec{v},\vec{h}_i)[S(\vec{x},\vec{h}_i) - {}^3S(\vec{x},\vec{h}_i) + \varepsilon K(\vec{x},\vec{h}_i)\sum_j K(\vec{h}_j,\vec{h}_j)] \\
&\quad - K(\vec{h}_i,{}^3R(\vec{h}_i,\vec{x})\vec{v}) - K(\vec{h}_i,{}^3R(\vec{h}_i,\vec{v})\vec{x}) + K_2(\vec{x},\vec{h}_i;\vec{h}_i,\vec{v})
\end{aligned}$$

$$+ K_2(\vec{v},\vec{h}_i;\vec{x},\vec{h}_i) - K_2(\vec{x},\vec{v};\vec{h}_i,\vec{h}_i) - K_2(\vec{h}_i,\vec{h}_i;\vec{x},\vec{v})\}\vec{n}$$

$$+ \{\nabla_{\vec{n}}(\Pi S)(\vec{x},\vec{v}) - \frac{1}{D^\perp}\vec{x}D^\perp S(\vec{n},\vec{v}) - \frac{1}{D^\perp}\vec{v}D^\perp S(\vec{x},\vec{n})\}\vec{n} \quad , \qquad (2.39)$$

$$\nabla_{\vec{n}}R(\vec{n},\vec{v})\vec{n} = \sum_{i,j}\{K(\vec{h}_j,\vec{v})[S(\vec{h}_i,\vec{h}_j) - {}^3S(\vec{h}_i,\vec{h}_j)$$

$$+ 2\varepsilon K(\vec{h}_i,\vec{h}_j)\sum_k K(\vec{h}_k,\vec{h}_k)] - K(\vec{h}_i,\vec{v})[S(\vec{h}_j,\vec{h}_j) - {}^3S(\vec{h}_j,\vec{h}_j)$$

$$+ 2\varepsilon K(\vec{h}_j,\vec{h}_j)\sum_k K(\vec{h}_k,\vec{h}_k)] + K(\vec{h}_i,\vec{h}_j)[S(\vec{h}_j,\vec{v}) - {}^3S(\vec{h}_j,\vec{v})]$$

$$- K(\vec{h}_j,\vec{h}_j)[S(\vec{h}_i,\vec{v}) - {}^3S(\vec{h}_i,\vec{v})] + K(\vec{h}_j,{}^3R(\vec{h}_j,\vec{h}_i)\vec{v})$$

$$+ K(\vec{h}_j,{}^3R(\vec{h}_j,\vec{v})\vec{h}_i) + K_2(\vec{h}_j,\vec{h}_j;\vec{h}_i,\vec{v}) - K_2(\vec{h}_i,\vec{h}_j;\vec{h}_j,\vec{v})$$

$$- K_2(\vec{v},\vec{h}_j;\vec{h}_i,\vec{h}_j) + K_2(\vec{h}_i,\vec{v};\vec{h}_j,\vec{h}_j)\}\vec{h}_i$$

$$- \varepsilon\sum_i\{\nabla_{\vec{n}}(\Pi S)(\vec{h}_i,\vec{v}) - \frac{1}{D^\perp}\vec{h}_i D^\perp S(\vec{n},\vec{v}) - \frac{1}{D^\perp}\vec{v}D^\perp S(\vec{h}_i,\vec{n})\}\vec{h}_i \quad . \qquad (2.40)$$

with $S(\vec{n},\vec{v})$ defined by (2.26). In both of these equations, the new data involved is the (deformation dependent) field $\nabla_{\vec{n}}(\Pi S)$. By noting that

$$\nabla_{\vec{n}}S(\vec{u},\vec{v}) = \vec{n}(S(\vec{u},\vec{v})) - S(\nabla_{\vec{n}}\vec{u},\vec{v}) - S(\vec{u},\nabla_{\vec{n}}\vec{v})$$

