Quantum of area and its sepectroscopy

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1- Quantum of area

2- Quantum black hole

3- New properties

   a) generic degeneracy

   b) Symmetry

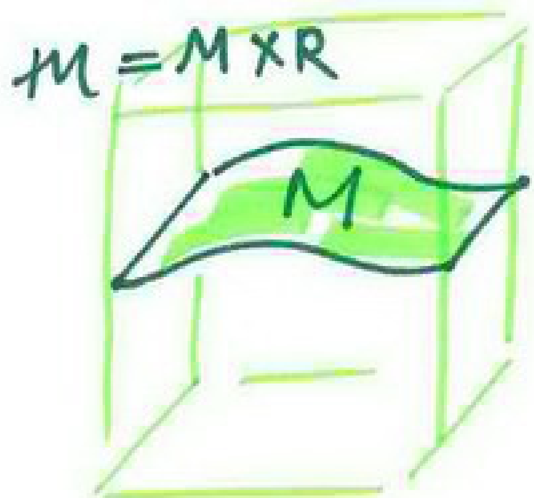
4 - Spectroscopy

May 2007, Perimeter Institute

Introduction

On a 3-spatial manifold M

- classical configuration space : $\mathcal{A}/\mathcal{G}$



connections modulo gauge transformations

Conjugate E has geometrical interpretations (triads)

- Quantum configuration space : $\overline{\mathcal{A}/\mathcal{G}}$

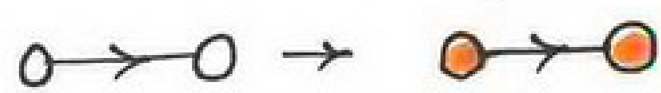
✓ SU(2)-valued holonomies
$h(p, A) = \mathcal{P} e^{-\int_p A}$ ← (generalized connections modulo generalized gauge transformations

- $h^{-1}(\rightarrow) = h(\leftarrow)$
- $h(\rightarrow\!\uparrow) = h(\rightarrow) \cdot h(\uparrow)$

local gauge transformation

on connection : $A \rightarrow \lambda A \lambda^{-1} + \lambda d \lambda^{-1}$

✓ on holonomies : $h \rightarrow \lambda(v_i)^{-1} \cdot h \cdot \lambda(v_f)$



Definitions:

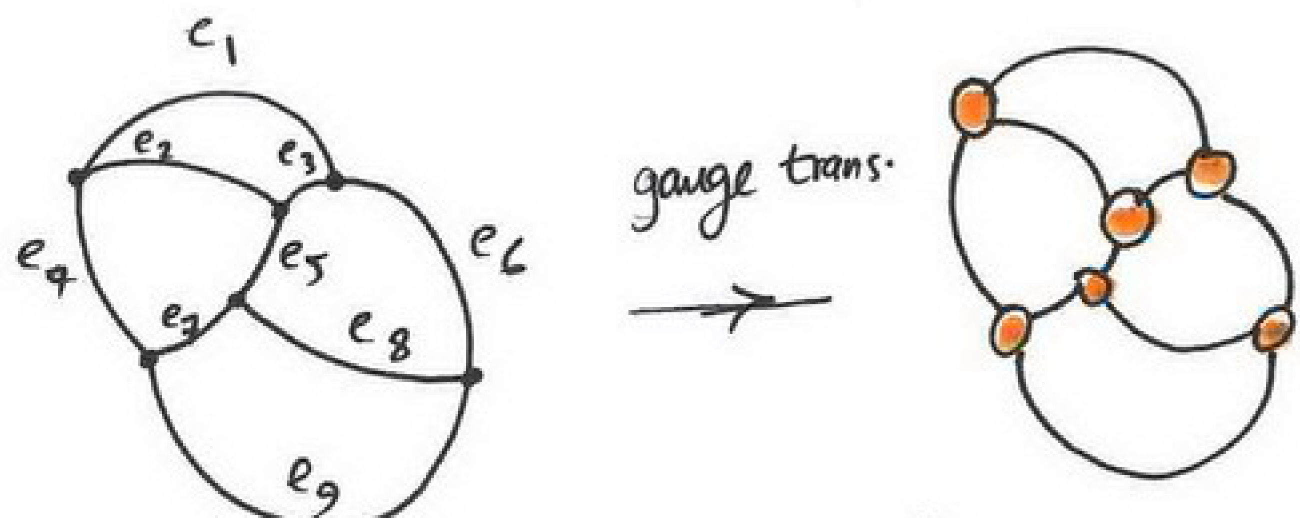
edge — oriented 1d submanifold of M
with 2 boundaries
analytical everywhere
parametrized by s running from 0 to 1
$s \in [0, 1]$

vertex — the two boundaries

graph — a collection of edges meeting at vertices.

example:

γ:



$$\Psi_\gamma = \psi\left(h(e_1), h(e_2), \dots, h(e_9)\right)$$

is $[SU(2)]^9$-valued generalized connection $\in \bar{\mathcal{A}}$

$\bar{G}$: isomorphic with $[SU(2)]^6$

$$\psi(h(e_1), \dots, h(e_9)) \longrightarrow \psi(\lambda^{-1}(v_i) \cdot h(e_1) \cdot \lambda(v_f), \dots)$$

physics comes after imposing some restrictions

- gauge – invariance
- diffeo – invariance
- Hamiltonian – Constraint – invariance

↓

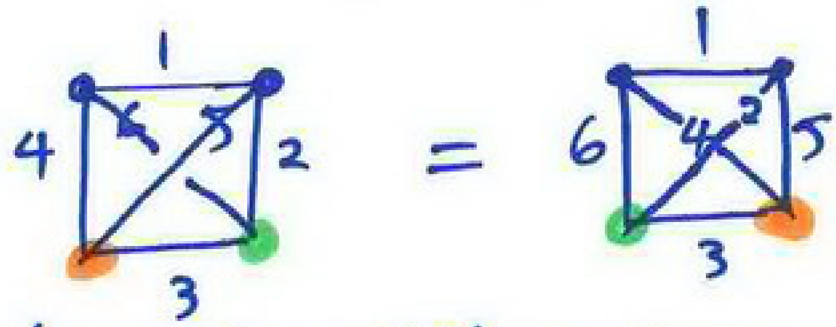
...?

suggests multiloops basis

(Lattice spin network)

Spin networks S and S′ are orthogonal unless their basis graphs belong to the same knot class

Diff ·



- This makes the Hilbert space separatable.
- The states are distinguished by coloring of links and nodes.

1- Quantum of area

- $\hat{A}^i_a \Psi = A^i_a \Psi$
- $\hat{E}^a_i \Psi = \frac{8\pi G\gamma}{i} \frac{\delta}{\delta A^i_a} \Psi$
- $\hat{A}_S \Psi = (\text{area of the state}) \Psi$

$$\hat{A}_S := \int d^2x \sqrt{n_a \hat{E}^a_i(x)\, n_b \hat{E}^b_i(x)}$$

we need it to be (a function of x dependency)

- (an operator acting on internal space)

Smolin, Rovelli (95)
Ashtekar, Lewandowski (96)
Rovelli, et. al. (96)

Strategy:

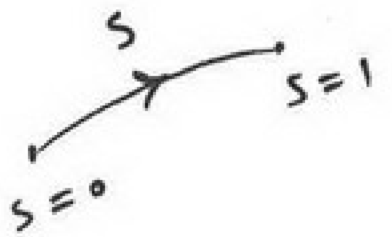
$S: \; x^3 = \text{const} = 0, \quad \vec{n} = \hat{x}^3$

- point-splitting technique:

<u>Def:</u> $\forall\, x, y \in S \quad \lim_{\epsilon \to 0} f_\epsilon(x,y) = \delta(x,y)$

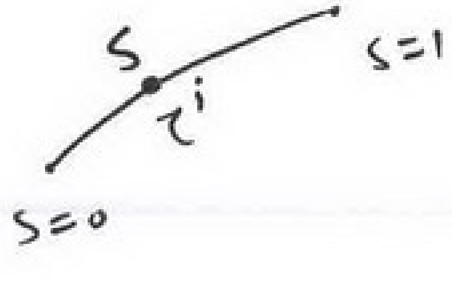
- $E^3_i(x)\Big|_\epsilon := \lim_{\epsilon \to 0} \int d^2y \; f_\epsilon(x,y) \; E^3_i(y)$

- $\hat{E}^3_i(x)\Big|_\epsilon \cdot h(e,A) = \frac{\hbar 4\gamma}{i} \int d^2y \; f_\epsilon(x,y) \frac{\delta h}{\delta A^i_3(y)}$



$$h(e,A) = \mathcal{P}\, e^{-\int_0^1 \tau^i A^i_a(e(s))\, \dot{e}^a(s)\, ds}$$

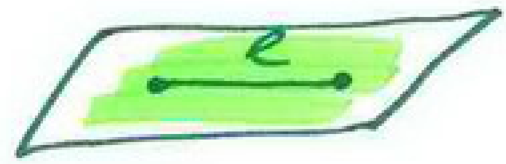
$$\frac{\delta h(e,A)}{\delta A^i_3(y)} = \int_0^1 ds \; \dot{e}^3(s) \;\; h(1,s;A)\, \tau^i\, h(s,0;A) \cdot \delta(y^1, e^1(s)) \cdot \delta(y^2, e^2(s)) \cdot \delta(0, e^3(s))$$



$\hat{E}$ "grasps" holonomy at s by τ^i

Classification:

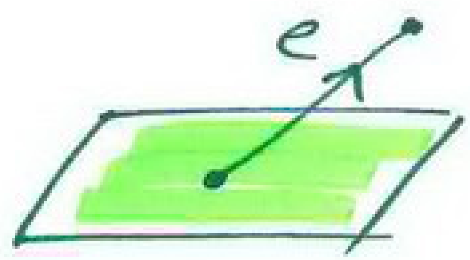
1- If e is contained in S



$$\dot{e}^3 \equiv 0$$

$$\hat{E}^3_i(x)\Big|_\epsilon \cdot h(e) \equiv 0$$

2- If e starts from S

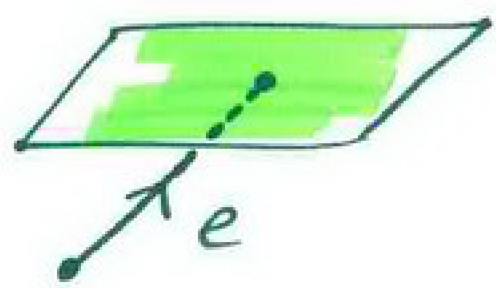


$$\dot{e}^3 > 0$$

$$\hat{E}^3_i(x)\Big|_\epsilon \cdot h(e) = \cancel{\frac{\hbar\kappa}{i}\int d^2y \; f_\epsilon(x,y)}$$

$$= \frac{\gamma \ell_P^2}{2i} f_\epsilon(x, e(0)) \; h(1,0)\tau^i$$

3- If e ends at S



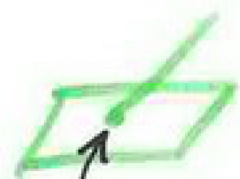
$$\dot{e}^3 < 0$$

$$\hat{E}^3_i(x)\Big|_\epsilon \cdot h(e) =$$

$$\frac{\gamma \ell_P^2}{2i} f_\epsilon(x, e(1)) \; \left(-\tau^i h(1,0)\right)$$

