$\boxed{2.0}$

Motivations:

① One way to obtain a quantum group is by taking a nontrivial formal deformation (to be defined (involving $\hbar$)) of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$

OR of the algebra of regular functions $\mathcal{O}_G$ on a semisimple algebraic group G

② Last time, we talked about the dual pair of Hopf algebras coming from a finite group

$$k[G] \longleftarrow G \text{ finite group} \longrightarrow (k[G])^*$$

Now, what if G is not finite?

The above construction would yield ∞-dim'l $k[G]$, and in general, it's not clear what $k[G]^*$ is. (Can be huge!)

It turns out that for G an (semisimple) algebraic group, the "right" construction is

$$\mathcal{U}(\mathfrak{g}) \longleftarrow G \text{ algebraic group} \longrightarrow \mathcal{O}_G$$

filtered algebras

have a countable basis (not too big)

Note: When the ground field k is "not smooth" (eg. if k is discrete), one would need to work with a purely algebraic definition of $\mathfrak{g}$ (the "Lie algebra" associated to the algebraic group G).

Universal Enveloping Algebra of a Lie Algebra

Lie Algebra $(\mathfrak{g}, [\,,\,])$

$\mathfrak{g}$ vector space

$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bilinear s.t.

antisymmetry $[x,y] = -[y,x]$

Jacobi identity $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

$\forall x,y,z \in \mathfrak{g}$

Lie bracket

Note: One could think of a Lie algebra $\mathfrak{g}$ as the tangent space at 1_G of its corresponding Lie group. (if $k = \mathbb{R}$ or $\mathbb{C}$)

Universal Enveloping Algebra $(\mathcal{U}(\mathfrak{g}), i_\mathfrak{g})$

Motivations: ① Often nicer to work with an associative algebra rather than a Lie algebra ($\because$ it has the additional structure of multiplication, and is well-studied)

② rep'ns of $\mathfrak{g}$ $\longleftrightarrow$ modules over $\mathcal{U}(\mathfrak{g})$

Given a Lie algebra $\mathfrak{g}$, define its universal enveloping algebra $(\mathcal{U}(\mathfrak{g}), i_\mathfrak{g})$ as

$\boxed{\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g}) / I(\mathfrak{g})}$ where $\boxed{T(\mathfrak{g}) := k \oplus \mathfrak{g} \oplus (\mathfrak{g}\otimes\mathfrak{g}) \oplus \cdots}$ tensor algebra

$I(\mathfrak{g})$ is the algebra ideal of $T(\mathfrak{g})$ generated by $\{x\otimes y - y\otimes x - [x,y]\}_{x,y\in\mathfrak{g}}$

together with a Lie algebra inclusion

$i_\mathfrak{g} : \mathfrak{g} \hookrightarrow L(\mathcal{U}(\mathfrak{g}))$ where for an associative algebra A, define a Lie algebra $\boxed{L(A)}$ s.t. $\begin{cases} L(A) = A \text{ as vector spaces} \\ [x,y] := xy - yx \ (\in A)\ \forall x,y \in A. \end{cases}$

Universal Property of $\mathcal{U}(\mathfrak{g})$

We have, via $\mathcal{U}$ and L, the following maps

{Lie algebras} {associative algebras}

$\mathfrak{g} \longmapsto \mathcal{U}(\mathfrak{g})$

$L(A) \longmapsfrom A$

Note: In general, neither $\mathfrak{g} = L(\mathcal{U}(\mathfrak{g}))$ nor $A = \mathcal{U}(L(A))$

They satisfy

$$\boxed{\mathrm{Hom}_{Lie}(\mathfrak{g}, L(A)) \overset{\text{bij.}}{\cong} \mathrm{Hom}_{Alg}(\mathcal{U}(\mathfrak{g}), A) \quad \text{for any associative algebra } A}.$$

i.e. For any Lie algebra morphism $f : \mathfrak{g} \to L(A)$, $\exists$ unique algebra morphism $\varphi : \mathcal{U}(\mathfrak{g}) \to A$ s.t. $\varphi \circ i_\mathfrak{g} = f$.

$\hookrightarrow$ linear map of the underlying vector spaces which preserves $[\,,\,]$

Regular Functions on an Algebraic Group

Motivation: The resulting Hopf algebra $\mathcal{O}_G$ is, in some cases, "dual" to the Hopf algebra $\mathcal{U}(\mathfrak{g})$ (where $\mathfrak{g}$ is the Lie algebra of G).

