Module $\leftrightarrow$ Comodule Duality (over finite-dim'l algebra/coalgebra)

4.a

$$
\begin{tikzcd}
V^* \arrow[r, "\Delta_{V^*}"] \arrow[d, "{}_V\mu^*"'] & V^* \otimes H^* \arrow[d, "{}_{V^*}\mu"] \arrow[dl, "\bar\lambda"'] \\
(V \otimes H)^* \arrow[r, "\Delta_V^*"'] & V^*
\end{tikzcd}
$$

$$\bar\lambda(\alpha \otimes \beta)(a \otimes b) = \alpha(a)\,\beta(b)$$

$\bar\lambda$ is an isomorphism of vector spaces

$\Leftrightarrow$ H or V is finite-dim'l.

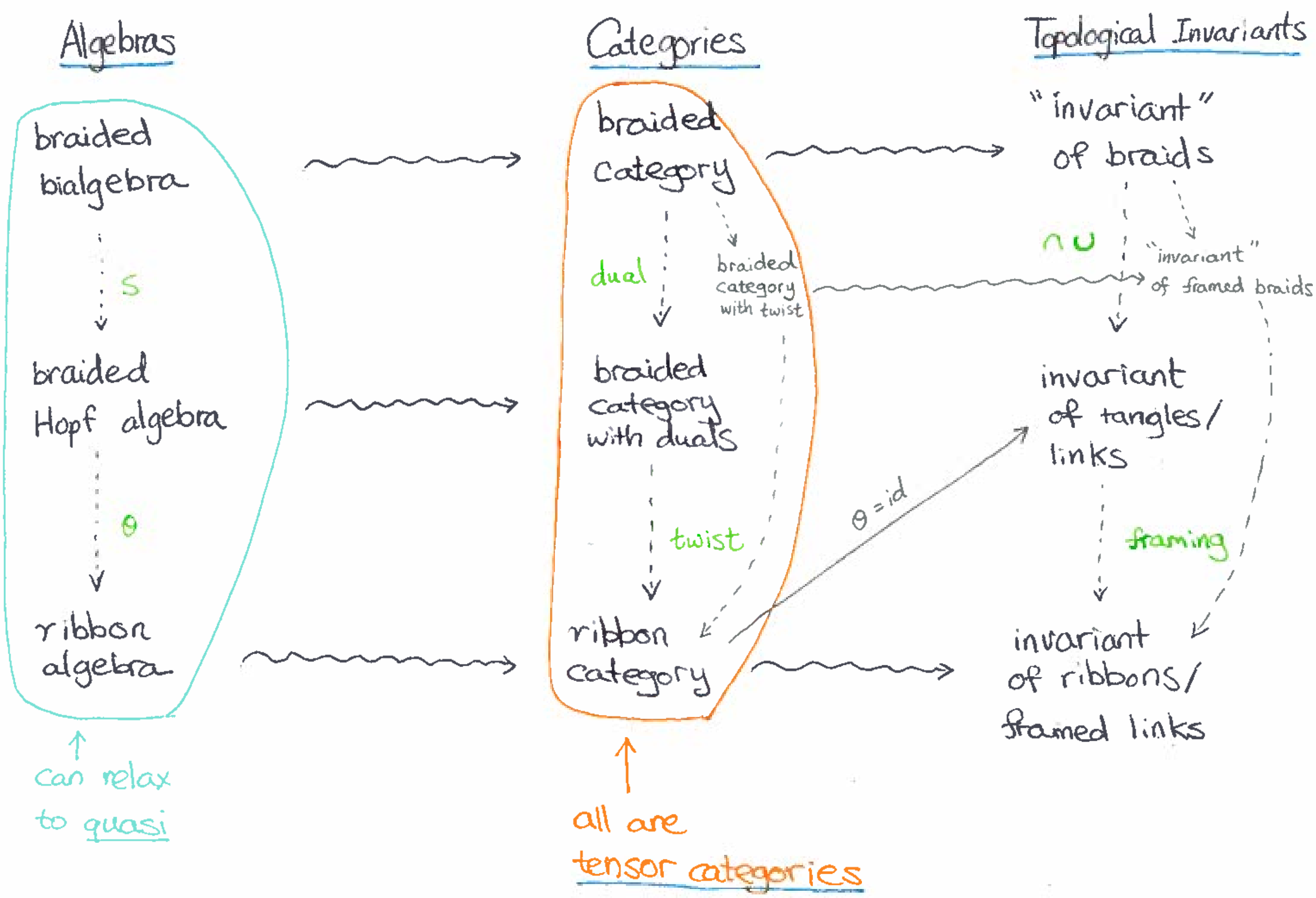
H left-comodule (V, Δ_V) $\rightsquigarrow$ H^* right-module $(V^*, {}_{V^*}\mu := \Delta_V^* \circ \bar\lambda)$