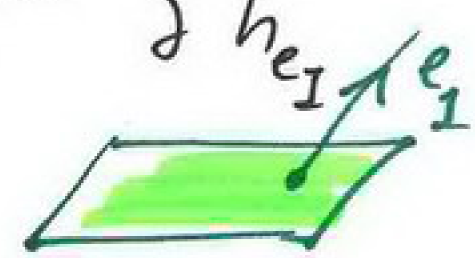
$$\cdot\ \hat{E}^3_i(x)\Big|_\epsilon \cdot \Psi\big(h(e_1), h(e_2), \dots, h(e_N)\big)$$

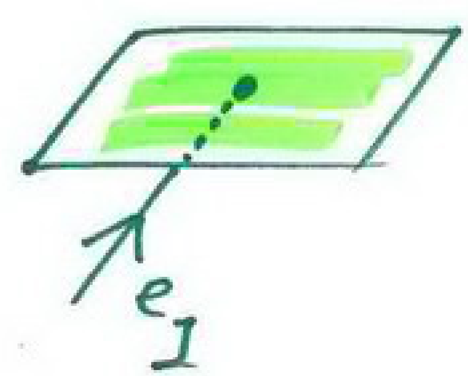
$$= \frac{\gamma \ell_p^2}{2i} \int d^2y \; f_\epsilon(x,y) \sum_{I = \text{all edges}} \frac{\delta h(e_I)}{\delta A^i_3(y)} \frac{\partial \Psi}{\partial h(e_I)}$$



$$= \frac{\gamma \ell_p^2}{2i} \sum_{I = \text{all edges}} f_\epsilon(x,\ e_I \cap S)\ X^i_{e_I} \cdot \Psi$$

$$= \quad '' \qquad '' \quad \begin{cases} h(e_I)\,\tau^i \dfrac{\partial \psi}{\partial h_{e_I}} \\[2ex] -\tau^i\, h(e_I) \dfrac{\partial \psi}{\partial h(e_I)} \end{cases}$$





$$\cdot\ \hat{E}^3_i(x)\Big|_\epsilon\ \hat{E}^{3i}(x)\Big|_\epsilon \cdot \Psi$$

$$= \frac{\gamma^2 l_p^4}{4} \sum_{\substack{I,J \\ \text{all edges}}} k_I k_J \, f_\epsilon(x, e_I \cap S)\, f_\epsilon(x, e_J \cap S) \cdot X^i_{(e_I)} X^i_{(e_J)} \Psi$$

→ If ϵ is so small that identifies $e_I \cap S$ and $e_J \cap S$ (the joint vertex between e_I & e_J)

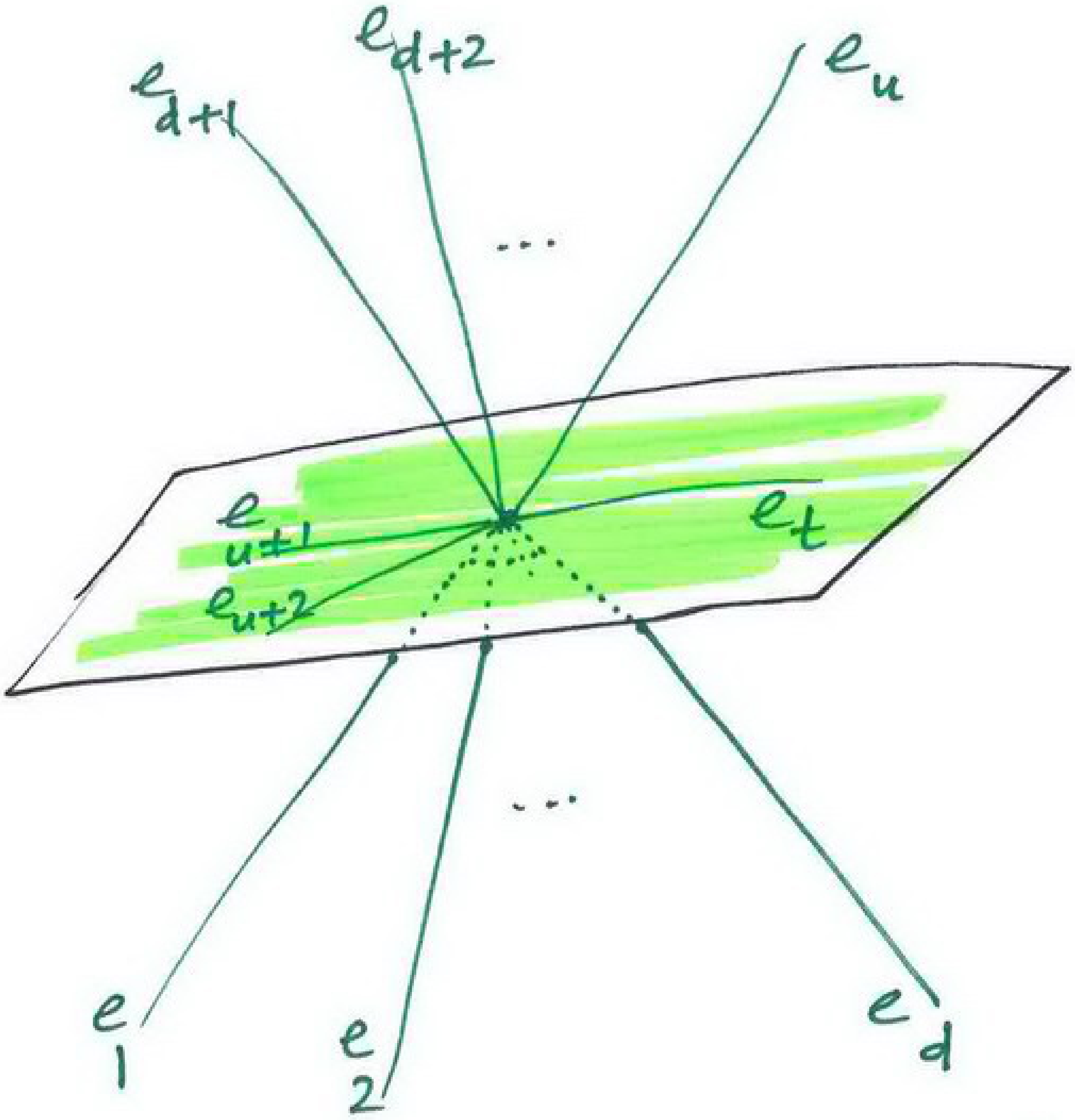
$$= \frac{\gamma^2 l_p^4}{4} \sum_{I,J} k_I k_J \underbrace{\left[f_\epsilon(x, v) \right]^2}_{\substack{\text{does not depent} \\ \text{on the edges} \\ \text{it depends only} \\ \text{on the vertices} \\ \text{resrding on } S}} X^i_{(e_I)} X^i_{(e_J)} \Psi$$

$\epsilon \to 0$ such that $f_\epsilon \to \delta$

$$\hat{E}^3_i(x)\Big|_\epsilon \; \hat{E}^{3i}(x)\Big|_\epsilon \cdot \Psi$$

$$= \frac{\gamma^2 l_p^4}{4} \sum_{\substack{\text{all}\\ \text{vertices}\\ \alpha}} \delta^2(x, v_\alpha) \cdot \underbrace{\sum_{\substack{\text{all}\\ \text{edges}\\ I_\alpha, J_\alpha}} k_{I_\alpha} k_{J_\alpha} X^i_{(I_\alpha)} X^j_{(J_\alpha)} \Psi}_{\substack{\text{vertex operator}\\ \Delta_{v_\alpha}\\ \text{defined at each vertex } \alpha}}$$

- $$A_s \cdot \Psi = \frac{l_p^2 \gamma}{2} \sum_{\substack{\text{all}\\ \text{vertices}\\ \alpha}} \sqrt{\Delta_{v_\alpha}} \cdot \Psi$$



$$J^i_{(d)} := -i\,(X^i_1 + X^i_2 + \cdots + X^i_d)$$

$$J^i_{(u)} := -i\,(X^i_{d+1} + \cdots + X^i_u)$$

$$J^i_{(t)} := -i\,(X^i_{u+1} + \cdots + X^i_t)$$

$$J^i_{(u+d)} := J^i_{(u)} + J^i_{(d)}$$

recall:

$$X^i \cdot h = \begin{cases} h \cdot \tau^i \\ -\tau^i \cdot h \end{cases}$$

$$-\Delta_v = \overbrace{\left(J^i_{(d)} - J^i_{(u)}\right)}^{K_I}\overbrace{\left(J^i_{(d)} - J^i_{(u)}\right)}^{K_J}$$

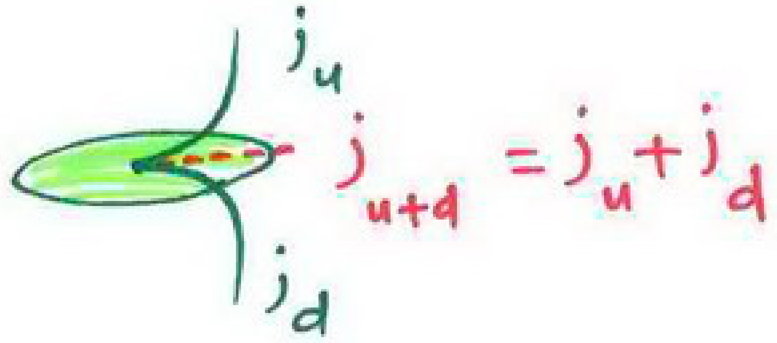
$$= J^i_d J^i_d - J^i_d J^i_u - J^i_u J^i_d + J^i_u J^i_u$$

