

# Hopf Algebras

**Motivations:** Quantum groups are a special class of Hopf algebras.  
 The category of modules over a Hopf algebra has some nice properties.  
 ↳ tensor category (:: compatible coalg. structure) "tensor product  $\mathcal{H}_1 \otimes \mathcal{H}_2$  composite system"  
 ↳ with left duality (:: antipode  $S$ ) "pair creation & annihilation"

**Ingredients:** Hopf algebra  $(H, \mu, \eta, \Delta, \epsilon, S)$

vector space  $H$  over  $k$  (always)

- ① algebra structure  $(H, \mu, \eta)$
  - ② coalgebra structure  $(H, \Delta, \epsilon)$
  - ③ antipode  $S$
- compatibility condition  $\left\{ \begin{array}{l} \mu, \eta \text{ are coalg. maps} \\ \Delta, \epsilon \text{ are alg. maps} \end{array} \right.$   
 defn uses  $\mu$  and  $\Delta$ , but not their compatibility

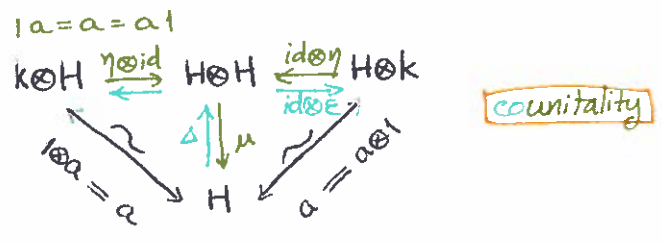
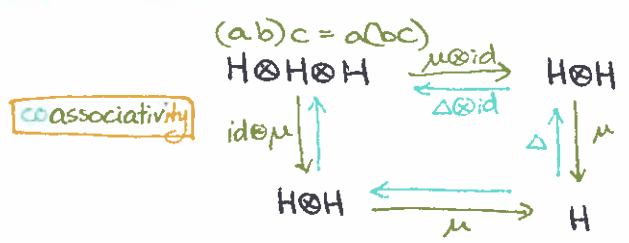
Now, I just have to explain what ①-③ mean.

- ① Algebra  $(H, \mu, \eta)$
- ② Coalgebra  $(H, \Delta, \epsilon)$

$H$  vector space  $\rightarrow$  addition & scalar multiplication

**comultiplication**  $\Delta$   $\mu: H \otimes H \rightarrow H$   
**counit**  $\epsilon$   $\eta: k \rightarrow H$

} linear maps st.



③ Algebra morphism  $f: (A, \mu, \eta) \rightarrow (A', \mu', \eta')$

means  $\left\{ \begin{array}{l} f: A \rightarrow A' \text{ linear} \\ f \text{ respects } \mu \quad \mu' \circ (f \otimes f) = f \circ \mu \quad \text{OR} \quad f(a)f(b) = f(ab) \quad \forall a, b \in A \\ f \text{ respects } \eta \quad f \circ \eta = \eta' \quad \text{OR} \quad f(1_A) = 1_{A'} \end{array} \right.$

④ Coalgebra morphism  $f: (C, \Delta, \epsilon) \rightarrow (C', \Delta', \epsilon')$

means  $\left\{ \begin{array}{l} f: C \rightarrow C' \text{ linear} \\ f \text{ respects } \Delta \quad (f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{OR} \quad \sum_{(x,y)} f(x') \otimes f(y'') = \sum_{(f(x))} f(x)' \otimes f(y'') \quad \forall x \in C \\ f \text{ respects } \epsilon \quad \epsilon = \epsilon' \circ f \quad \text{OR} \quad \epsilon(x) = \epsilon'(f(x)) \quad \forall x \in C \end{array} \right.$

③  $\Leftrightarrow$  ④  
 clear once you write out the 4 eqn's (in each case)

- ③ Since  $\Delta, \epsilon$  map between  $k, H, H \otimes H$ , to understand what it means for them to be algebra morphisms, we need to define alg. structures on  $k$  and  $H \otimes H$ .  
 $(k, \cdot, 1) \quad (H \otimes H, (\mu \otimes id) \circ (id \otimes \tau_{H,H} \otimes id), \eta \otimes \eta)$   
 $\uparrow$  induced by  $(H, \mu, \eta)$  alg. structure
- ④ Since  $\mu, \eta$  map between  $k, H, H \otimes H$ , ... coalg. structures on  $k$  and  $H \otimes H$ .  
 $(k, \Delta_0, \epsilon_0) \quad \Delta_0(1) = 1 \otimes 1, \epsilon_0(1) = 1$   
 $(H \otimes H, (id \otimes \tau_{H,H} \otimes id) \circ (\Delta \otimes \Delta), \epsilon \otimes \epsilon) \leftarrow$  induced by  $(H, \Delta, \epsilon)$  coalg. structure