H right-module $(V, {}_V\mu)$ $\xrightarrow[\text{finite-dim'l}]{\text{if } H \text{ (or V)}}$ H^* left-comodule $(V^*, \Delta_{V^*} := \bar\lambda^{-1} \circ {}_V\mu^*)$

Categorical Language

Motivations: ① Categorical description of modules over algebras provides a link between algebra and topological invariants.

② Categorical description leads to generalization of the algebraic objects.



Category Theory Basics

Category $\mathcal{C} = (Ob(\mathcal{C}), Hom(\mathcal{C})\ ;\ s, b, \circ, id)$

$Ob(\mathcal{C})$ the class (("set", for us)) of objects of $\mathcal{C}$

$Hom(\mathcal{C})$ the class of morphisms (between objects) of $\mathcal{C}$

ASIDE

Representing a category as a (big) gra

object V $\quad \overset{V}{\bullet}$

morphism $f: V \to W$ $\quad \overset{V}{\bullet} \xrightarrow{f} \overset{W}{\bullet}$

(compare to concrete categories)

4 maps

source $\quad s: Hom(\mathcal{C}) \to Ob(\mathcal{C})$

target $\quad b: Hom(\mathcal{C}) \to Ob(\mathcal{C})$

For $f \in Hom(\mathcal{C})$, let $V := s(f)$, $W := b(f)$. Then we also write $f: V \to W$.

composition $\quad \circ: Hom(\mathcal{C}) \times_{Ob(\mathcal{C})} Hom(\mathcal{C}) \to Hom(\mathcal{C})$

$Hom(\mathcal{C}) \times_{Ob(\mathcal{C})} Hom(\mathcal{C})$ denotes the class of pairs of composable morphisms in the category

$(f, g) \in Hom(\mathcal{C}) \times_{Ob(\mathcal{C})} Hom(\mathcal{C})$ if $b(f) = s(g)$.

$U \xrightarrow{f} V \xrightarrow{g} W$

s.t. $(h \circ g) \circ f = h \circ (g \circ f)$ $\forall\ f, g, h \in Hom(\mathcal{C})$ for which the expression makes sense

(Composition is associative)

identity $\quad id: Ob(\mathcal{C}) \to Hom(\mathcal{C})$ s.t. for any $V \in Ob(\mathcal{C})$,

$$\begin{cases} id_V : V \to V \\ id_{b(f)} \circ f = f = f \circ id_{s(f)} \quad \forall f \in Hom(\mathcal{C}) \end{cases}$$

(id behaves like an algebraic identity)

Functor (between categories) $\quad F: \mathcal{C} \to \mathcal{C}'$

roughly speaking, they are morphisms in the category of categories.

$$F: \begin{cases} Ob(\mathcal{C}) \to Ob(\mathcal{C}') \\ Hom(\mathcal{C}) \to Hom(\mathcal{C}') \end{cases}$$

s.t. F respects s, b, $\circ$, id

i.e.

$s(F(f)) = F(s(f))$

$b(F(f)) = F(b(f))$

$F(g \circ f) = F(g) \circ F(f)$

$F(id_V) = id_{F(V)}$

$$\begin{array}{c} V \\ \downarrow f \\ W \end{array} \xrightarrow{F} \begin{array}{c} F(V) \\ \downarrow F(f) \\ F(W) \end{array}$$

$$\begin{array}{c} \downarrow f \\ \downarrow g \end{array} \xrightarrow{F} \begin{array}{c} \downarrow F(f) \\ \downarrow F(g) \end{array}$$

$$V \circlearrowleft^{id} \xrightarrow{F} F(V) \circlearrowleft^{id}$$

Natural Transformation (between functors) $\quad \eta: F \to G$ (where $F, G: \mathcal{C} \to \mathcal{C}'$)

Isomorphism

is a family of morphisms (isomorphisms) in $\mathcal{C}'$ indexed by objects of $\mathcal{C}$.