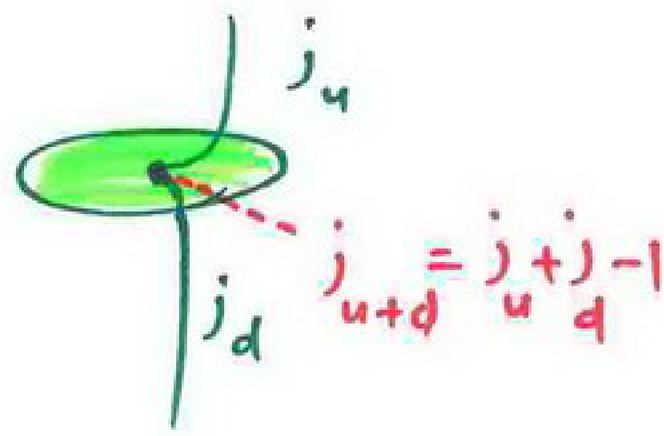
$$= 2 J^i_{(d)} J^i_{(d)} + 2 J^i_{(u)} J^i_{(u)} - J^i_{(u+d)} J^i_{(u+d)}$$

eigenvalues in diagonal rep:

$$\lambda_\Delta = 2 j_d (j_d + 1) + 2 j_u (j_u + 1) - j_{d+u} (j_{d+u} + 1)$$

$$j_{d+u} \in \{ |j_d - j_u|, \ldots, (j_d + j_u) \}$$





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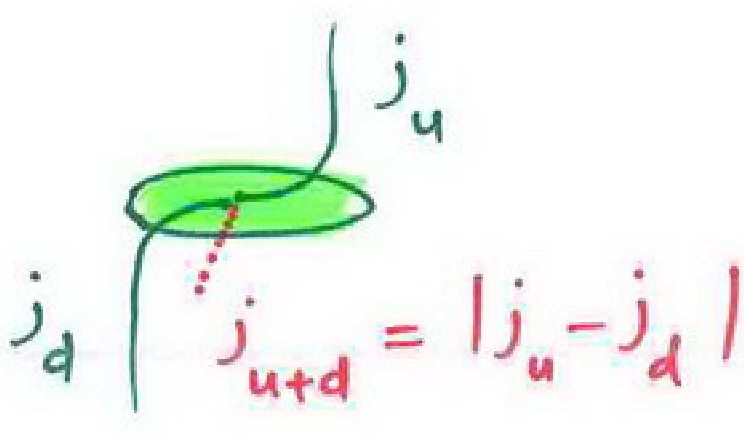
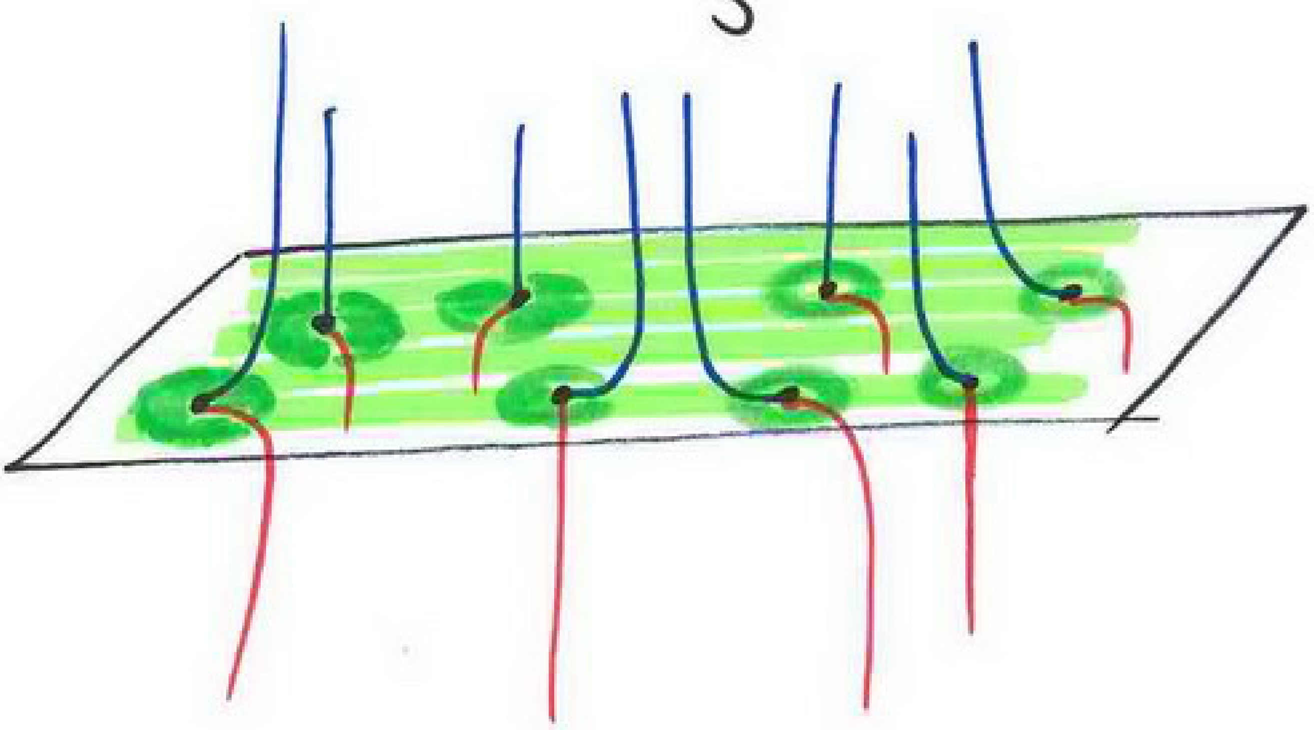


Summery

quantum of area

$$a_S = \frac{l_p^2 \gamma}{2} \sum_{\substack{\text{all} \\ \text{vertices } \alpha \\ \text{residing} \\ \text{on} \\ S}} \sqrt{2 j_d^{(\alpha)} (j_d^{(\alpha)}+1) + 2 j_u^{(\alpha)} (j_u^{(\alpha)}+1) - j_{u+d}^{(\alpha)} (j_{u+d}^{(\alpha)}+1)}$$



an important subset:

$$j_{u+d} = 0 \,, \quad j_u = j_d = j$$

$$a_s = \frac{l_p^2 \gamma}{2} \sqrt{j(j+1)}$$

First was discovered by Smolin and Rovelli

2- Isolated horizons

strategy to define a localized horizon is :

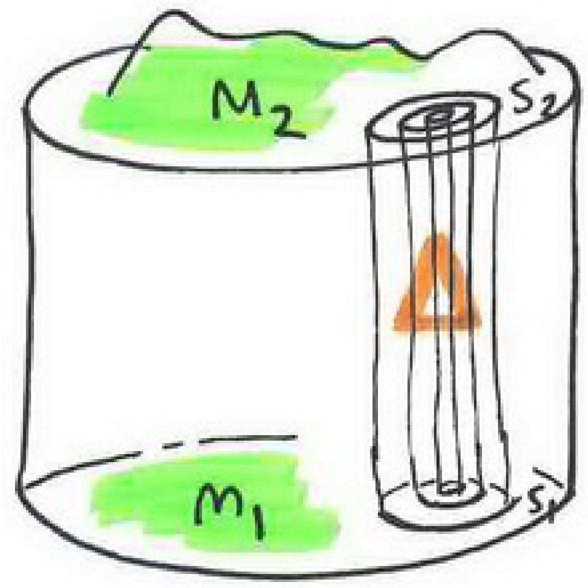
Firstly provide a sector of spacetime with some restrictions such that the sector behave the black hole mechanics .

Secondly pull-back Ashtekar variables to the sector and define boundary conditions.

Thirdly quantize it on spin networks

* Consequence:
  Some independent degrees of freedom appear on the horizon .

$\mathcal{M} = M \times R$



$\Delta : S^2 \times R$

Δ null

each S time-independent

each S non-rotating

- pull-back A to Δ

  A is left with two freedom

  one represent area of S cross-sections

  one a 1-form with elements in $U(1)$-bundle

  boundary condition

$$F\big|_{\Delta} \cong \cdots \underleftarrow{E} + \cdots \underleftarrow{n_a}$$

- further pull-back to S

  A is left with only V freedom

  boundary condition

$$F\big|_{S} \cong \cdots \underleftarrow{E}$$

In action: $$S \overset{?}{=} \int_{\mathcal{M}} \mathrm{Tr}(E \wedge F) - \int_{\mathcal{T}} \mathrm{Tr}(E \wedge A) + \int_{\Delta} \mathrm{Tr}(E \wedge A)$$

$$\delta S_\Delta = \int_\Delta Tr(E \wedge \delta A)$$
$$= \int_\Delta Tr(F \wedge \delta A)$$
$$= \delta \int_\Delta Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$
$$= \delta\,(\text{Chern-Simons action})$$

Quantization

$j, m \qquad m = \{-j, \dots, j\}$

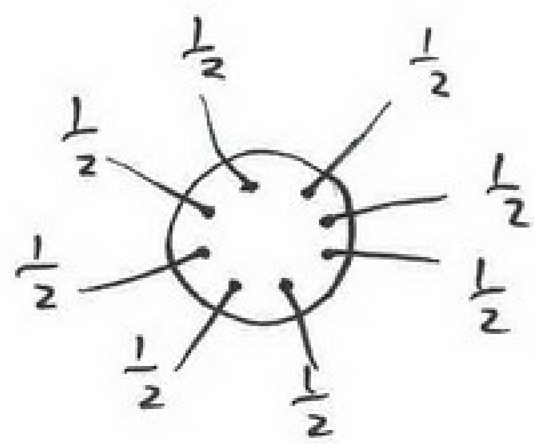
SU(2) Connection

puncture, not vertex

$$\text{area} = 8\pi\gamma \ell_p^2 \sum_{\substack{\text{all} \\ \text{punctures} \\ i}} j_i (j_i + 1) = \text{horizon area}$$

$$\text{flux} = 8\pi\gamma \ell_p^2 \sum_{\substack{\text{all} \\ \text{punctures} \\ i}} m_i = 0$$

$\Psi_{\text{Isolated horizon}} = |m_1, \dots, m_N\rangle$



Number of microstates $\simeq 2^N$

area $= N a_0$

$$\Rightarrow \boxed{S \simeq A}$$

Major Limitations

1 - In classical GR, metric field extends through a black hole (via the junction conditions).

However in quantum isolated horizon picture the spin network states even donot extend through horizon!

2 - The horizon is defined by classical notion of localization. However, the notion of quantum localization is completely different. A quantum horizon must be localized as a 'quantum boundary' of its interior states.

