Ribbon Algebra ⇝ Ribbon Category ⇝ Invariant of Framed Tangles

Ribbon Algebra $(H, \mu, \eta, \Delta, \varepsilon, S, R, \theta)$ p361

is a braided Hopf algebra (S not necessarily invertible)
with an element θ in H s.t. θ is central

$$\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta)$$
$$\varepsilon(\theta) = 1$$
$$S(\theta) = \theta$$

perhaps should have been "Ribboned"

Ribbon Category $(\mathcal{C}, \otimes, I, c, b, d, \theta)$ p349

$(\mathcal{C}, \otimes, I, c, b, d)$ a strict braided category with left duality

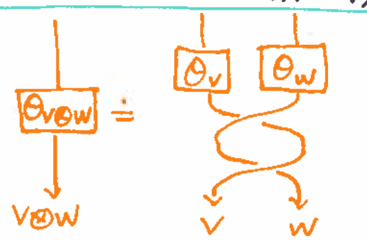
twist:
- θ is a natural isomorphism $\theta_V : V \xrightarrow{\sim} V$ (indexed by $V \in Ob(\mathcal{C})$)

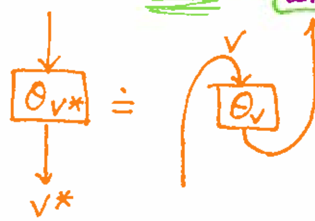
$$\begin{array}{ccc} V & \xrightarrow{\theta_V} & V \\ f\downarrow & & \downarrow f \\ V' & \xrightarrow{\theta_{V'}} & V' \end{array}$$

- s.t. $\forall V, W \in Ob(\mathcal{C})$ $\boxed{\theta_{V\otimes W} = (\theta_V \otimes \theta_W)\, c_{W,V}\, c_{V,W}}$ and $\theta_{V^*} = (\theta_V)^*$

identity might violate this

In general, in a tensor category with left duality, if $f: V \to W$ in $Hom(\mathcal{C})$
$f^* := (d_V \otimes id_{W^*})(id_{V^*} \otimes f \otimes id_{W^*})(id_{V^*} \otimes b_W)$
and $f^*: W^* \to V^*$ in $Hom(\mathcal{C})$

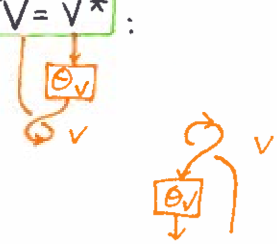




Important Property of Ribbon Category

Using the braiding, the left duality and the twist, we can equip $\mathcal{C}$ with a natural right duality structure where $\boxed{{}^*V = V^*}$:

$\forall V \in Ob(\mathcal{C})$
$$b'_V := (id_{V^*} \otimes \theta_V)\, c_{V,V^*}\, b_V$$
$$d'_V := d_V\, c_{V,V^*}\, (\theta_V \otimes id_{V^*})$$



Proof: Omitted

The Category of Ribbons $\mathcal{R}$ (or of Framed Tangles) p271

As a strict tensor category, $\mathcal{R}$ has the same generating morphisms and relations as those of $\mathcal{T}$ except with (RI) replaced by

(RI')

Note: $\mathcal{R}$ is a ribbon category, with $\theta_{(+)} :=$ (blackboard framing) & $\theta_{(-)} :=$
$\theta_{(\pm)} :=$ (blackboard framing)

Note / Def: Morphisms of $\mathcal{R}$ correspond to isotopy classes of framed tangles/ribbons.

Ribbon Algebra $\rightsquigarrow$ Ribbon Category p361

THM Let H be a braided Hopf algebra.

(H, θ) is a ribbon algebra $\underset{\text{if } H \text{ finite-dim'l}}{\overset{\Rightarrow}{\Leftarrow}}$ H-Mod_f (already shown to be braided with left duality) is a ribbon category.

$\theta := \theta_{\mathcal{T}}(1)^{-1}$ $\qquad$ $\theta_V := \theta^{-1}$

Proof: Exercise.

