Hopf Algebras $\mathcal{U}(\mathfrak{sl}(2))$ and $\mathcal{O}_{SL(2)}$

Let's apply the Hopf algebra constructions we did last time to the algebraic group $SL(2)$.

$\mathfrak{g} = \mathfrak{sl}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_{2\times 2}(k) \mid \alpha + \delta = 0 \right\}$

Take basis of $\mathfrak{g}$: $\left\{ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

$$\mathcal{U}(\mathfrak{sl}(2)) = T(\mathfrak{sl}(2)) \Big/ \begin{matrix} XY - YX = H \\ HX - XH = 2X \\ HY - YH = -2Y \end{matrix} = T(k\langle X, Y, H\rangle)/\sim \ \underset{\text{product in } \mathcal{U}(\mathfrak{g}) \text{ is } \otimes}{\overset{\text{use tensor basis}}{=}} \ k\{X, Y, H\}/\sim$$

$G = SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(k) \mid ad - bc = 1 \right\}$, $M_{2\times 2}(k) = k^4$

$$\mathcal{O}_{SL(2)} = k[a,b,c,d] \big/ \{ f \in k[a,b,c,d] : f|_{SL(2)} \equiv 0 \}$$
$$= k[a,b,c,d] / (ad - bc - 1)$$

$\because$ $(ad-bc-1)$ radical ideal $\because$ $ad-bc-1$ is irreducible poly

$$\boxed{\mathcal{U}(\mathfrak{sl}(2))} \leftarrow G = SL(2) \text{ algebraic group} \rightarrow \boxed{\mathcal{O}_{SL(2)}}$$

(follows from $\mathcal{U}(\mathfrak{g})$) (follows from $\mathcal{O}_G$)

algebra
μ tensor product in $\mathcal{U}(\mathfrak{sl}(2))$ (mod Lie bracket rel'ns)
$\eta(1_k) = 1_{\mathcal{U}(\mathfrak{sl}(2))} = 1_k$

coalgebra
$\Delta(x) = 1 \otimes x + x \otimes 1 \quad \forall x \in \mathfrak{sl}(2)$
$\varepsilon(x) = 0 \quad \forall x \in \mathfrak{sl}(2)$

antipode
$S(x) = -x \quad \forall x \in \mathfrak{sl}(2)$

coalgebra (enough to define on alg. generators)

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a\otimes a + b \otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{pmatrix} \overset{!!}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\because$ eg $\lambda(\Delta(a))(g\otimes h) = a(g\cdot h)$ should pick out the (1,1)-entry of $g \cdot h$; on the other hand, $\lambda(a\otimes a + b\otimes c)(g\otimes h)$ does exactly this.
(λ is an isomorphism in this case.)

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

algebra
μ polynomial multiplication (mod rel'ns)
$\eta(1_k) =$ constant polynomial 1

antipode
$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(Think: Heisenberg picture vs. Schrödinger picture

Modules and Comodules

Motivations: ① Modules are basically synoym for representations, and we all know why rep'ns are important.

Comodules are perhaps a bit less natural, but they are the dual objects to modules. We need them to make the theory nice.

② Modules and comodules are key to definition of Hopf algebra duality.

③ They give rise to categorical descriptions of the underlying (∞-dim'l) algebra/coalgebra.

① Left – Module (V, μ_V) over algebra (H, μ, η)

② Right – Comodule (V, Δ_V) over coalgebra (H, Δ, ε)

V vector space

Δ_V $\mu_V : H\otimes V \rightleftarrows V$ linear map s.t.

$$\begin{tikzcd}
H\otimes H\otimes V \arrow[r, shift left, "\mu\otimes id"] \arrow[d, shift right, "id\otimes\mu_V"'] & H\otimes V \arrow[l, shift left, "\Delta\otimes id"] \arrow[d, shift left, "\mu_V"] \\
H\otimes V \arrow[u, shift right, "id\otimes\Delta_V"'] \arrow[r, shift right, "\mu_V"'] & V \arrow[u, shift left, "\Delta_V"] \arrow[l, shift right, "\Delta_V"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k\otimes V \arrow[r, shift left, "\eta\otimes id"] \arrow[dr, "\sim"', "1\otimes v \mapsto v"] & H\otimes V \arrow[l, shift left, "\varepsilon\otimes id"] \arrow[d, shift left, "\mu_V"] \\
 & V \arrow[u, shift left, "\Delta_V"]
\end{tikzcd}$$

$(a\,b).v = a.(b.v)$ $\qquad$ $1.v = v$

③ Module Morphism $f : (V, \mu_V) \longrightarrow (V', \mu_{V'})$

v.sp. map $f: V \to V'$ linear

respects μ_V $f\circ\mu_V = \mu_{V'}\circ(id\otimes f) \; : H\otimes V \longrightarrow V$

④ Comodule Morphism $f:(V, \Delta_V) \longrightarrow (V', \Delta_{V'})$

v.sp. map $f: V \to V'$ linear

respects Δ_V $(id\otimes f)\circ\Delta_V = \Delta_{V'}\circ f$

⑤ Example of Modules

(a) An algebra is a module over itself : $\mu_V := \mu$, compare commutative diagrams

(b) Representation of finite group $G \leftrightarrow$ module over $k[G]$: compare module axioms with group action axioms

• (c) Representation of Lie algebra $\mathfrak{g} \leftrightarrow$ module over $\mathcal{U}(\mathfrak{g})$

if H is a bialgebra:

(d) Trivial H-module structure on any v.sp. V : $\boxed{\mu_V(a\otimes v) \overset{\text{write as}}{=} a.v} := \varepsilon(a)v$, $\because \varepsilon$ is an alg. morph.
↑ counit of the bialg. H

(e) Tensor H-module structure on $U\otimes V$ where U, V are H-modules : $a.(u\otimes v) := \bar{\lambda}(\Delta(a))(u\otimes v)$ $\because \Delta$ is an alg. morph. and $U\otimes V$ is an $A\otimes A$-mod
↑ comultiplication of bialg. H

⑥ Examples of Comodules

(a) A coalgebra is a left/right-comodule over itself : $\Delta_V := \Delta$, compare commutative diagrams

if H is a bialgebra:

(b) Trivial H-comodule structure on any v.sp. V : $\Delta_V(v) := \eta(1)\otimes v$
↑ unit of bialg. H

(c) Tensor H-comodule Structure on $U\otimes V$: $\Delta_V := (\mu\otimes id_U\otimes id_V)\circ(id\otimes\tau_{U,H}\otimes id_V)\circ(\Delta_U\otimes\Delta_V)$ where U, V are H-comodules
↑ multip. of bialg. H