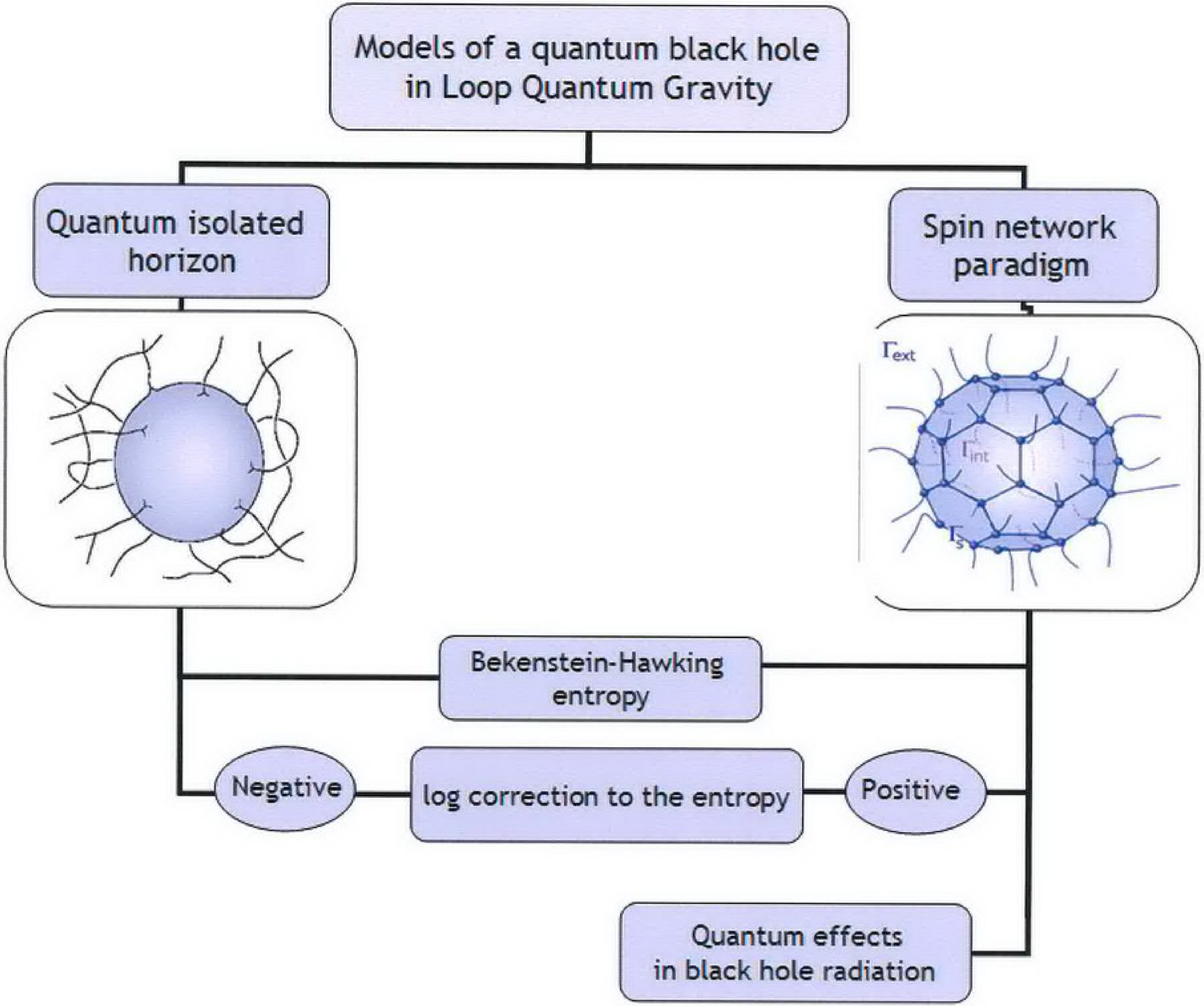
3 — The bulk edges may bend at the horizon and allow SU(2)-valued tangent vectors on the horizon. In fact for a rotating black hole (as well as a non-homogenuous one) this can be the case even by the use of Isolated strategy.

A new picture and paradigm:

- Let all excluded quanta of area are now included.

  ↳ Let tangential edges reside on horizon.

  ↳ Let the connection field of horizon be SU(2)-valued. not U(1)-valued.


Models of a quantum black hole in Loop Quantum Gravity
Quantum isolated horizon
Spin network paradigm
Γ_{ext}
Γ_{int}
Γ_s
Bekenstein-Hawking entropy
Negative
log correction to the entropy
Positive
Quantum effects in black hole radiation

- A classical horizon is closed ($\partial S = 0$)

- " " divides the underlying 3-manifold M into M_{in}, S, M_{out}.

  a) $M_{in} \cup S \cup M_{out} = M$

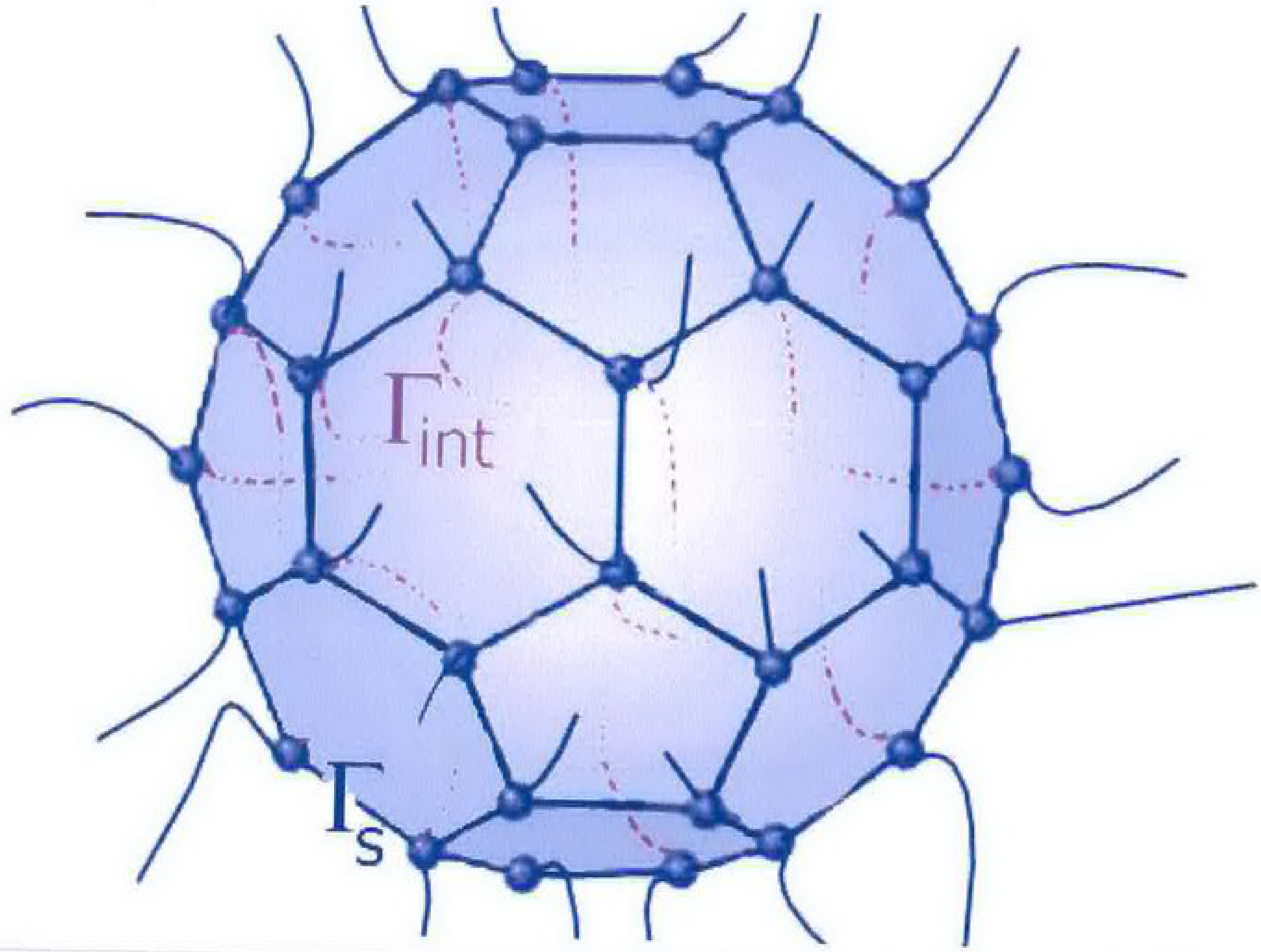
  b) $M_{in} \cap M_{out} = 0$

- Any embedded graph into M is partitioned into Γ_{out}, Γ_S, Γ_{in}.

  i) $\forall\, v_\alpha \in \Gamma_{out}$, $v_\alpha \in M_{out}$

  ii) $\forall\, v_\alpha \in \Gamma_S$, $v_\alpha \in S$

  iii) $\forall\, v_\alpha \in \Gamma_{in}$, $v_\alpha \in M_{in}$



$$\Psi_{\gamma_S} = \psi\,(h(e_1), \cdots, h(e_N))$$

is $[SO(2)]^N$-valued generalized connection.

physics of black hole appears after:

Kinematics
- gauge-invariance
- surface diffeo-invariance

null non-expanding dynamics
⋮
- Hamiltonian-invariance or Hamiltonian eigenvector?

- The area operator should map only the elements of a gauge-invariant subspace of the full Hilbert space unto itself.

- If the underlying surface S is closed and divides the 3-manifold into two disjoint subsets,

  a few additional vertices are needed to close the surface. This put the restrictions on the states corresponding to closed surfaces:

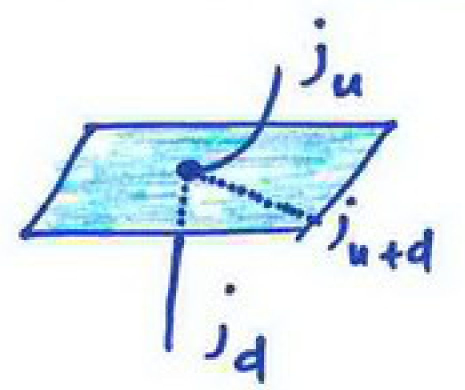
  a) $\displaystyle\sum_{\substack{\text{all}\\ \text{upwards}\\ \text{edges}}} j_{(u)} \in \mathbb{Z}^+$ ✓

  b) $\displaystyle\sum_{\substack{\text{all}\\ \text{downward}\\ \text{edges}}} j_{(d)} \in \mathbb{Z}^+$ ✓

- If a horizon area constitutes the complete spectrum of area, how the wave function associated with it (Ψ) is degenerate?