$$= \nabla_{\vec{n}}(\Pi S)(\vec{u},\vec{v}) + \Pi S(\nabla_{\vec{n}}\vec{u},\vec{v}) - S(\nabla_{\vec{n}}\vec{u},\vec{v}) + \Pi S(\vec{u},\nabla_{\vec{n}}\vec{v}) - S(\vec{u},\nabla_{\vec{n}}\vec{v})$$

$$= \nabla_{\vec{n}}(\Pi S)(\vec{u},\vec{v}) - \frac{1}{D^\perp}\vec{u}D^\perp S(\vec{n},\vec{v}) - \frac{1}{D^\perp}\vec{v}D^\perp S(\vec{u},\vec{n}) \quad , \qquad (2.41)$$

all of the deformation dependent terms may be combined into the one physical (deformation independent) field $\Pi(\nabla_{\vec{n}}S)$, yielding results completely analogous to (2.29) and (2.30).

The final outcome of these rather lengthy calculations is that if we wish to characterize the geometry of space-time from within a given space-like hypersurface, say $e(\mathcal{S})$, then the only a priori independent fields that we must specify on $e(\mathcal{S})$ are the hypersurface metric, ${}^3g$, the extrinsic curvature, K , the projection, ΠS , of the space-time Ricci tensor onto $e(\mathcal{S})$, and the hypersurface components of the covariant derivatives along $\vec{n}$, to all orders, of the space-time Ricci tensor: $\Pi(\nabla_{\vec{n}} S)$, $\Pi(\nabla_{\vec{n}}\nabla_{\vec{n}} S - \nabla_{\nabla_{\vec{n}}\vec{n}} S)$, etc.. Equation (2.10) shows that K is just the derivative of ${}^3g$ along $\vec{n}$, and (2.25) indicates that ΠS is (roughly speaking) the derivative of K along $\vec{n}$; so the independent data is effectively ${}^3g$ and all of its (normalized) time derivatives.

Although all of these fields are well defined and a priori independent of each other, physical space-time is such that only a finite number of them need be specified in order to determine the complete set (cf. (1.1.9)). The <u>initial data</u> on $e(\mathcal{S})$ then consists of ${}^3g$ and those derivatives of ${}^3g$ along $\vec{n}$ up to some finite order, say m , that cannot be obtained as functionals of the others. By implication, the $(m+1)$th and higher derivatives of ${}^3g$ can be obtained as explicit functionals of the initial data fields and their hypersurface derivatives.

3. Gravitational Field Equations

The physical assumptions that I have made so far are insufficient to determine at what differential order the initial data cuts off and dynamical equations begin. I shall assume, therefore, that, as above, the initial data includes derivatives of ^3g along $\vec{n}$ up to and including the m-th order. If this data is known on $e(S)$, then on an infinitesimally close hypersurface, $D_{\delta t}(S)$, the metric is given by (cf. (2.1))

$$^3g_{\delta t}(\vec{u},\vec{v}) = {}^3g_0(\vec{u},\vec{v}) + \frac{d}{dt}({}^3g_t(\vec{u},\vec{v}))\Big|_{t=0}\cdot\delta t \quad , \tag{3.1}$$

and the time derivatives up to (m-1)th order are given by similar expressions. By iterating this process, the hypersurface metric can be carried forward m infinitesimal steps in time, but $m\cdot\delta t$ is still infinitesimal. In order to be able to integrate ahead a finite distance in time, we must carry all m derivatives forward onto each successive hypersurface, thus making it equivalent to its predecessor. This can be done only if the (m+1)th time derivative of ^3g is assumed to be an explicit functional of ^3g and the lower derivatives, on each of the hypersurfaces, with the functional form being the same on all hypersurfaces. Once the (m+1)th derivative has been determined on $e(S)$, with the use of these <u>dynamical equations</u>, the m-th derivative can be constructed on $D_{\delta t}(S)$, and the process can be repeated ad infinitum.

