Hopf Algebras

Motivations: Quantum groups are a special class of Hopf algebras.
The category of modules over a Hopf algebra has some nice properties.
→ tensor category ($\because$ compatible coalg. structure) "tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ composite system"
→ with left duality ($\because$ antipode S) "pair creation & annihilation"

Ingredients: Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$

vector space H over k (always)

① algebra structure (H, μ, η)
② coalgebra structure (H, Δ, ε)
} compatibility condition { ④ μ, η are coalg. maps ⇕ (OR) ③ Δ, ε are alg. maps }
⑤ antipode S defn uses μ and Δ, but not their compatibility

Now, I just have to explain what ①-⑤ mean.

① Algebra (H, μ, η) ② Coalgebra (H, Δ, ε)

H vector space ⇝ addition & scalar multiplication

comultiplication Δ $\quad \mu: H\otimes H \rightleftarrows H$
counit ε $\quad \eta: k \rightleftarrows H$
} linear maps s.t.

$(ab)c = a(bc)$ $\qquad\qquad$ $1a = a = a1$

coassociativity

$$\begin{tikzcd}
H\otimes H\otimes H \arrow[r, shift left, "\mu\otimes id"] \arrow[d, shift left] & H\otimes H \arrow[l, shift left, "\Delta\otimes id"] \arrow[d, shift left, "\mu"] \\
H\otimes H \arrow[u, shift left, "id\otimes\mu"] \arrow[r, shift right, "\mu"'] & H \arrow[u, shift left, "\Delta"] \arrow[l, shift right]
\end{tikzcd}
\qquad
\begin{tikzcd}
k\otimes H \arrow[r, shift left, "\eta\otimes id"] \arrow[dr, "\sim", "1\otimes a \,=\, a"'] & H\otimes H \arrow[l, shift left] \arrow[r, shift right] \arrow[d, shift left, "\mu"] & H\otimes k \arrow[l, shift right, "id\otimes\eta"'] \arrow[l, shift left, "id\otimes\varepsilon"] \arrow[dl, "\sim"', "a \,=\, a\otimes 1"] \\
& H \arrow[u, shift left, "\Delta"] &
\end{tikzcd}$$

counitality

③ Algebra morphism $f: (A, \mu, \eta) \longrightarrow (A', \mu', \eta')$

means $\begin{cases} f: A \to A' \text{ linear} \\ \mu' \circ (f\otimes f) = f\circ\mu \\ f\circ\eta = \eta' \end{cases}$

f respects μ: $\mu'\circ(f\otimes f) = f\circ\mu$ OR $f(a)f(b) = f(ab)$ $\forall a, b \in A$

f respects η: $f\circ\eta = \eta'$ OR $f(1_A) = 1_{A'}$

④ Coalgebra morphism $f: (C, \Delta, \varepsilon) \longrightarrow (C', \Delta', \varepsilon')$

means $\begin{cases} f: C \to C' \text{ linear} \\ (f\otimes f)\circ\Delta = \Delta'\circ f \\ \varepsilon = \varepsilon'\circ f \end{cases}$

f respects Δ: $(f\otimes f)\circ\Delta = \Delta'\circ f$ OR $\sum_{(x)} f(x')\otimes f(x'') = \sum_{(f(x))} f(x)'\otimes f(x)''$ $\forall x \in C$

f respects ε: $\varepsilon = \varepsilon'\circ f$ OR $\varepsilon(x) = \varepsilon'(f(x))$ $\forall x \in C$

③ ⇔ ④ clear once you write out the 4 eqn's (in each case)

③ Since Δ, ε map between $k, H, H\otimes H$, to understand what it means for them to be algebra morphisms, we need to define alg. structures on k and $H\otimes H$. $(k, \cdot, 1)$ $\quad (H\otimes H, (\mu\otimes\mu)\circ(id\otimes\tau_{H,H}\otimes id), \eta\otimes\eta)$ ← induced by (H, μ, η) alg. structure

④ Since μ, η map between $k, H, H\otimes H$, ... coalg. structures on k and $H\otimes H$
$(k, \Delta_0, \varepsilon_0)$ $\quad \Delta_0(1) = 1\otimes 1$, $\varepsilon_0(1) = 1$
$(H\otimes H, (id\otimes\tau_{H,H}\otimes id)\circ(\Delta\otimes\Delta), \varepsilon\otimes\varepsilon)$ ← induced by (H, Δ, ε) coalg. structure

⑤ Antipode S (for bialgebra H)

$(\mathrm{End}(H), *, 1 \mapsto \eta\circ\varepsilon)$ forms an algebra. ← can check

