Braided bialgebra $\rightsquigarrow$ Braided category $\rightsquigarrow$ "Invariant" of braids

later

Braided bialgebra $(H, \mu, \eta, \Delta, \varepsilon, R)$

$(H, \mu, \eta, \Delta, \varepsilon)$ bialgebra

$R \in H \otimes H$ s.t. R is invertible $\sim c$ isomorphism

$\forall x \in H$, $\Delta^{op}(x) = R\,\Delta(x)\,R^{-1}$ $\sim c$ natural $\sim c$ A-linear

$(\Delta \otimes id_H)(R) = R_{13} R_{23}$ $\sim c$ satisfies (H1)

$(id_H \otimes \Delta)(R) = R_{13} R_{12}$ $\sim c$ satisfies (H2)

} $\boxed{\text{universal } R\text{-matrix}}$

ASIDE: $\Delta^{op}(x) := \tau_{H,H} \circ \Delta(x)$. For any coalgebra $(H, \Delta, \varepsilon$ … $(H, \Delta^{op}, \varepsilon)$ is also a coa… called the $\boxed{\text{opposite coalge…}}$

(Recall:)
Braided category $(\mathcal{C}, \otimes, I, a, \ell, r, c)$

$(\mathcal{C}, \otimes, I, a, \ell, r)$ tensor category

c is a natural isomorphism $c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ } c $\boxed{\text{commutativity constraint}}$

$$\begin{tikzcd} V \otimes W \arrow[r, "c_{V,W}"] \arrow[d, "f\otimes g"] & W \otimes V \arrow[d, "g \otimes f"] \\ {} & {} \end{tikzcd}$$

s.t.

$$\begin{tikzcd}[column sep=small] & (U\otimes V)\otimes W \arrow[dl, "a"'] \arrow[dr, "c\otimes id"] & \\ U\otimes(V\otimes W) \arrow[d, "c"'] & & (V\otimes U)\otimes W \arrow[d, "a"] \\ (V\otimes W)\otimes U \arrow[dr, "a"'] & & V\otimes(U\otimes W) \arrow[dl, "id\otimes c"] \\ & V\otimes(W\otimes U) & \end{tikzcd}$$

(H1)

$$\begin{tikzcd}[column sep=small] & (U\otimes V)\otimes W \arrow[dl, "a"'] \arrow[dr, "c^{-1}\otimes id"] & \\ U\otimes(V\otimes W) \arrow[d, "c^{-1}"'] & & (V\otimes U)\otimes W \arrow[d, "a"] \\ (V\otimes W)\otimes U \arrow[dr, "a"'] & & V\otimes(U\otimes W) \arrow[dl, "id\otimes c^{-1}"] \\ & V\otimes(W\otimes U) & \end{tikzcd}$$

(H2)

} c $\boxed{\text{braiding}}$

Hexagon Axiom

A braided (tensor) category is strict if $a, \ell, r = id$.

In a strict tensor category,


(H1) $\Leftrightarrow$ $c_{U, V\otimes W} = (id_V \otimes c_{U,W}) \circ (c_{U,V} \otimes id_W)$


(H2) $\Leftrightarrow$ $c_{V\otimes W, U} = (c_{V,U} \otimes id_W) \circ (id_V \otimes c_{W,U})$

} $\Rightarrow$ Take $U = V = W$, $c = c_{V,V}$.

$$\boxed{\begin{aligned}(c\otimes id_V)(id_V\otimes c)(c\otimes id_V) \\ = (id_V\otimes c)(c\otimes id_V)(id_V\otimes c)\end{aligned}} \quad \boxed{c \in Aut(V\otimes V)}$$

i.e. $c = c_{V,V}$ is a solution of the Yang-Baxter equation

} c $\boxed{\text{R-matr…}}$

Proof: omitted.
Use (H1), (H2) and naturality of c.

k-Vector spaces as a braided (tensor) category $\big(\underset{C}{\mathrm{Vect}(k)}, \underset{\otimes}{\otimes}, \underset{I}{k}, \underset{a}{id}, \underset{l}{id}, \underset{r}{id}, \underset{c}{\tau}\big)$

$C = \mathrm{Vect}(k)$ vector spaces over k (strictly speaking, after some identifications)

$\otimes$ the usual tensor product for vector spaces

$I = k$ the ground field

$$\left.\begin{aligned} a_{u,v,w}((u\otimes v)\otimes w) &= u\otimes(v\otimes w) \\ l_v(1\otimes v) = v &= r_v(v\otimes 1) \\ c_{v,w}(v\otimes w) = \tau_{v,w}(v\otimes w) &= w\otimes v \end{aligned}\right\} \forall u \in U, v \in V, w \in W$$

Check: a, l, r, c are natural isomorphisms
a satisfies ⬠ axiom.
l, r, a satisfy Δ axiom.
c, a satisfy (H1) & (H2).

"Comultiplication" & "counit" $\xrightarrow{\text{on algebra } A}$ induced A-module structure on $\otimes$ & k

A an (associative unital k-)algebra.

$\Delta : A \to A\otimes A$ alg. morph. $\boxed{\text{"comultiplication"}}$
$\varepsilon : A \to k$ alg. morph. $\boxed{\text{"counit"}}$

assumed A-module structure for these objects in A-Mod

Using alg. morph. Δ, we can define an A-module structure on $\underset{\text{vector space } \otimes}{U\otimes V}$ (U, V A-modules) by
$$a.(u\otimes v) = \Delta(a)(u\otimes v)$$ $\boxed{\text{"Tensor A-Module Structure"}}$

Using alg. morph. ε, we can define an A-module structure on k by
$$a.\lambda = \varepsilon(a)\lambda$$ $\boxed{\text{"Trivial A-Module Structure"}}$

$\because \Delta, \varepsilon$ are alg. morph.'s

THM: ($\overset{A}{\text{bialg}} \leftrightarrow \overset{\text{A-Mod}}{\text{tensor}}$ subcategory of Vect(k) ; $\underset{\text{braided}}{A} \leftrightarrow \underset{\text{braided}}{\text{A-Mod}}$)

Let A be an algebra with "comultiplication" Δ and "counit" ε.

p285 a) (A, Δ, ε) is a bialgebra $\Leftrightarrow$ the category A-Mod is a tensor subcategory of Vect(k) (i.e. A-Mod is a tensor category with $a, l, r = id$)

p318 b) $(A, \Delta, \varepsilon, \underset{R := \tau_{H,H}(c_{H,H}(1\otimes 1))}{\ })$ is a braided bialg. $\Leftrightarrow$ the category A-Mod is a braided tensor category with $a, l, r = id$.

$c^R_{V,W} := \tau_{V,W} \circ R$

Proof: Next time.