At first glance, it might seem as though any functional of the $(m+1)$ hypersurface fields that comprise the initial data, and their hypersurface derivatives to all orders, should yield a consistent set of dynamical field equations. However, things aren't quite that simple. Let us suppose, for the moment, that $m = 2$. The initial data on $e(S)$ is then ${}^3g$, K, and ΠS, and the dynamical field equations give $\Pi(\nabla_{\vec{n}}S)$ as a functional of the initial data:

$$\Pi(\nabla_{\vec{n}}S) = \Pi(\nabla_{\vec{n}}S)[{}^3g, K, \Pi S, {}^3R, {}^3\nabla K, {}^3\nabla(\Pi S), {}^3\nabla{}^3R, \ldots] \quad . \tag{3.2}$$

If this functional is known, then, with the use of (2.41) and (2.26), we can determine $\nabla_{\vec{n}}(\Pi S)$ as a functional of the initial data and the deformation vector field, $\vec{D}$; and by repeatedly applying the techniques demonstrated in (2.31) we can compute the functional form of the covariant derivative along $\vec{n}$ of each of the fields upon which $\Pi(\nabla_{\vec{n}}S)$ depends. The dynamical field equations, (3.2), thus determine their own derivative:

$$\nabla_{\vec{n}}(\Pi(\nabla_{\vec{n}}S)) = \nabla_{\vec{n}}(\Pi(\nabla_{\vec{n}}S))[{}^3g, K, \Pi S, \ldots; \vec{D}] \quad . \tag{3.3}$$

But we also have the general result:

$$\nabla_{\vec{n}}(\Pi(\nabla_{\vec{n}}S))(\vec{u},\vec{v}) = \vec{n}(\Pi(\nabla_{\vec{n}}S)(\vec{u},\vec{v})) - \Pi(\nabla_{\vec{n}}S)(\nabla_{\vec{n}}\vec{u},\vec{v})$$

$$- \Pi(\nabla_{\vec{n}}S)(\vec{u},\nabla_{\vec{n}}\vec{v})$$

$$= \vec{n}(\nabla_{\vec{n}}S(\vec{u},\vec{v})) - \nabla_{\vec{n}}S(\nabla_{\vec{n}}\vec{u},\vec{v}) - \nabla_{\vec{n}}S(\vec{u},\nabla_{\vec{n}}\vec{v}) + \frac{1}{D^{\perp}}\vec{u}D^{\perp}\nabla_{\vec{n}}S(\vec{n},\vec{v})$$

$$+ \frac{1}{D^{\perp}}\vec{v}D^{\perp}\nabla_{\vec{n}}S(\vec{u},\vec{n})$$

$$= \{\nabla_{\vec{n}}\nabla_{\vec{n}}S(\vec{u},\vec{v}) - \nabla_{\nabla_{\vec{n}}\vec{n}}S(\vec{u},\vec{v})\} - \varepsilon\frac{1}{D^{\perp}}\sum_i \vec{h}_i D^{\perp}\nabla_{\vec{h}_i}S(\vec{u},\vec{v})$$

$$+ \frac{1}{D^{\perp}}\vec{u}D^{\perp}\nabla_{\vec{n}}S(\vec{n},\vec{v}) + \frac{1}{D^{\perp}}\vec{v}D^{\perp}\nabla_{\vec{n}}S(\vec{u},\vec{n}) \quad . \tag{3.4}$$

Taken together, the two terms in parentheses on the right hand side of (3.4) constitute the fourth (normalized) time derivative of ^3g , a quantity which must not depend in any way on the deformation, $\vec{D}$. However, when (3.4) is subtracted from (3.3) the resulting equation may be solved to give an expression for $\{\nabla_{\vec{n}}\nabla_{\vec{n}}S - \nabla_{\nabla_{\vec{n}}\vec{n}}S\}$ which does depend explicitly on $\vec{D}$. This apparent contradiction is resolved by constraining the initial data to satisfy functional relations which make the deformation dependence of $\{\nabla_{\vec{n}}\nabla_{\vec{n}}S - \nabla_{\nabla_{\vec{n}}\vec{n}}S\}$ vanish identically. If the functional form of $\Pi(\nabla_{\vec{n}}S)$ (Equation (3.2)) has been chosen appropriately, then the associated constraint equations will be sufficiently weak that ΠS may still be considered as part of the initial data (i.e. ΠS may not be obtained as a functional of ^3g , K , and their hypersurface derivatives).