- The answer is hidden in the quantum of area formula:

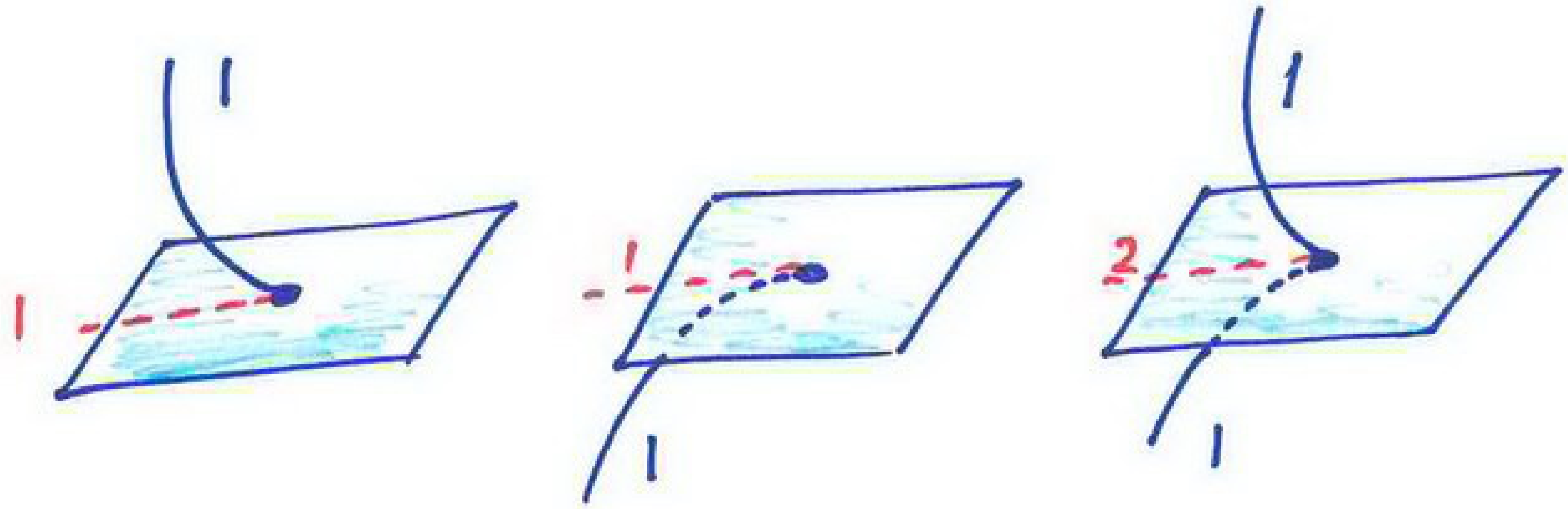
$$\hat{A}\,|j_u, j_d, j_{u+d}\rangle = a\,|j_u, j_d, j_{u+d}\rangle$$



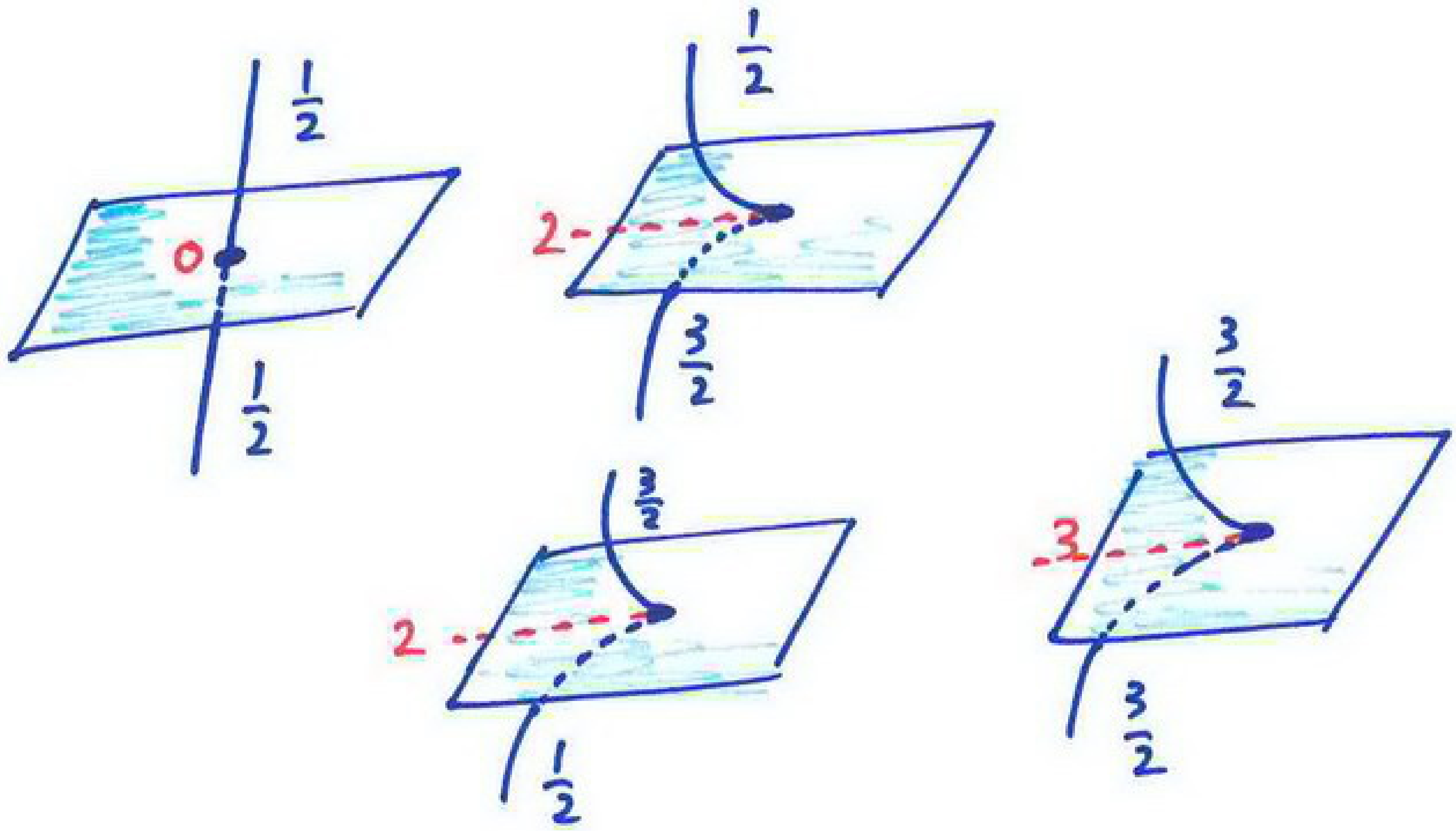
$$a = \underbrace{4\pi l_p^2 \gamma}_{a_0} \sqrt{2j_u(j_u+1) + 2j_d(j_d+1) - j_{u+d}(j_{u+d}+1)}$$

Consider

$$\left.\begin{array}{l} |j_u=0,\ j_d=1,\ j_{u+d}=1\rangle \\ |j_u=1,\ j_d=0,\ j_{u+d}=1\rangle \\ |j_u=1,\ j_d=1,\ j_{u+d}=2\rangle \end{array}\right\} \quad a = \overset{\sqrt{2}}{\cancel{\tfrac{\sqrt{3}}{2}}}\, a_0$$

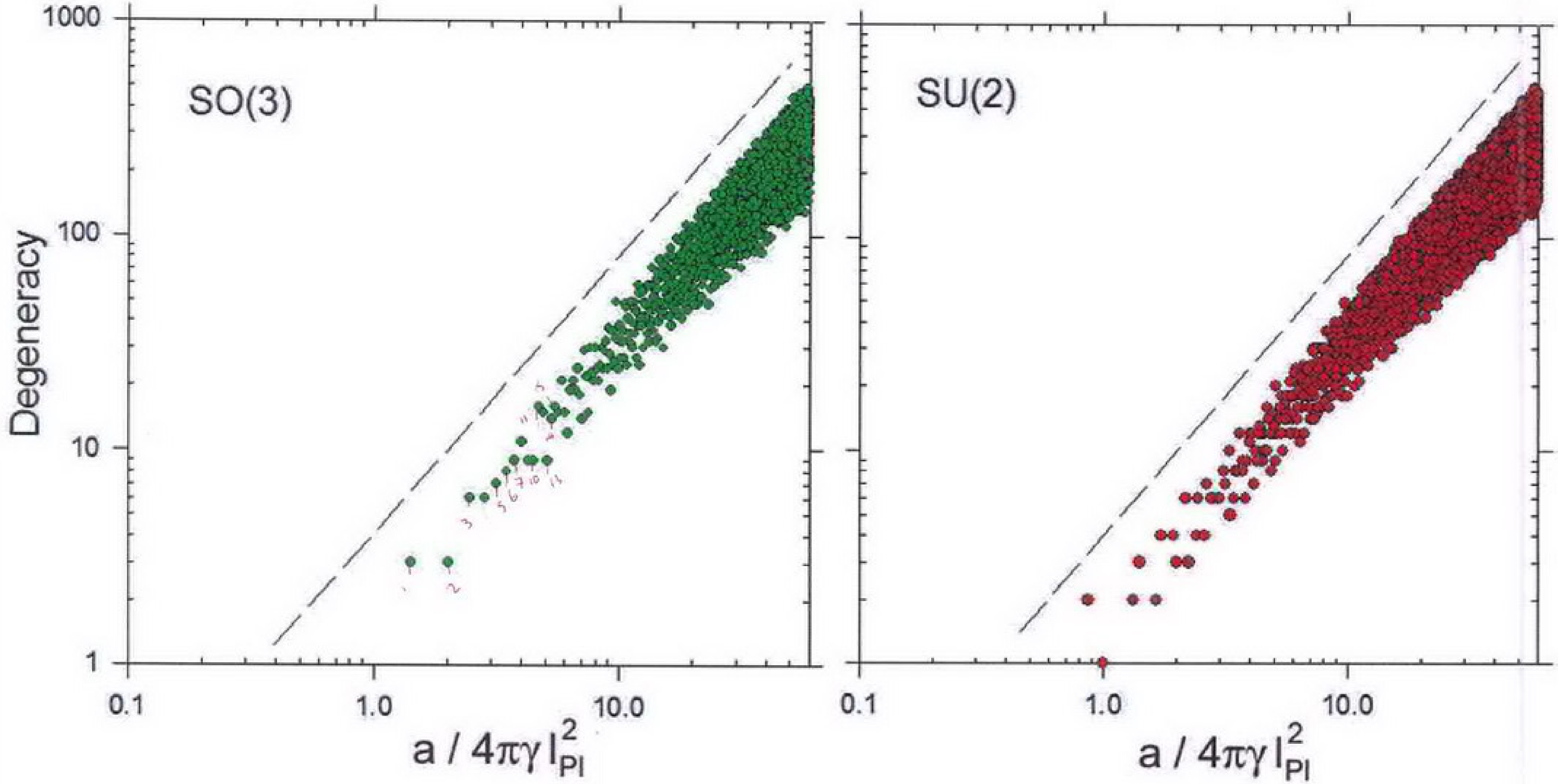


Another example :



all correspond to eigen value

$$a_6 = \sqrt{3}\, a_0$$


Degeneracy
SO(3)
SU(2)
1000
100
10
1
0.1
1.0
10.0
a / 4πγ l_Pl^2

Entropy

- Easy to see in SU(2) and SO(3) rep., the kinematic entropy is

$$S \sim A$$

- In SO(3) rep., we can fine-tune the Immirzi parameter to be

$$\gamma = \frac{\ln 3}{\sqrt{2}\,\pi}$$

this gives us Bekenstein - Hawking entropy

$$S = \frac{A}{4}$$

Spectroscopy

Ladder Symmetry in area spectrum

In SO(3) rep.