Ribbon Category $\mathcal{C}$ $\overset{\text{picking vector(s)}}{\rightsquigarrow}$ Invariant of Framed Tangles (Ribbons) p358-360

Invariant of Ribbons (a sensable definition)

is a strict tensor functor from $\mathcal{R}$ to another ribbon category preserving the braiding (ie. braided), the left duality and the twist.

(It's then easy to see that the right duality is automatically preserved.)

Construction of Invariant

Given a ribbon category $\mathcal{C}$ and an object V of $\mathcal{C}$.

Define $F((+)) := V$.

Then we can extend F uniquely to an invariant of ribbons

(i.e. to a strict braided tensor functor $\mathcal{R} \to \mathcal{C}$ preserving left duality and twis

Proof: Exercise / Omitted

In particular,

$F(\times) = c_{V,V}$ $\qquad$ $F(\times) = c_{V,V}^{-1}$

$F(\cup) = b_V$ $\qquad$ $F(\cap) = d_V$

$F(\cup) = b_V'$ $\qquad$ $F(\cap) = d_V'$

$F(\ell) = \theta_V$ $\qquad$ $F(\ell) = \theta_V^{-1}$

Discuss: ① Role of θ

② Universality of $\mathcal{R}$

Quantum Trace and Quantum Dimension

Quantum Trace $tr_q(f)$

Let $\mathcal{C} = (\mathcal{C}, \otimes, I, c, b, d, \theta)$ be a ribbon category, with right duality b' and d' constructed as before.

For any $V \in Ob(\mathcal{C})$ and $f \in End(V)$,

$tr_q(f) := d_V'(f \otimes id_{V^*})\, b_V$  $\in End(I)$.

Quantum Dimension $dim_q(V)$

Let $\mathcal{C}$ be a ribbon category as above.

For any $V \in Ob(\mathcal{C})$,

$dim_q(V) := tr_q(id_V) = d_V' b_V$  V $=$ $\in End(I)$ invariant of the trivial link constructed using $\mathcal{C}$ and $V \in Ob(\mathcal{C})$

Properties of Quantum Trace

For $f, g \in Hom(\mathcal{C})$, $\mathcal{C}$ a ribbon category.

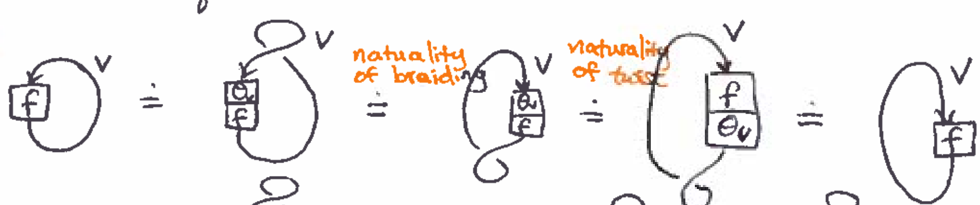
(a)  $\doteq tr_q(f)$

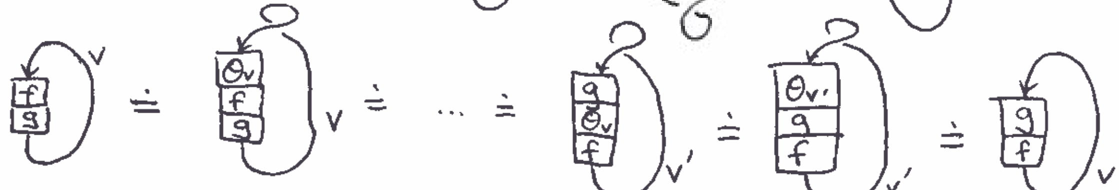
(b) $tr_q(f \circ g) = tr_q(g \circ f)$ whenever

(c) $tr_q(f \otimes g) = tr_q(f) \circ tr_q(g)$

(d) $tr_q(f) = tr_q(f^*)$

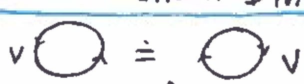
Proof:

(a) 

(b) 

(c) & (d): Long and omitted.

Properties of Quantum Dimension (Corollaries)

(a')  $= dim_q(V)$

(c') $dim_q(V \otimes W) = dim_q(V) \circ dim(W)$

(d') $dim_q(V^*) = dim_q(V)$ $\because$ $id_{V^*} = (id_V)^*$ in a strict tensor category with left duality