Once the primary constraint equations (introduced in the previous paragraph) have been found, they may be used to aid in the construction of the next covariant derivative of S along $\vec{n}$: $\{\nabla_{\vec{n}}\nabla_{\vec{n}}\nabla_{\vec{n}}S - 3\nabla_{\vec{n}}\nabla_{\nabla_{\vec{n}}\vec{n}}S + 2\nabla_{\nabla_{\vec{n}}\nabla_{\vec{n}}\vec{n}}S\}$. As with $\{\nabla_{\vec{n}}\nabla_{\vec{n}}S - \nabla_{\nabla_{\vec{n}}\vec{n}}S\}$, this will also depend explicitly on $\vec{D}$, and new secondary constraints on the initial data must be chosen to make the deformation dependent terms vanish. The calculations are exactly analogous to those for the primary constraints, but longer due to the extra derivative.

Finally, the fourth derivative of S along $\vec{n}$ may be computed with the use of the dynamical equations and the primary and secondary constraints, and once again new constraints must be imposed on the initial data to eliminate the explicit deformation dependence. Still higher derivatives will automatically be independent of $\vec{D}$.

Looking back at (3.4), we see that the primary constraints place restrictions on $\nabla_{\vec{w}} S(\vec{u},\vec{v})$ and $\nabla_{\vec{n}} S(\vec{u},\vec{n})$. Equations analogous to (3.4) for the higher derivatives would show that the secondary constraints restrict $\nabla_{\vec{w}} S(\vec{u},\vec{n})$ and $\nabla_{\vec{n}} S(\vec{n},\vec{n})$, and that the tertiary constraints (which are often also called secondary) restrict $\nabla_{\vec{w}} S(\vec{n},\vec{n})$. Thus the complete set of dynamic plus constraint equations determines the form of all the components of ∇S , as functionals of the initial data. Moreover, since ∇S is a space-time tensor field that is completely independent of the choice of hypersurface, $e(\mathcal{S})$, or deformation, $\vec{D}$, the functionals that make up its components must fit together to form a space-time tensor field that is also independent of e or $\vec{D}$, but which is nonetheless constructed from the initial data on $e(\mathcal{S})$.

When we look at the initial data, though, we see that it is itself derived from g and its space-time derivatives, so any space-time tensor field that is constructed from the initial data (and is independent of e and $\vec{D}$) must ultimately be a functional of g , R , ∇R , etc.. Of these latter fields, only g and R can be constructed directly from the initial data on $e(S)$, in the particular case that we are considering ($m = 2$). The complete set of geometrical field equations (dynamical equations and constraints) must therefore take the form:

$$\nabla S = \nabla S[g,R] \quad . \qquad (3.5)$$

Aside from the requirement that it yield a tensor field ∇S of the correct form (i.e. third rank, covariant, symmetric in the last two indices, and satisfying the contracted Bianchi identities), no further restrictions are placed on this functional by the physical assumptions made so far.

The calculations for other values of m ($m \geqslant 1$) are very much the same as for $m = 2$. If the initial data is assumed to include ^3g and its invariant derivatives along $\vec{n}$ up to and including m-th order, then the dynamical equations give the $(m+1)$th derivative as a functional of the initial data. Higher derivatives of ^3g are then obtained by differentiating the field equations, and the deformation dependence of the invariant terms (space-time tensors) is eliminated by imposing constraints on the initial data. Although the sequence may terminate earlier, there are, in general, $m + 1$ orders of constraint equations. When the dynamical equations and the constraints are all satisfied, derivatives of g to all orders may be computed, and the system of equations is integrable.

The purpose of the constraints is to guarantee that the predicted geometry of any future hypersurface, $D_t(S)$, depends only on the initial data defined on $e(S)$, and not on the sequence of intermediate hypersurfaces used in the time integration. Their net effect, however, is to supplement the dynamical equations, building them up into a set of covariant equations in the space-time fields, g , R , ∇R , etc., in which the highest derivative of g is of $(m+1)$th order and enters linearly (cf. (3.5)).