For $A, B \in \mathrm{End}H$, $H \xrightarrow{\varepsilon} k \xrightarrow{\eta} H$

$$\underbrace{(A * B)}_{\mathrm{End}(H)}\underbrace{(h)}_{H} = \mu((A\otimes B)(\Delta(h))) \in H$$

← Remember multiplication $* : \mathrm{End}(H)\otimes\mathrm{End}(H) \to \mathrm{End}(H)$, so it is enough to define on pure tensors; bilinearity of '$*$' follows from bilinearity of μ

$S \in \mathrm{End}(H)$ s.t. $S * id_H = \eta\circ\varepsilon = id_H * S$

Since $(\mathrm{End}(H), *, 1 \mapsto \eta\circ\varepsilon)$ is only an algebra, such S might not exist.
However, if such S exists, then it is unique.

Example: (of Hopf algebras)

$\boxed{\text{Group algebra of } G}$ ⟵ G finite group ⟶ $\boxed{\text{Algebra of Functions on } G}$

$\boxed{k[G]}$ vector space over k with basis $\{g_1, \dots, g_n\} = G$

$\boxed{(k[G])^*}$ dual vector space of $k[G]$

$\mu(g_i \otimes g_j) = g_i g_j \in G \hookrightarrow k[G]$ $\forall g_i, g_j \in G$ ⟷ $\bar\lambda(\Delta(f))(x\otimes y) = f(\overset{\mu}{xy})$ $\forall x, y \in k[G]$

$\eta(1_k) = 1_G$ ⟷ $\varepsilon(f) = f(1_G)$ $\quad \forall f \in (k[G])^*$

$\left.\begin{array}{l}\Delta(g_i) = g_i \otimes g_i \\ \varepsilon(g_i) = 1_k\end{array}\right\} \forall g_i \in G$ ⟷ $\mu(f\otimes g)(g_i) = \bar\lambda(f\otimes g)(\Delta(g_i)) = f(g_i)g(g_i)$ $\forall g_i \in G$ $\forall f, g \in (k[G])^*$

⟷ $\eta(1_k) = \varepsilon$

compatibility:

$\Delta(g_i g_j) = g_i g_j \otimes g_i g_j$
$= (g_i \otimes g_i)(g_j \otimes g_j) = \Delta(g_i)\Delta(g_j)$ ← multiplication in $k[G]\otimes k[G]$

$\varepsilon(g_i g_j) = 1 = 1\cdot 1 = \varepsilon(g_i)\varepsilon(g_j)$ $\quad$ also compatible

$\Delta(1_G) = 1_G \otimes 1_G$ ← unit in $k[G]\otimes k[G]$

$\varepsilon(1_G) = 1$

$S(g_i) = g_i^{-1}$ $\forall g_i \in G$ $\quad\quad$ $S(f)(g_i) = f(S(g_i)) = f(g_i^{-1})$ $\forall g_i \in G$ $\forall f \in k[G]^*$

$\boxed{\text{Algebra} \leftrightarrow \text{Coalgebra Duality}}$ $\xrightarrow{\text{induces}}$ $\boxed{\text{Hopf algebra Duality}}$

$$\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^*\otimes H^* \arrow[dl, "\bar\lambda"'] \arrow[d, "\mu_{H^*}"] \\
(H\otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}$$

$\bar\lambda(\alpha\otimes\beta)(a\otimes b) = \alpha(a)\beta(b)$

◺ coalgebra $(H, \Delta_H, \varepsilon_H)$ ⟶ algebra structure on H^* $(H^*, \mu_{H^*}, \eta_{H^*} := \varepsilon_H^*)$

◸ algebra (H, μ_H, η_H) $\xrightarrow[\text{finite-dim}]{\text{if } H}$ coalgebra $(H^*, \Delta_{H^*}, \varepsilon_{H^*} := \eta_H^*)$

Corollary bialgebra $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$ $\xrightarrow[\text{finite-dim}]{\text{if } H}$ bialgebra $(H^*, \mu_{H^*}, \eta_{H^*}, \Delta_{H^*}, \varepsilon_{H^*})$

Proof Check compatibility on H ⟹ compatibility on H^*.

Corollary Hopf algebra $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H, S_H)$ $\xrightarrow[\text{finite-dim}]{\text{if } H}$ Hopf algebra $(H^*, \mu_{H^*}, \eta_{H^*}, \Delta_{H^*}, \varepsilon_{H^*}, S_{H^*} := S_H^*)$

Proof Check S_H^* is an antipode for $(H^*, \dots)$