Algebraic Group $(G, \cdot, 1_G, (\ )^{-1})$

$(G, \cdot, 1_G, (\ )^{-1})$ satisfies the usual axioms for a group.

G is an (affine) algebraic variety. ← $G \subset k^n$ (for some n) s.t. $\exists$ polynomials (over k) in n variables $p_1, \cdots, p_m$ for which

$$G = \{(z_1, \cdots, z_n) \in k^n : p_i(z_1, \cdots, z_n) = 0 \ \forall i \in \{1, \cdots, m\}\}$$

multiplication $\cdot : G \times G \to G$
inverse $(\ )^{-1} : G \to G$
} are regular maps

regular maps: maps where each coordinate function (domain variety $\to k$) is a regular function

Regular Functions on G $\ \mathcal{O}_G$

(same definition if G is merely an (affine) algebraic variety)

Basically, they are polynomial (regular) functions in the coordinates $(z_1, \cdots, z_n)$ if we view $G \subset k^n$; two functions f & g are the same as functions on G when $f|_G = g|_G$ or $(f-g)|_G = 0$

Therefore, formally

$$\boxed{\mathcal{O}_G := k[z_1, \cdots, z_n] \big/ \{ f \in k[z_1, \cdots, z_n] : f|_G \equiv 0 \}}$$

Motivation: For an algebraic group (variety), we can study the algebraic object $\mathcal{O}_G$ instead of the geometric object G without losing information.

Hopf Algebra Structures on $\mathcal{U}(\mathfrak{g})$ and $\mathcal{O}_G$

Motivations: ① More examples of Hopf algebras ② Will "deform" to produce families of quantum groups. $(\hbar \neq 0)$

$\boxed{\mathcal{U}(\mathfrak{g})}$ ⇝ G algebraic group ⇝ $\boxed{\mathcal{O}_G}$

(compare with $k[G]$) (Compare with $(k[G])^*$) straight-forward correspondence

$\mathcal{U}(\mathfrak{g})$:

filtered algebra:
μ tensor product (mod ...)
$\eta(1_k) = 1_k \in k \hookrightarrow \mathcal{U}(\mathfrak{g})$

coalgebra:
$\Delta(x) = 1 \otimes x + x \otimes 1 \quad \forall x \in \mathfrak{g}$

$$\Rightarrow \Delta(x_1 \cdots x_n) = 1 \otimes x_1 \cdots x_n + \sum_{p=1}^{n-1} \sum_{\substack{\sigma \\ (p,q)\text{-shuffles}}} \underbrace{x_{\sigma(1)} \cdots x_{\sigma(p)}}_{\in\, \mathcal{U}(\mathfrak{g})} \otimes \underbrace{x_{\sigma(p+1)} \cdots x_{\sigma(n)}}_{\in\, \mathcal{U}(\mathfrak{g})} + x_1 \cdots x_n \otimes 1 \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

$\varepsilon(x) = 0 \quad \forall x \in \mathfrak{g} \qquad (\varepsilon(1) = 1_k)$

antipode: $S(x) = -x \quad \forall x \in \mathfrak{g}$

$\Rightarrow S(x_1 \cdots x_n) = (-1)^n x_n \cdots x_1$

$\because$ S is an alg. antimorphism (Recall, $S : H \to H$) — $\mathcal{U}(\mathfrak{g})$ here $= H$

i.e. S is an alg. morph. from H to H^{op}

$\boxed{H^{op} = (H, \mu^{op}, \eta)}$ is the opposite algebra of the algebra (H, μ, η)

where $\mu^{op}(a \otimes b) := \mu(b \otimes a) \quad \forall a, b \in H$

$\mathcal{O}_G$:

coalgebra:
$$\left.\begin{array}{l} \bar{\lambda}(\Delta(f))(g \otimes h) = f(g \cdot h) \quad \forall g, h \in G \\ \varepsilon(f) = f(1_G) \end{array}\right\} \forall f \in \mathcal{O}_G$$

filtered algebra:
μ polynomial product (mod ...)
$\eta(1_k) =$ constant polynomial 1

antipode: $\underbrace{S(f)}_{\in\, \mathcal{O}_G}(g) = f(g^{-1}) \quad \forall g \in G \quad \forall f \in \mathcal{O}_G$.