4. The Einstein Vacuum Equations

Throughout modern physics it is assumed that dynamical systems are characterized completely by their instantaneous "coordinates" and "velocities", with their "accelerations" being determined by dynamical equations. For the geometrical field theory being discussed in this chapter, the coordinates are the components of ^3g on $e(S)$, and the velocities are the components of K ; so in this section I shall investigate the class of theories for which $m = 1$.

As outlined above, the dynamical equations must take the form

$$\Pi S = \Pi S[{}^3g, K, {}^3R, {}^3\nabla K, \ldots] \quad . \tag{4.1}$$

Once this functional has been chosen, it may be used with (2.25) to find $\nabla_{\vec{n}}K$ in terms of the initial data and the deformation vector field $\vec{D}$, and with (2.36),(2.37),(2.38), and other similar equations to compute $\nabla_{\vec{n}}$ of each of the other fields upon which ΠS depends. Knowing all these derivatives, we can use the chain rule to compute $\nabla_{\vec{n}}(\Pi S)$.

On the other hand, though, equation (2.41) gives

$$\nabla_{\vec{n}}(\Pi S)(\vec{u},\vec{v}) = \nabla_{\vec{n}}S(\vec{u},\vec{v}) + \frac{1}{D^{\perp}}\vec{u}D^{\perp}S(\vec{n},\vec{v}) + \frac{1}{D^{\perp}}\vec{v}D^{\perp}S(\vec{u},\vec{n}) \quad . \quad (4.2)$$

Subtracting from this the expression obtained from (4.1) for $\nabla_{\vec{n}}(\Pi S)(\vec{u},\vec{v})$ yields an equation that may be solved for $\nabla_{\vec{n}}S(\vec{u},\vec{v})$:

$$\nabla_{\vec{n}}S(\vec{u},\vec{v}) = \nabla_{\vec{n}}S[{}^3g,K,{}^3R,{}^3\nabla K,\ldots;\vec{D}](\vec{u},\vec{v}) \quad . \quad (4.3)$$

Because $\nabla_{\vec{n}}S$ may be constructed directly from g , it is clear that the right hand side of (4.3) must actually be independent of $\vec{D}$; but an examination of the terms in (2.25),(2.36),(2.37), and (2.38) that depend explicitly on $\vec{D}$ shows that no matter how the functional (4.1) is chosen, its derivative, $\nabla_{\vec{n}}(\Pi S)$, will not have (explicitly) deformation dependent terms of the form $\frac{1}{D^{\perp}}\vec{u}D^{\perp}T(\vec{v})$ and $\frac{1}{D^{\perp}}\vec{v}D^{\perp}T(\vec{u})$ (with T independent of $\vec{D}$) capable of cancelling the last two terms in (4.2). The only way in which the deformation dependence in (4.3) can be eliminated is thus to constrain the initial data to satisfy

$$S(\vec{n},\vec{v}) \equiv \varepsilon\sum_i\{{}^3\nabla_{\vec{v}}K(\vec{h}_i,\vec{h}_i) - {}^3\nabla_{\vec{h}_i}K(\vec{v},\vec{h}_i)\} = 0 \quad , \quad (4.4)$$

and to choose the functional (4.1) so that its derivative, $\nabla_{\vec{n}}(\Pi S)$, is completely independent of $\vec{D}$.

The constraints (4.4) serve to limit the configurations of the initial data fields on $e(S)$. However, there is nothing special about this particular space-like hypersurface, so equations (4.4) should also be satisfied on each subsequent hypersurface, $D_t(S)$. To this end I require that

$$\vec{n}(S(\vec{n},\vec{v})) = 0 \quad , \tag{4.5}$$

which leads, through a straightforward calculation, to

$$\nabla_{\vec{n}}S(\vec{n},\vec{v}) + \frac{1}{D^{\perp}}\sum_i \vec{h}_i D^{\perp}\{g(\vec{v},\vec{h}_i)S(\vec{n},\vec{n}) - \varepsilon S(\vec{v},\vec{h}_i)\} = 0 \quad . \tag{4.6}$$