$$m = \frac{1}{2}\left(2a(a+1) + 2b(b+1) - c(c+1)\right)$$

for all $a, b, c \in \mathbb{Z}^+$ $\quad c \in \{|a-b|, \dots, a+b\}$

is an integer.

$\{m\}$, modulo repetitions, is identical with the Natural number set.

~~proof~~ proof:

- suppose $a = b+n$ for any $n \in \mathbb{Z}^+$
- consider the subset of $c = a+b$

$$m = \frac{1}{2}\left(2(b+n)(b+n+1) + 2b(b+1) - (2b+n)(2b+n+1)\right)$$

$$= \frac{n(n+1)}{2} + b$$

$=$ triangular number (integer) $+$ any arbitrary positive integer

So $\{m^*\} \equiv \mathbb{Z}^+$

Since m is integer all other subsets fit into $\mathbb{Z}^+$, thus

$$\{m\} \equiv \mathbb{Z}^+$$

so

$$a_n = 4\pi\gamma \ell_p^2 \sqrt{2}\sqrt{n}$$

Theorem: $\{\mathbb{N}\} \equiv \bigcup_{\zeta \in \mathbb{A}} \{\zeta \mathbb{N}^2\}$

for $\mathbb{A}$ = square-free numbers

- square-free number: its prime number factors contains no repeated factor. Example: $15 = 3 \times 5$

proof:

$$\forall b \in \mathbb{N},\ b = p_1^{n_1} \times p_2^{n_2} \times \cdots \times p_i^{n_i}$$

for $p_1, p_2, \cdots, p_i$ all different prime numbers.

for $n_1, n_2, \cdots, n_i$ all positive integers.

$$n_i := 2m_i + k_i$$

$$= \begin{cases} \text{even} ; & k_i = 0 \\ \text{odd} ; & k_i = 1 \end{cases}$$

$$b = \underbrace{\left(p_1^{m_1} \times p_2^{m_2} \times \cdots \times p_i^{m_i}\right)^2}_{\text{square}} \times \underbrace{\left(p_1^{k_1} \times p_2^{k_2} \times \cdots \times p_i^{k_i}\right)}_{\text{square-free}}$$

---

example:

$$80 = 2^4 \times 5$$

$$= \text{square} \times \text{square-free}$$

In general, one can prove:

$$a_{\xi,n} = \left(4\pi\gamma \ell_p^2 \chi\right) \sqrt{\xi}\, n$$

SO(3) :	$\sqrt{2}$	Square-free $\in \mathbb{A}$	integer $\in \mathbb{N}$
SU(2) :	$\frac{1}{2}$	discriminants of all quadratic positive definit forms $\in \mathbb{B}$	integer $\in \mathbb{N}$

$$\underset{\mathbb{A}}{\text{Square-free}} = \{1, 2, 3, 5, 6, 7, 10, 11, 13, \dots\}$$

$$\mathbb{B} = \{3, 4, 7, 8, 11, 15, \dots\}$$

χ : group characteristic parameter.

ξ : generation representative.

- for a fixed ξ_1, the set of areas are

$$\left(4\pi \ell_p^2 \gamma \;\; x \;\; \sqrt{\xi_1}\right)\{1,2,3,4,5,\cdots\}$$

we call $\{a_{\xi_1,n}\}$ the generation ξ_1.

Lemma 1 : $a_{\xi_1,n}$ and $a_{\xi_2,m}$; $\xi_1 \neq \xi_2$

$\nexists\ n,m \in \mathbb{N}$ such that $a_{\xi_1,n} = a_{\xi_2,m}$

Lemma 2 : $a_{\xi_1,n}$ and $a_{\xi_2,m}$; $\xi_1 \neq \xi_2$

$$a_{\xi_1,n} \pm a_{\xi_2,m} \neq a_{\xi_3,\ell}$$

Radiation

If S is a black hole horizon

$$A = \frac{16\pi G^2}{c^4} M^2$$

Quantum theory says:

$$A = A_{n,\xi}$$

$$= \left(4\pi l_p^2 \gamma \times \sqrt{\xi}\right) n$$

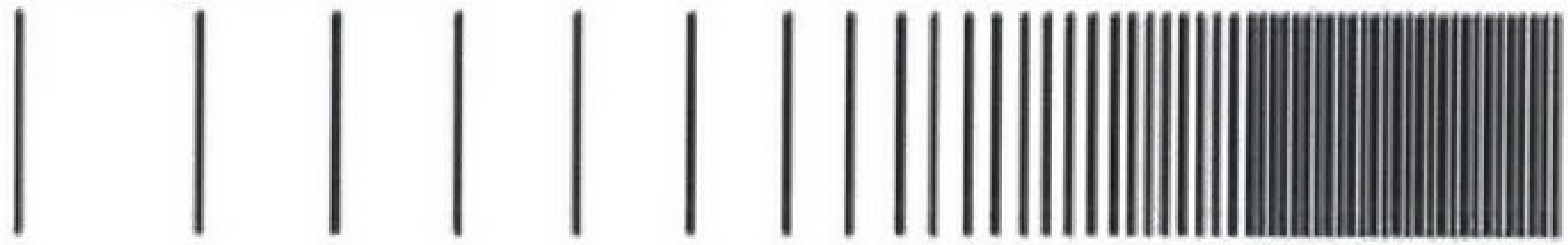
$$= \alpha(\xi) \; n \; l_p^2$$

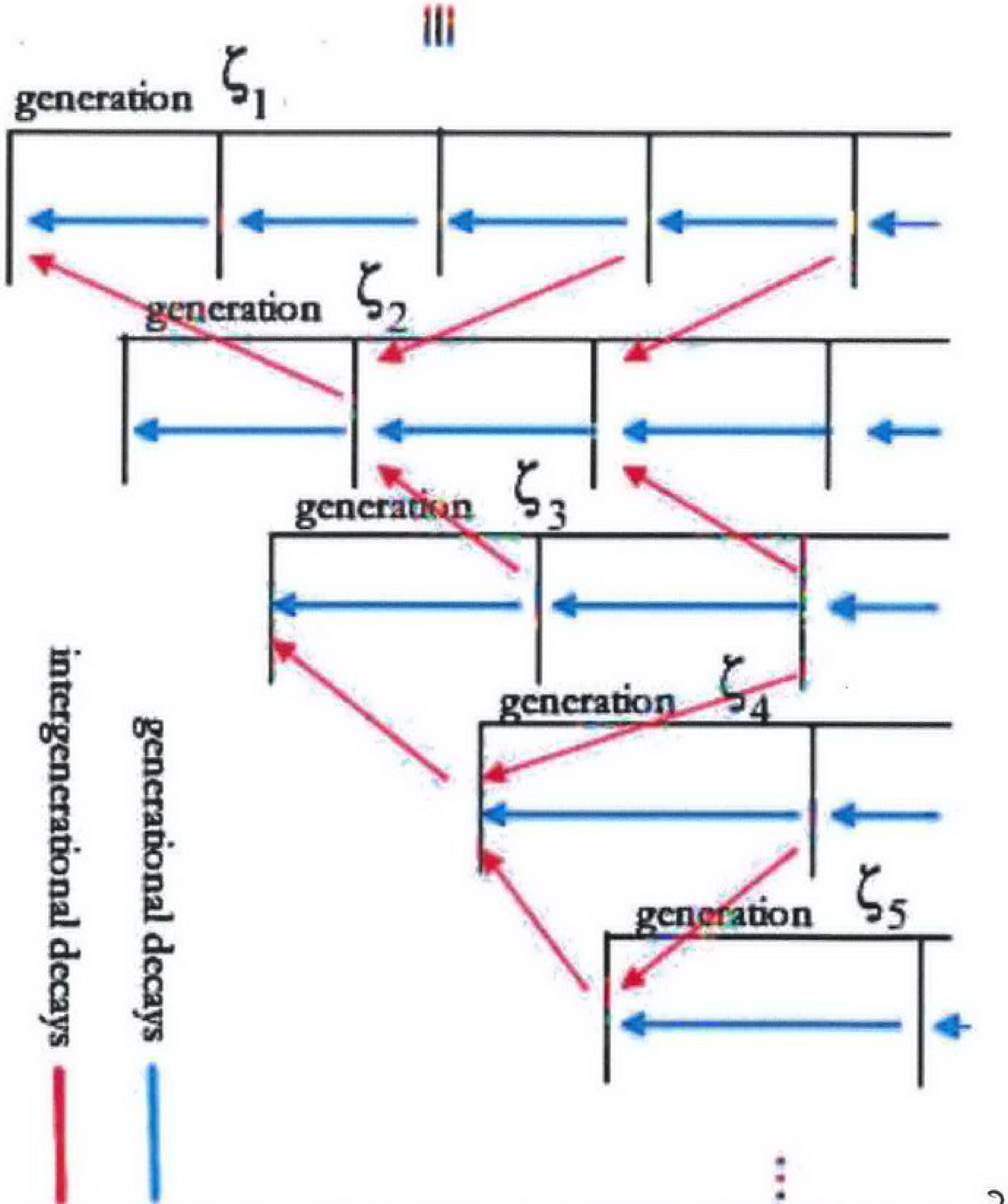
$$\Rightarrow \quad \delta M = \frac{\alpha_\xi \, \delta n}{32\pi M} M_{pl}$$

Assume : Black hole mass does not change during quantum emissions,

$$\delta M \ll M$$

Area eigenvalues





- The fundamental radiation frequency of the generation ξ is

$$\bar{\omega}(\xi) = \frac{\gamma c^3}{8GM} \chi \sqrt{\xi}$$

$$= (\gamma \, \omega_0 \, \chi) \sqrt{\xi}$$

$$\frac{10^{16}}{M_{kg}} (eV)$$

for $M \sim 10^{12}$ kg, $\omega_0 \sim 10$ keV

$$\bar{\omega}(\xi) = (10 \gamma \chi) \cdot \sqrt{\xi} \, (keV)$$

- Harmonic frequencies are

$$\omega(\xi, n) = n \, \bar{\omega}(\xi)$$

- There are also non-harmonics.

