

# Hopf Algebras $U(\mathfrak{sl}(2))$ and $\mathcal{O}_{SL(2)}$

3.a

Let's apply the Hopf algebra constructions we did last time to the algebraic group  $SL(2)$ .

$$\mathfrak{g} = \mathfrak{sl}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_{2 \times 2}(k) \mid \alpha + \delta = 0 \right\}$$

Take basis of  $\mathfrak{g}$   $\{X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$

$$U(\mathfrak{sl}(2)) = T(\mathfrak{sl}(2)) \Big/ \begin{matrix} XY - YX = H \\ HY - YH = 2X \\ HX - XH = 2Y \end{matrix} = T(k\langle X, Y, H \rangle) \Big/ \begin{matrix} \text{use tensor} \\ \text{basis} \\ \downarrow \\ k\langle X, Y, H \rangle \\ \uparrow \\ \text{product in } U(\mathfrak{g}) \\ \text{is } \otimes \end{matrix}$$

$$G = SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(k) \mid ad - bc = 1 \right\}$$

"k"

$$\mathcal{O}_{SL(2)} = k[a, b, c, d] \Big/ \{f \in k[a, b, c, d] : f|_{SL(2)} = 0\}$$

$$= k[a, b, c, d] \Big/ (ad - bc - 1)$$

:: radical ideal ::  $ad - bc - 1$  is irreducible poly.

$U(\mathfrak{sl}(2))$

(follows from  $U(\mathfrak{g})$ )

$G = SL(2)$   
algebraic group

$\mathcal{O}_{SL(2)}$

(follows from  $\mathcal{O}_a$ )

algebra  $\mu$  tensor product in  $U(\mathfrak{sl}(2))$  (mod Lie bracket rel's)  
 $\eta(1_k) = 1_{U(\mathfrak{sl}(2))} = 1_k$

coalgebra  $\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}$

enough to define on alg. generators

!!  
 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\therefore$  eg.  $\lambda(\Delta(a))(g \otimes h) = a(g \cdot h)$  should pick out the (1,1)-entry of  $g \cdot h$ ;  
on the other hand,  
 $\lambda(a \otimes a + b \otimes c)(g \otimes h)$  does exactly that  
( $\lambda$  is an isomorphism in this case.)

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

coalgebra  $\Delta(x) = 1 \otimes x + x \otimes 1 \quad \forall x \in \mathfrak{sl}(2)$   
 $\varepsilon(x) = 0 \quad \forall x \in \mathfrak{sl}(2)$

algebra  $\mu$  polynomial multiplication (mod rel's)  
 $\eta(1_k) = \text{constant polynomial } 1$

antipode  $S(x) = -x \quad \forall x \in \mathfrak{sl}(2)$

antipode  $S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(Think: Heisenberg picture vs. Schrödinger picture)

# Modules and Comodules

3.6

**Motivations:** ① Modules are basically synonym for representations, and we all know why reps are important.

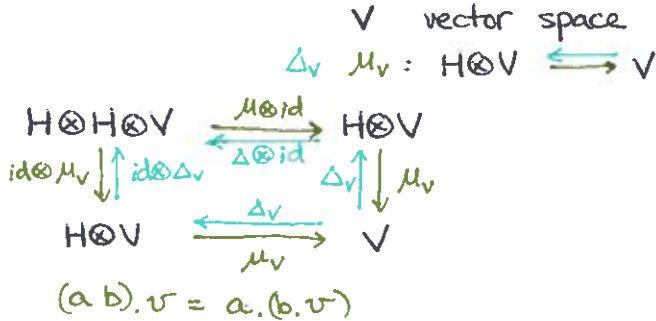
Comodules are perhaps a bit less natural, but they are the dual objects to modules. We need them to make the theory nice.

② Modules and comodules are key to definition of Hopf algebra duality.

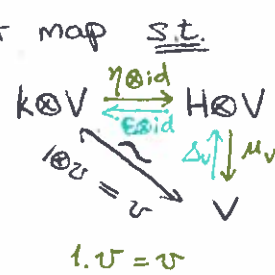
③ They give rise to categorical descriptions of the underlying (co-)algebra/coalgebra.

Left-

① Module  $(V, \mu_V)$   
over algebra  $(H, \mu, \eta)$



② Comodule  $(V, \Delta_V)$   
over coalgebra  $(H, \Delta, \epsilon)$



③ Module Morphism  $f: (V, \mu_V) \rightarrow (V', \mu_{V'})$

v.sp. map  $f: V \rightarrow V'$  linear

respects  $\mu_V$   $f \circ \mu_V = \mu_{V'} \circ (\text{id} \otimes f) : H \otimes V \rightarrow V'$

④ Comodule Morphism  $f: (V, \Delta_V) \rightarrow (V', \Delta_{V'})$

v.sp. map  $f: V \rightarrow V'$  linear

respects  $\Delta_V$   $(\text{id} \otimes f) \circ \Delta_V = \Delta_{V'} \circ f$

⑤ Example of Modules

(a) An algebra is a module over itself:  $\mu_V = \mu$ , compare commutative diagrams

(b) Representation of finite group  $G \leftrightarrow$  module over  $k[G]$ : compare module axioms with group action axioms.

(c) Representation of Lie algebra  $\mathfrak{g} \leftrightarrow$  module over  $U(\mathfrak{g})$

if  $H$  is a bialgebra { (d) Trivial  $H$ -module structure on any v.sp.  $V$ :  $\mu_V(a \otimes v) = \epsilon(a)v$ ,  $\because \epsilon$  is an alg. morph. write as  $a \cdot v$   $\uparrow$  counit of the bialg.  $H$

(e) Tensor  $H$ -module structure on  $U \otimes V$  where  $U, V$  are  $H$ -modules:  $a \cdot (u \otimes v) = \epsilon(\Delta(a))(u \otimes v)$   $\because \Delta$  is an alg. morph.  $\uparrow$  comultiplication of bialg.  $H$  and  $U \otimes V$  is an  $A \otimes A$ -mod

⑥ Examples of Comodules

(a) A coalgebra is a comodule over itself:  $\Delta_V = \Delta$ , compare commutative diagrams

if  $H$  is a bialgebra { (b) Trivial  $H$ -comodule structure on any v.sp.  $V$ :  $\Delta_V(v) = \eta(1) \otimes v$   $\uparrow$  unit of bialg.  $H$

(c) Tensor  $H$ -comodule structure on  $U \otimes V$ :  $\Delta_V := (\mu \otimes \text{id}_U \otimes \text{id}_V) \circ (\text{id}_H \otimes \tau_{U, H} \otimes \text{id}_V) \circ (\Delta_U \otimes \Delta_V)$   $\uparrow$  mult. of bialg.  $H$