Since this must be satisfied for all choices of $\vec{D}$, I find that

$$\nabla_{\vec{n}}S(\vec{n},\vec{v}) = 0 \tag{4.7}$$

and

$$\Pi S(\vec{u},\vec{v}) = \varepsilon\, {}^3g(\vec{u},\vec{v})S(\vec{n},\vec{n}) \quad . \tag{4.8}$$

With the use of (2.25) and (2.27), equation (4.8) can be solved to give

$$\Pi S(\vec{u},\vec{v}) = \tfrac{1}{2}\, {}^3g(\vec{u},\vec{v})\{\sum_i {}^3S(\vec{h}_i,\vec{h}_i) - \varepsilon \sum_{i,j} [K(\vec{h}_i,\vec{h}_i)K(\vec{h}_j,\vec{h}_j) - K(\vec{h}_i,\vec{h}_j)K(\vec{h}_i,\vec{h}_j)]\} \quad . \tag{4.9}$$

Using (2.25) and (2.38), it can then be shown that

$$\nabla_{\vec{n}}(\Pi S)(\vec{u},\vec{v}) = 0 \quad (= \nabla_{\vec{n}}S(\vec{u},\vec{v})) \quad . \tag{4.10}$$

Derivatives of S in directions parallel to the hypersurface are given by

$$
\begin{aligned}
&{}^3\nabla_{\vec{w}}(\Pi S)(\vec{u},\vec{v}) \qquad (= \nabla_{\vec{w}} S(\vec{u},\vec{v})) \\
&= \tfrac{1}{2}\ {}^3g(\vec{u},\vec{v})\{\vec{w}\,{}^3S - 2\varepsilon \sum_{i,j} [{}^3\nabla_{\vec{w}} K(\vec{h}_i,\vec{h}_i)K(\vec{h}_j,\vec{h}_j) \\
&- {}^3\nabla_{\vec{w}} K(\vec{h}_i,\vec{h}_j)K(\vec{h}_i,\vec{h}_j)]\} \quad ;
\end{aligned} \tag{4.11}
$$

however, the contracted Bianchi identities for S tell us that these must vanish:

$$
\begin{aligned}
0 &= \sum_i \nabla_{\vec{h}_i} S(\vec{h}_i,\vec{v}) + \varepsilon \nabla_{\vec{n}} S(\vec{n},\vec{v}) - \tfrac{1}{2}\vec{v}S \\
&= \sum_i \nabla_{\vec{h}_i} S(\vec{h}_i,\vec{v}) - \tfrac{1}{2}\vec{v}\{(4/3)\sum_i S(\vec{h}_i,\vec{h}_i)\} \\
&= \sum_i \{{}^3\nabla_{\vec{h}_i}(\Pi S)(\vec{h}_i,\vec{v}) - (2/3)\,{}^3\nabla_{\vec{v}}(\Pi S)(\vec{h}_i,\vec{h}_i)\} \\
&= \sum_i \{-\tfrac{1}{2}\vec{v}\,{}^3S + \varepsilon\sum_j [{}^3\nabla_{\vec{v}} K(\vec{h}_i,\vec{h}_i)K(\vec{h}_j,\vec{h}_j) - {}^3\nabla_{\vec{v}} K(\vec{h}_i,\vec{h}_j)K(\vec{h}_i,\vec{h}_j)]\} \\
&= -\sum_i {}^3\nabla_{\vec{v}}(\Pi S)(\vec{h}_i,\vec{h}_i) \quad .
\end{aligned} \tag{4.12}
$$

Now it is well known that any symmetric, second rank tensor with vanishing covariant derivative must be proportional to the metric tensor, so we finally obtain the dynamical equations

$$
(\Pi S)(\vec{u},\vec{v}) = -\lambda\,{}^3g(\vec{u},\vec{v}) \quad , \tag{4.13}
$$

with λ a constant on $e(S)$. Equation (4.10) indicates that λ must also be a constant in time; and (4.8) now reduces to the secondary constraint:

$$S(\vec{n},\vec{n}) = -\varepsilon\lambda \quad . \tag{4.14}$$

When collected together, equations (4.4),(4.13), and (4.14) are quickly recognized as Einstein's vacuum equations for the gravitational field (with the cosmological term):

$$S(\vec{U},\vec{V}) = -\lambda g(\vec{U},\vec{V}) \quad . \tag{4.15}$$

For the restricted set of initial data fields, ^3g and K , they are the only field equations capable of unambiguously propagating the metric of space forward in time.