For instance:

$$A\ (\text{meter}^2) = 2.77 \times 10^{-53}\ M^2\ (\text{kg}^2)$$

$$T\ (\text{k}) = \frac{1.23 \times 10^{23}}{M\ (\text{kg})}$$

$$M = 10^{12}\ \text{kg}$$

$$A = 2.77 \times 10^{-29}\ (\text{meter}^2)$$

$$T = 1.228 \times 10^{11}\ \text{k}$$

Such a horizon is 40 order of magnitude larger than a quantum of area.

=> Quantum amplification effect is strong enough to make a discrimination in Black hole radiation on certain frequencies.

- probability of time-order decays

→ probability of a jump (of no matter what frequency) in the course of time Δt is $P_{\Delta t}(1)$.

→ probability of no jump is $p_{\Delta t}(0)$

easy to see $$p_{2\Delta t}(0) = p_{\Delta t}(0) \times p_{\Delta t}(0)$$

general solution: $$p_{\Delta t}(0) = e^{-\Delta t/\tau}$$

also $$p_{2\Delta t}(1) = p_{\Delta t}(0) \times p_{\Delta t}(1) + p_{\Delta t}(1) \times p_{\Delta t}(0)$$

therefore $$p_{\Delta t}(1) = \frac{\Delta t}{\tau} e^{-\Delta t/\tau}$$

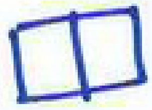
also $$p_{2\Delta t}(2) = 2 p_{\Delta t}(0)\, p_{\Delta t}(2) + \left[p_{\Delta t}(1)\right]^2$$

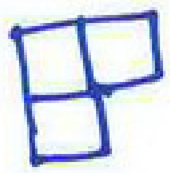
$$p_{\Delta t}(j) = \frac{1}{j!}\left(\frac{\Delta t}{\tau}\right)^j e^{-\frac{\Delta t}{\tau}}$$

Mukhanov, Bekenstein

probability of a decay

a_1  $g(a_1)$

a_2  $g(a_2) + [g(a_1)]^2$

a_3  $g(a_3) + g(a_2)\cdot g(a_1) + [g(a_1)]^3$

$\vdots$

a_N $\quad g(a_N) = [g(a_1)]^N$

each one of the decays has the probability

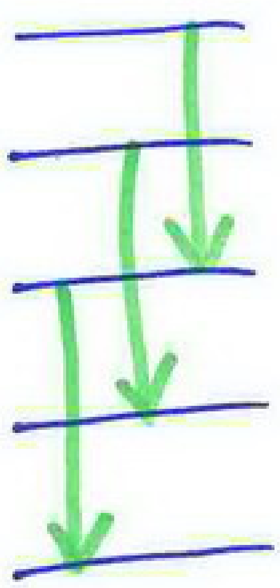
$$p = \frac{1}{[g(a_1)]^N}$$

probability of the decay $a_n - a'_m$ is

$$p = \frac{[g'(a_1)]^m}{[g(a_1)]^n}$$

- From each harmonic frequency there are many copies.

- From each non-harmonics there is only 1.



$N_0 = 5$

$\rho(2\omega_0) = \frac{3}{5}$

Define population weight: $\rho(\omega_n) = \frac{N}{N_0} \sim \frac{1}{\sqrt{5}}$

$$\rho(\omega') \equiv \frac{1}{N_0}$$

Assumption 3: The density matrix elements for quantum transitions are uniform.

$$P_{\Delta t}(\{\omega\}|1) = \frac{1}{C}\,\rho(\omega)\,e^{-\Lambda\omega}$$

Λ : depends on g_1, M

C : normalization parameter

$$\rho(\omega) = \frac{N}{N_0}$$

N → number of quanta

N_0 → number of levels

for large black holes $N_0 \gg 1$

$$\rho(\text{non-harmonics}) = \frac{1}{N_0} \simeq 0$$

$$P_{\Delta t}(\{\omega_m(\xi)\}|1) = \frac{1}{C}\,\rho(\omega_m)\,q_\xi^{-m}$$

$$q_\xi := e^{\pi\gamma\chi\sqrt{\xi}}$$

$$\rho(\omega_m) \simeq \frac{\dfrac{A}{a_{0,\xi}}}{\dfrac{A}{a_{min}}} \sim \frac{1}{\sqrt{\xi}}$$

- $$P_{\Delta t}(\{\omega_n\}) = \frac{\Delta t}{c\tau} e^{-\Delta t/\tau} \rho(\zeta)\, q(\zeta)^{-n}$$

$$\rightarrow c = \sum_{\substack{\text{generations} \\ \zeta}} \frac{\rho(\zeta)}{q(\zeta) - 1}$$

- probability of a sequence of emissions

$$P_{\Delta t}(\{\omega_1, \omega_2, \cdots, \omega_j\}) = \frac{1}{j!}\left(\frac{\Delta t}{c\tau}\right)^j e^{-\frac{\Delta t}{\tau}} \cdot \prod_{i=1}^{j} \rho(\zeta_i)\, q(\zeta_j)^{-j}$$

- probability of k quanta at the same frequency in a sequence of j dimension

- probability of k quanta at the same frequency in all possible sequences of any dimension $> k$

$$p_{\Delta t}\left(k \mid \omega_n(\zeta)\right) = \frac{1}{k!}\left(x_n(\zeta)\right)^k e^{-x_n(\zeta)}$$

where $x_n(\zeta) = \frac{\Delta t}{c\tau}\, \rho(\zeta)\, q(\zeta)^{-n}$

- Intensity

the intensity of $\omega_n(\zeta)$ is the total energy emitted at this frequency per unit time per unit area.

$$\bar{k} = \sum_{k=1}^{\infty} k \, P_{\Delta t}\left(k, \omega_n(\zeta)\right)$$

$$= \frac{\Delta t}{2\tau} \, \rho(\zeta) \, q(\zeta)^{-n}$$

$$\Rightarrow \boxed{I\left(\omega_n(\zeta)\right) = I_0 \, \omega_n(\zeta) \, \rho(\zeta) \, q(\zeta)^{-n}}$$

This distribution is equivalent to the distribution of quanta in a black body if

$$\boxed{T := \frac{\hbar c^3}{8\pi G M k_B}}$$

Width of lines

The mean value of emitting <u>frequencies</u>

$$\langle \omega \rangle := \sum_{\zeta} \sum_{n} \omega_n(\zeta) \; P_{\Delta t}(\omega_n(\zeta)|1)$$

$$= \frac{1}{C} \sum_{\zeta} \overline{\omega}(\zeta) \; \frac{\rho(\zeta)\, q(\zeta)}{(q(\zeta)-1)^2}$$

$$\sim \frac{\omega_0 \gamma \chi}{C} \underbrace{\sum_{\zeta} \frac{q(\zeta)}{(q(\zeta)-1)^2}}_{\substack{\text{Convergent} \\ \eta}}$$

$C = 2 \leftarrow SU(2) \rightarrow \eta \sim 9,$

$C = 0.9 \leftarrow SO(3) \rightarrow \eta \sim 1.7$

- Also the mean value of the number of quanta (no matter of what frequency) is emitted from a black hole is: $\frac{\Delta t}{\tau}$

- The mean decrease of black hole mass during Δt is

$$\frac{\Delta \bar{M}}{\Delta t} = - \frac{\hbar \langle \omega \rangle}{c^2 \tau}$$

- On the other hand, if black hole is a black body, the Stefan-Boltzmann law says:

$$\frac{\Delta \bar{M}}{\Delta t} = - \frac{\hbar c^4}{15360 \pi G^2 M^2}$$

$$\Rightarrow \boxed{\tau = \frac{1920 \ \pi \eta \gamma X}{c \omega_0}}$$

On average the time elapsed before a decay is

$$\bar{t} = \int_{t=0}^{\infty} t \, P_t (j=1) \, dt = 2\tau$$

The uncertainty of elapsing time before a decay is

$$(\Delta t)^2 = \int_{t=0}^{\infty} (t-\tau)^2 P_t (j=1) \, dt = 3\tau^2$$

$$\Delta E \, \Delta t \sim \frac{\hbar}{2} \quad \Rightarrow \boxed{\Delta \omega \sim \frac{1}{\tau}}$$