[ie. $\eta: Ob(\mathcal{C}) \to Hom(\mathcal{C}')$ $Iso(\mathcal{C}')$]

s.t.

$$\begin{tikzcd} V \arrow[d, "f"] & F(V) \arrow[r, "\eta_V"] \arrow[d, "F(f)"] & G(V) \arrow[d, "G(f)"] \\ W & F(W) \arrow[r, "\eta_W"] & G(W) \end{tikzcd}$$

$\forall f \in Hom(\mathcal{C})$

If η is a natural isomorphism

For any $f \in Hom(\mathcal{C})$, one can use η to recover $G(f)$ from $F(f)$, or vice versa.

Tensor Category $(\mathcal{C}, \otimes, I, a, \ell, r)$

$\mathcal{C} = (Ob(\mathcal{C}), Hom(\mathcal{C}); s, b, \circ, id)$ a category

tensor product: $\otimes$ a functor from $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$

would need to define category structure on $\mathcal{C}\times\mathcal{C}$
objects of $\mathcal{C}\times\mathcal{C}$ $(V, W) \in Ob(\mathcal{C}) \times Ob(\mathcal{C})$
morphisms of $\mathcal{C}\times\mathcal{C}$ $(f, g) \in Hom(\mathcal{C}) \times Hom(\mathcal{C})$
$s(f,g) := (s(f), s(g))$
$b(f,g) := (b(f), b(g))$
$(f', g') \circ (f, g) := (f' \circ f, g' \circ g)$ for composable morphisms
$id_{(V,W)} := (id_V, id_W)$

More explicitly written out:

$(V, W) \in Ob(\mathcal{C}) \times Ob(\mathcal{C}) \mapsto V \otimes W \in Ob(\mathcal{C})$

$(f, g) \in Hom(\mathcal{C}) \times Hom(\mathcal{C}) \mapsto f \otimes g \in Hom(\mathcal{C})$

s.t. $s(f \otimes g) = s(f) \otimes s(g)$

$b(f \otimes g) = b(f) \otimes b(g)$

$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$

$id_{V \otimes W} = id_V \otimes id_W$

unit: $I \in Ob(\mathcal{C})$ is a special object of $\mathcal{C}$ (see ℓ and r)

associativity constraint: a is a natural isomorphism

$$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$$
$$\downarrow (f \otimes g) \otimes h \qquad \downarrow f \otimes (g \otimes h)$$

s.t. Pentagon Axiom

$$\begin{tikzcd}
 & ((U \otimes V) \otimes W) \otimes X \arrow[dl, "a \otimes id"'] \arrow[dr, "a"] & \\
(U \otimes (V \otimes W)) \otimes X \arrow[d, "a"'] & & (U \otimes V) \otimes (W \otimes X) \arrow[ddl, "a"] \\
U \otimes ((V \otimes W) \otimes X) \arrow[dr, "id \otimes a"'] & & \\
 & U \otimes (V \otimes (W \otimes X)) &
\end{tikzcd}$$

left constraint: ℓ a natural isomorphism

$$\ell_V : I \otimes V \xrightarrow{\sim} V$$
$$\downarrow id_I \otimes f \qquad \downarrow f$$

s.t. Triangle Axiom

$$\begin{tikzcd}
(V \otimes I) \otimes W \arrow[rr, "a"] \arrow[dr, "r \otimes id"'] & & V \otimes (I \otimes W) \arrow[dl, "id \otimes \ell"] \\
 & V \otimes W &
\end{tikzcd}$$

right constraint: r a natural isomorphism

$$r_V : V \otimes I \xrightarrow{\sim} V$$
$$\downarrow f \otimes id_I \qquad \downarrow f$$