A much shorter, but less instructive, derivation of these same equations follows from the conclusions of Section 3 of this chapter. They indicate that the Ricci tensor S must be an explicit functional of tensors formed from g and its first derivatives (since $m = 1$). But it is impossible to form any tensor field from the first derivatives of g , so we must have

$$S = S[g] \quad , \tag{4.16}$$

which leads immediately to (4.15).

It is also interesting to note that even if we were willing to add ΠS to the initial data (cf. Section 3) we would be frustrated in all attempts to do so. Because g and R are both of even rank, any non-trivial functional of these fields must also be an even rank tensor. But ∇S is a third rank tensor, so the only solution to (3.5) is

$$\nabla S = 0 \quad , \tag{4.17}$$

which leads us once again to the field equations (4.15).

5. Metric Signature, and Causality

Throughout the foregoing discussions the sign ε of $g(\vec{n},\vec{n})$ has been left undetermined, but definite. Whether ε is +1 or −1 makes little difference to the form of the field equations. However, if g is to give rise to the (partial) time ordering of physical events that is demanded by (1.1.2), then we must make the standard assignment:

$$\varepsilon = -1 \quad . \tag{5.1}$$

More pragmatic reasons for making this choice are provided by the empirical successes of special relativity and Maxwell's theory of electromagnetism.

Our original motivation for introducing the field g was to induce a positive definite metric on each space-like hypersurface of space-time (the term "space-like" being defined in (1.1.3)), and if we had decided that $\varepsilon = +1$, then every hypersurface of space-time would have had such a metric. Starting with the initial space-like hypersurface, $e(S)$, <u>any</u> deformation vector field, $\vec{D}$, would have then led to a sequence of hypersurfaces $D_t(S)$ with induced Riemannian metrics. But with the locally Minkowskian metric of physical space-time, the field

$\vec{D}$ is severely restricted by the requirement that both $\vec{D}^{\parallel}$ and $\vec{D}^{\perp}$ be smooth. This is illustrated by the following example.

Let M be a space-time endowed with a locally Minkowskian metric g that satisfies some known set of (predictive) field equations; let M' be an open cell in M ; and let σ' be a space-like hypersurface of M' which extends to a space-like hypersurface σ of M . Then, just as in Section 1, it is always possible to generate a parameterized family of space-like hypersurfaces σ_t by deforming σ along a smooth vector field $\vec{D}$ on M . For the purposes of this example I shall choose $\vec{D}$ such that on σ' it satisfies $D^{\perp} > 0$, and on $\sigma\setminus\bar{\sigma}'$ its perpendicular part vanishes. I shall denote the portion of σ_t that does not coincide with σ by σ'_t ($\sigma'_t = (\sigma_t \cup \sigma)\setminus\sigma$). This is illustrated in Figure 5.1. It follows immediately from the above assumptions that for each point x of σ'_t , and for all t , every past (future) directed time-like path through x intersects σ' and each of the intermediate surfaces σ'_s , $0 < s < t$ (cf. (1.1.9)).

In conjunction with the field equations, the initial data induced on σ can be used to predict what the geometry of each of the subsequent hypersurfaces σ_t is. The fields predicted to exist on σ'_t , however, depend only on the initial data defined on σ' and are independent of the field configurations on $\sigma\setminus\sigma'$. Without this result, one could not make confident predictions about the future (or past) without first gathering