⑤ Antipode S (for bialgebra H)

$(\text{End}(H), *, 1 \mapsto \eta \circ \epsilon)$  forms an algebra.  $\leftarrow$  can check

For  $A, B \in \text{End}(H), H \xrightarrow{\epsilon} k \xrightarrow{\eta} H$

$(A * B)(h) = \underbrace{\mu((A \otimes B)(\Delta(h)))}_{\text{End}(H)} \in H$

$\leftarrow$  Remember multiplication  $*$ :  $\text{End}(H) \otimes \text{End}(H) \rightarrow \text{End}(H)$ .  
so it is enough to define on pure tensors  
bilinearity of '\*' follows from bilinearity of  $\mu$

$S \in \text{End}(H)$  st  $S * \text{id}_H = \eta \circ \epsilon = \text{id}_H * S$

Since  $(\text{End}(H), *, 1 \mapsto \eta \circ \epsilon)$  is only an algebra, such S might not exist.  
However, if such S exists, then it is unique.

Example: (of Hopf algebras)



$k[G]$  vector space over k with basis  $\{g_1, \dots, g_n\} = G$

$(k[G])^*$  dual vector space of  $k[G]$

$\mu(g_i \otimes g_j) = g_i g_j \in G \leftrightarrow k[G]$   
 $\eta(1_k) = 1_G$

$(\Delta(f))(x \otimes y) = f(x \cdot y) \quad \forall x, y \in k[G] \quad \forall f \in (k[G])^*$   
 $\epsilon(f) = f(1_G)$

$\Delta(g_i) = g_i \otimes g_i \quad \forall g_i \in G$   
 $\epsilon(g_i) = 1_k$

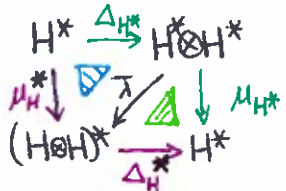
$\mu(f \circ g)(g_i) = (f \circ g)(\Delta(g_i)) = f(g_i) g_i \quad \forall g_i \in G \quad \forall f, g \in (k[G])^*$   
 $\eta(1_k) = \epsilon$

Compatibility  
 $\Delta(g_i g_j) = g_i g_j \otimes g_i g_j = (g_i \otimes g_j)(g_i \otimes g_j) = \Delta(g_i) \Delta(g_j)$   
 $\epsilon(g_i g_j) = 1 = 1 \cdot 1 = \epsilon(g_i) \epsilon(g_j)$   
 $\Delta(1_G) = 1_G \otimes 1_G \leftarrow$  unit in  $k[G] \otimes k[G]$   
 $\epsilon(1_G) = 1$

also compatible

$S(g_i) = g_i^{-1} \quad \forall g_i \in G$

$S(f)(g_i) = f(S(g_i)) = f(g_i^{-1}) \quad \forall g_i \in G \quad \forall f \in (k[G])^*$



coalgebra  $(H, \Delta_H, \epsilon_H) \rightarrow$  algebra structure on  $H^*$   
 $(H^*, \mu_H^*, \eta_H^* := \epsilon_H^*)$

algebra  $(H, \mu_H, \eta_H) \xrightarrow{\text{if } H \text{ finite-dim}} \text{coalgebra } (H^*, \Delta_{H^*}, \epsilon_{H^*} := \eta_H^*)$

$\bar{\lambda}(\alpha \otimes \beta)(a \otimes b) = \alpha(a) \beta(b)$

Corollary bialgebra  $(H, \mu_H, \eta_H, \Delta_H, \epsilon_H) \xrightarrow{\text{if } H \text{ finite-dim}} \text{bialgebra } (H^*, \mu_{H^*}, \eta_{H^*}, \Delta_{H^*}, \epsilon_{H^*})$

Proof check compatibility on H  $\Rightarrow$  compatibility on  $H^*$ .

Corollary Hopf algebra  $(H, \mu_H, \eta_H, \Delta_H, \epsilon_H, S_H) \xrightarrow{\text{if } H \text{ finite-dim}} \text{Hopf algebra } (H^*, \mu_{H^*}, \eta_{H^*}, \Delta_{H^*}, \epsilon_{H^*}, S_{H^*} := S_H^*)$

Proof Check  $S_{H^*}$  is an antipode for  $(H^*, \dots)$ .