$$\Delta\omega = 0.00029\,\omega_0 \quad \text{in } SU(2)$$

$$= 0.00009\,\omega_0 \quad \text{in } SO(3)$$

recall: $\omega_0 = \dfrac{c^3}{8\pi G M}$

$M \sim 10^{12}\,\mathrm{kg}$, $\omega_0 \sim 10\ \mathrm{keV}$

$\omega_n \sim 10n\ \mathrm{keV}$

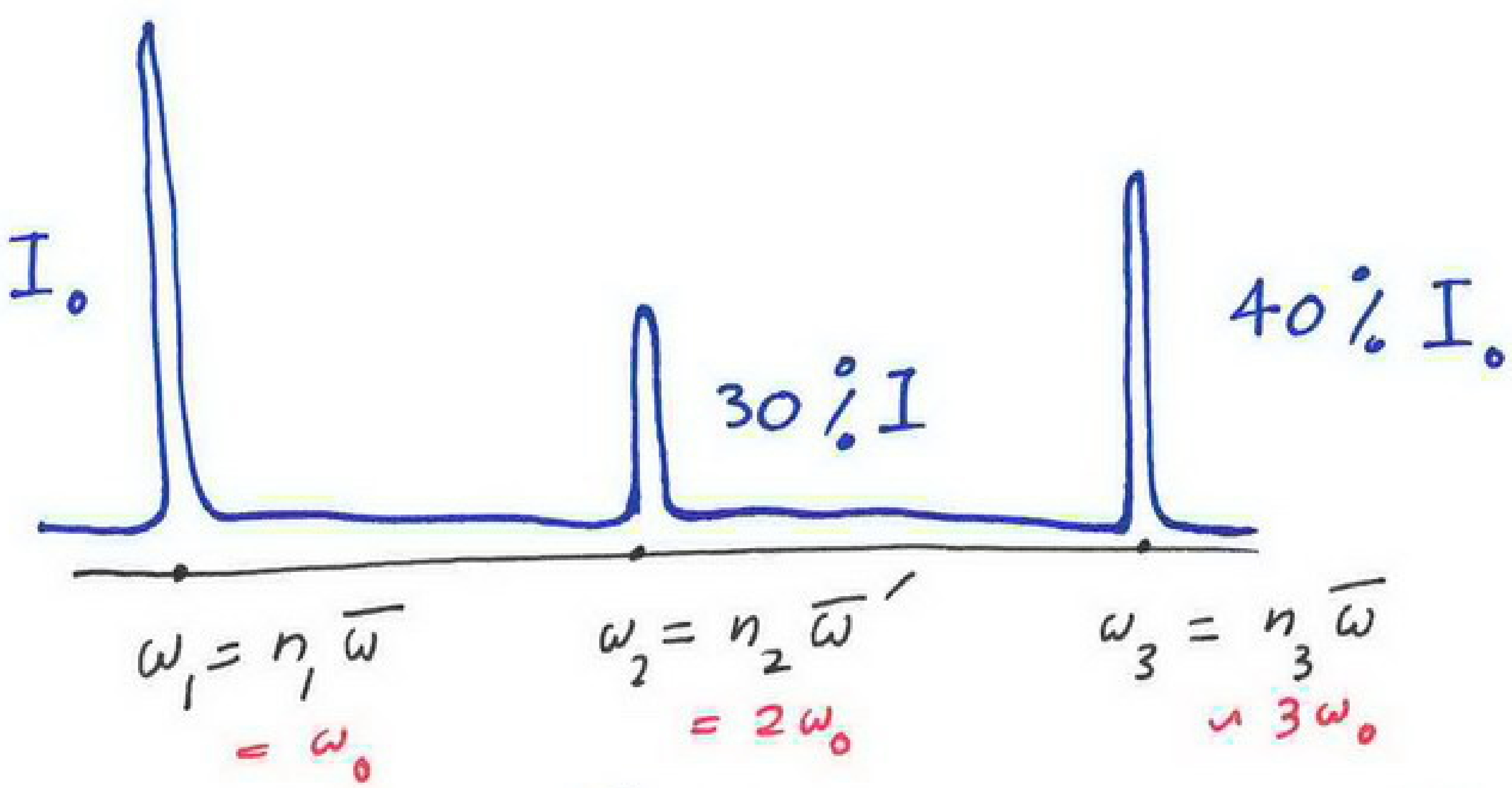
$\Delta\omega \sim 0.0001\ \mathrm{keV}$

The lines are quite sharp.

$$I(\omega_n(\zeta)) \sim \omega_n(\zeta)\, \rho(\zeta)\, e^{-\frac{\omega_n(\zeta)}{\omega_0}}$$

$$\sim n\,\bar{\omega}(\zeta)\, \frac{1}{\bar{\omega}(\zeta)}\, e^{-\frac{\omega_n(\zeta)}{\omega_0}}$$

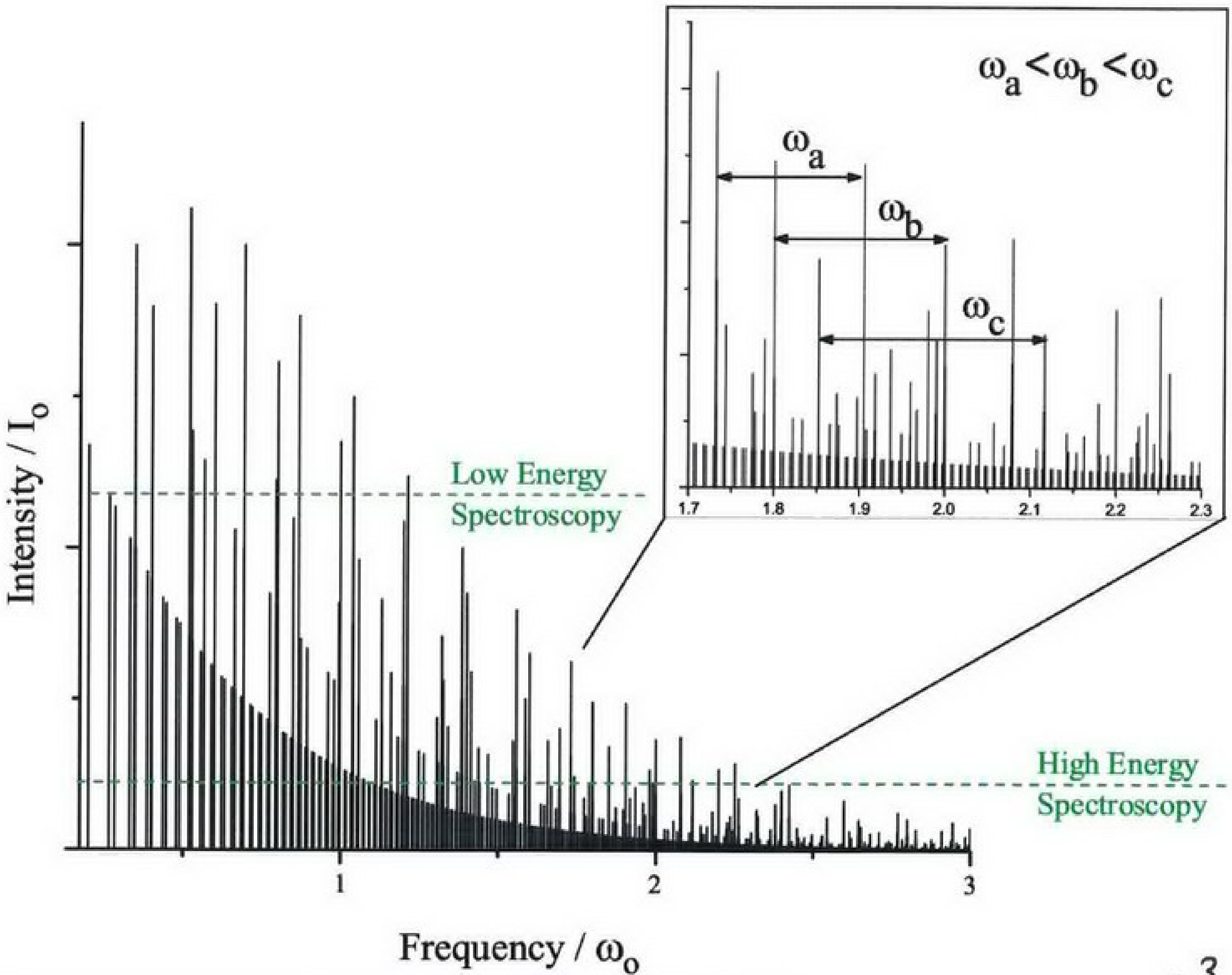
$$\sim n\, e^{-n\sqrt{\zeta}}$$

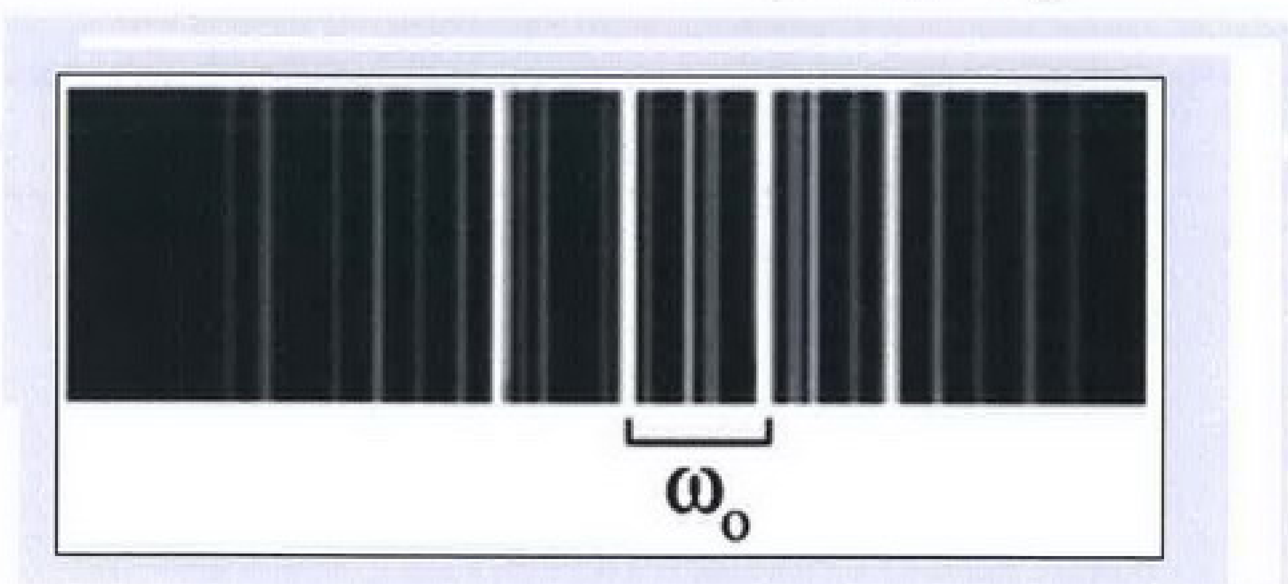


$\zeta = 1 \rightarrow \sqrt{\zeta} = 1$ $\qquad I_1 = I_0\, e^{-1}$

$\zeta' = 5 \rightarrow \sqrt{\zeta} \doteq 2$ $\qquad I_2 = I_0\, e^{-2}$

$n_1 = 1,\ n_2 = 1,\ n_3 = 3$ $\qquad I_3 = I_0\, 3\, e^{-3}$





$$\omega_o = \frac{c^3}{8GM} \times \sqrt{3}$$

Conclusion

1 - A different paradigm to defining a quantum black hole

2 - Its entropy

3 - quantum effect macroscopically observable at least for primordial black holes.

Ref :

Bekenstein and Mukhanov (95)

M. Ansari (06)