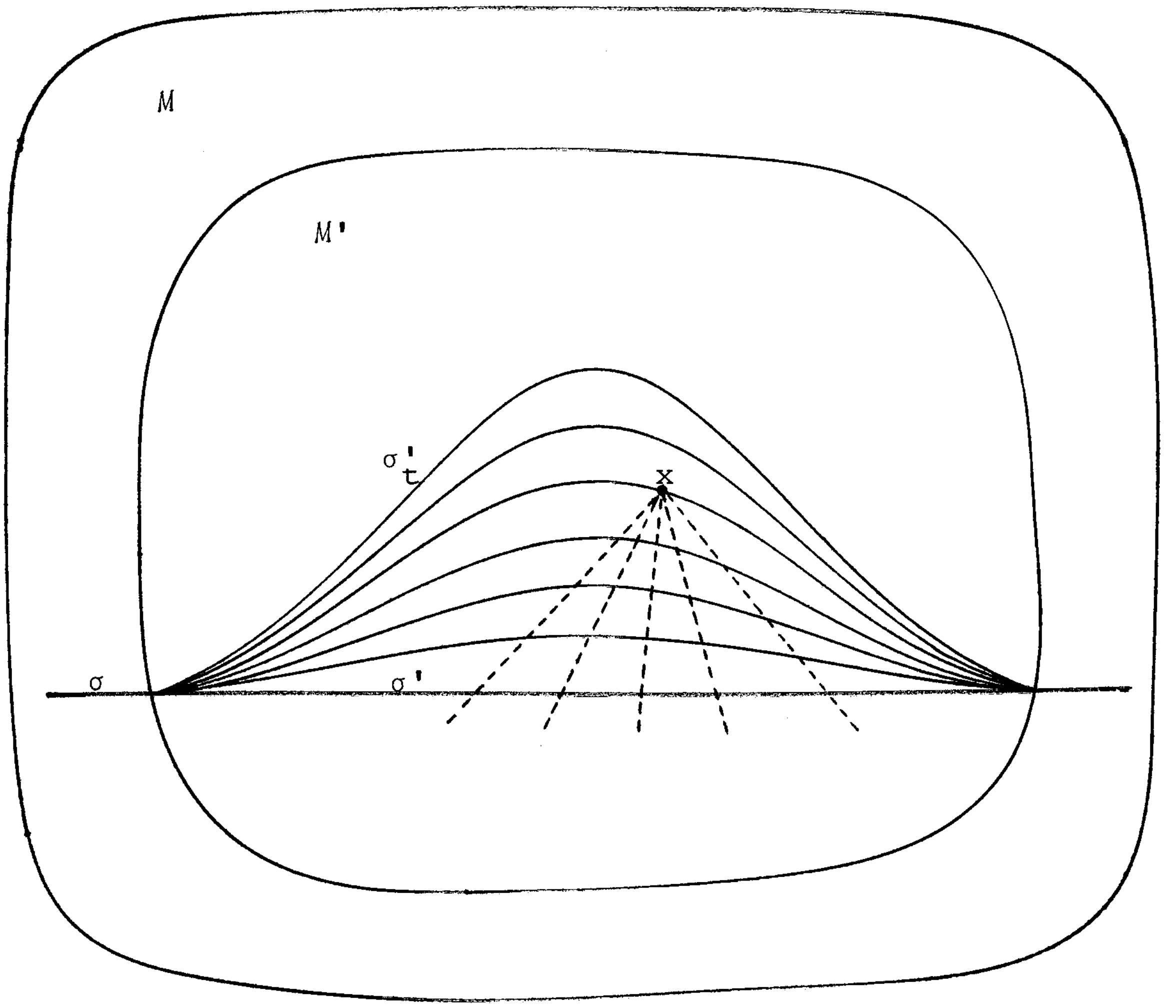

Figure 5.1 Every past directed time-like path through x (broken lines) intersects σ' and each of the intermediate space-like hypersurfaces σ'_s .

information about the entire present universe, rather than just some local neighbourhood.

Conversely, if the fields on σ'_t are to be independent of the initial data on $\sigma\backslash\sigma'$, then the deformation vector field $\vec{D}$ must (1) satisfy $D^{\perp} = 0$ on $\sigma\backslash\overline{\sigma}'$, and (2) leave each of the hypersurfaces σ_t space-like. This latter condition is assured by requiring that both $\vec{D}^{\parallel}$ and $D^{\perp}$ be smooth fields.